If $U$ is total energy in $[0,t_0)$, given by

$$U = \frac{1}{2}\int_B (\rho_0 \dot{u}_i \dot{u}_i + \rho_0 I_{ij}\dot{\varphi}_i\dot{\varphi}_j + \rho_0 k\dot{\nu}\dot{\nu} + 2W + a\theta^2)dV \tag{5.7}$$

then from (5.7),(5.6),(5.5) we have

$$\begin{aligned}\dot{U} - \int_B \rho_0(f_i\dot{u}_i + \ell_i\dot{\varphi}_i - \ell\dot{\nu} + \frac{1}{T_0}s\theta)dV - \int_{\partial B}(t_i\dot{u}_i + m_i\dot{\varphi}_i - \\ - h\dot{\nu} + \frac{1}{T_0}s\theta)d\sigma = -\frac{1}{T_0}\int_B q_i\theta_{,i}dV \le 0\end{aligned} \tag{5.8}$$

If the boundary value problem has two different solutions $(u_i, \varphi_i, \nu_i, T)$ corresponding to the same boundary conditions and initial values, then their difference will satisfy the same system with homogeneous boundary and initial conditions. If $U_1(t)$ is the total energy corresponding to the difference of the solutions we have

$$\dot{U}_1 - \int_B \rho_0(f_i^*\dot{u}_i^* + \ell_i^*\dot{\varphi}_i^* - \ell^*\nu^* + \frac{1}{T_0}s^*\theta^*)dV = -\frac{1}{T_0}\int_B q_i\theta^*_{,i}dV.$$

But $f_i^* = 0,\ \ell_i^* = 0,\ \ell^* = 0,\ s^* = 0$ and

$$\dot{U}_1^* \le 0 \tag{5.9}$$

that means that $U_1(t) \le U_1(0) = 0$. From (5.1)-(5.3) it follows that $U_1(t) = 0$, $\forall t \in [0,t_0)$, $u_i^* = 0$, $\varphi_i^* = 0$, $\nu^* = 0$, $\theta^* = 0$ and the proof is completed.

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# HERMITIAN OPERATORS ON $C^*$-ALGEBRAS

BY

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**1.Introduction.** We first introduce some basic notions with which we work.

**Definition 1.1.** *Let $E$ be an complex Banach algebra with unit element $I$ and dual space $E'$. An element $a$ of $E$ is called* **hermitian** *if its numerical range $V(E,a)$, where*

$$V(E,a) = \{f(a) : f \in E',\ f(I) = 1 = \|f\|\}$$

*is a subset of the real field $\mathbb{R}$, or equivalently, if $\|exp(i\alpha a)\| = 1$ for all $\alpha \in \mathbb{R}$.*

**Definition 1.2.** *A $C^*$-algebra is a complex Banach algebra $X$ with an involution $*$, satisfying $\|x^*x\| = \|x\|^2$ for all $x$ in $X$. $X$ is said to be an* **irreducible** *or a* **primitive $C^*$-algebra** *if there exists a faithful irreductible representation from $X$ into $BL(X)$, the Banach algebra of all bounded opertors on a suitable Hilbert space $K$. An ideal $P$ in $X$ is called* **primitive** *if it is the kernel of an irreductible representation. A* **structure $C^*$-algebra** *is a $C^*$-algebra $X$ such that the set, $Prim(X)$, of all primitive ideals in $X$ is a Hausdorff space when endowed with the Jacobson topology.*

For details, refer to [3],[10].

**Definition 1.3.** *A* **derivation** *on a $C^*$-algebra $X$ is a linear mapping $D$ of $X$ to itself such that $D(ab) = aDb + (Da)b$, for any $a, b \in X$. $D$ is called* **an inner derivation** *if there exists an element $c \in X$ such that $Da = ca - ac$, for all $a \in X$. A derivation $D$ is said to be a* **star derivation** *if $Da^* = -(Da)^*$, ($a^*$ is the adjoint of $a$).*

Throughout this paper, unless mentioned otherwise, $X$ denotes a complex unital $C^*$-algebra with all derivations inner and the algebraic element

$\alpha I$ is denoted by simply the scalar $\alpha$. In [5], we proved that every Hermitian operator $H$ on $BL(K)$ is of the form $L_x + R_y$, where $L_x$ is left multiplication by a Hermitian $x$ and $R_y$ is right multiplication by a Hermitian $y$ and that its norm is given by

$$\|H\| = \min\{\|x\| + \|y\| : \ x \text{ and } y \text{ Hermitians in } BL(K) \text{ and } H = L_x + R_y\}.$$

While we are able to represent every Hermitian operator $H$ on $X$ in the form $L_x + R_y$, it is quite natural to raise the question whether it is possible to choose Hermitians $x$ and $y$ in such a way that $\|H\| = \|x\| + \|y\|$. We give an affirmative answer to this question if $X$ is a primitive $C^*$-algebra, and show with a counter example that the result is not necessarily true in an abstract $C^*$-algebra. As a nice application of the result in a primitive $C^*$-algebra, we establish a close link of Hermitian operator with primitive ideals by

$$\|H\| = \sup\{\|x - z + P\| + \|y + z + P\| : \ P \in \text{Prim}(X)\},$$

where $z$ is in the centre of $X$.

**2.Preliminaries.** The following lemmas are useful and enrich the main results:

**Lemma 2.1.** (Dauns-Hofmann's Theorem, [6]) *Let $x$ be an element of $X$ and let $f$ be a bounded continuous scalar-valued function on Prim(X). Then there exists a unique element $z$ of $X$ such that $z+P=f(P)\ (x+P)$, for all $P \in$ Prim(X). Here, $z$ depends on $x$ and $f$, but not on $P$.*

**Lemma 2.2.** *Let $H$ be a Hermitian operator on X. Then, $H = L_x + R_y$, $x$ and $y$ being hermitians in X.*

**Proof.** We know that a linear operator $H$ on $X$ is Hermitian if and only if it is of the form

$$(i) \qquad Hz = hz + Dz$$

where $z$ and $h = h^*$ are in $X$ and $D$ is a star-derivation [13]. Again, since all derivations on $X$ are inner, there is an element $u$ in $X$ such that

$$(ii) \qquad Dz = uz - zu, \quad \forall z \in X.$$

Therefore, $Dz^* = uz^* - z^*u$. Further,

$$(Dz^*)^* = (uz^* - z^*u)^* = zu^* - u^*z = -Dz,$$

(since $D$ is a star-derivation). So,

$$(iii) \qquad Dz = u^*z - zu^*.$$



From (i) and (iii), we obtain

$$Dz = (\frac{u+u^*}{2})z - z(\frac{u+u^*}{2}) = vz - zv,$$

where $v = \frac{u+u^*}{2}$ is hermitian. Hence, from (i)

$$Hz = hz + vz - zv = L_{h+v}z + R_{-v}z.$$

Here, $h + v$ and $-v$ are hermitian in $X$, and so replacing them by $x$ and $y$ respectively, we obtain the result. Q.E.D.

If $x$ is a hermitian element in $X$, then for every $P \in \text{Prim}(X)$, $x + P$ is hermitian in the quotient space $X/P$. Therefore, the theory of numerical ranges [3] shows that there are two real numbers $a_P$ and $b_P$ such that the numerical range of $x + P$ is the closed interval $[a_P, b_P]$.

**Lemma 2.3.** *Let $X$ be a structure $C^*$-algebra. If $V(X/P, x + P) = [a_P, b_P]$, where $x = x^*$ in $X$ and $P$ in Prim$(X)$, then the maps $f : P \to a_P$ and $g : P \to b_P$ from Prim$(X)$ to the real line are continuous.*

**Proof.** Because $x + P$ is hermitian in $X/P$, Lemma 3 in [5] suggests that either $-\|x + P\|$ or $\|x + P\|$ belongs to $V(X/P, x + P)$. In other words, we have either $a_P = -\|x + P\|$ *or* $b_P = \|x + P\|$

**Case (i).** Suppose $a_P = -\|x+P\|$ and $b_P \le \|x+P\|$. Now for a fixed $x$ in $X$ the map $P \to \|x + P\|$ from Prim$(X)$ to the real line is continuous [12]; and so the map $f : P \to -\|x + P\| = a_P$ is also continuous. Next, choose $\lambda > 0$ sufficiently large (for example, $\lambda > \|x\|$) such that $\|x + \lambda + P\|$ belongs to $V(X/P, x + \lambda + P)$. Because $V(X/P, x + \lambda + P) = V(X/P, x + P) + \lambda$ we obtain $V(X/P, x + \lambda + P) = [a_P + \lambda, b_P + \lambda]$ and so $b_P + \lambda = \|x + \lambda + P\|$. Again, for fixed $x$ and $\lambda$, the map $P \to \|x + \lambda + P\|$ and the map $P \to -\lambda$ are continuous and consequently $g$ is continuous.

**Case (ii).** Next, suppose that $b_P = \|x + P\|$ and $a_P \ge -\|x + P\|$. Then

$$V(X/P, -x + P) = [-b_P, -a_P] = [-\|x + P\|, -a_P].$$

By similar arguments as in case (i), it can be shown that both maps $f$ and $g$ are continuous.

**Remark 2.4.** It can be shown with a counter example that for a general $C^*$-algebra in which the space Prim(X) is not Hausdorff, the maps $f$ and $g$ defined in the above lemma are not necessarily continuous.

**3.The main results.** Lemma 4 in [5] guarantees that every pair $(x, y)$ of hermitian elements in $X$ gives rise to a convex compact subset $M(x, y)$ of the real line. In other words, let $V(X, x) = [a, b]$, $V(X, y) = [c, d]$ and

$M(x,y) = \{\alpha : \|x-\alpha\| + \|y-\alpha\| = \inf\{\|x-\beta\| + \|y-\beta\| : \beta \in \mathbb{R}\}\}$. Then $M(x,y)$ equals the closed interval $[-\frac{c+d}{2}, \frac{a+b}{2}]$ or the closed interval $[\frac{a+b}{2}, -\frac{c+d}{2}]$ according as $-\frac{c+d}{2} < \frac{a+b}{2}$ or $-\frac{c+d}{2} > \frac{a+b}{2}$. With this wealth of ideas we proceed to prove Theorem 3.1 given below. This theorem could have been proved directly without using Theorem 3 in [5], but we provide a such shorter and simpler proof by applying that theorem:

**Theorem 3.1.** *Let $X$ be a primitive $C^*$-algebra. If $H$ is a hermitian operator on $X$, then there exist two hermitian elements $x$ and $y$ in $X$ such that*

*(i) $H = L_x + R_y$ and (ii) $\|H\| = \|x\| + \|y\|$.*

**Proof.** The first result (i) follows from Lemma 2.2. Now Lemma 2.2 provides hermitians $x_0$ and $y_0$ such that

$$(I) \qquad H = L_{x_0} + R_{y_0}$$

Then, for every $\alpha \in M(x_0, y_0)$ we obtain from (I)

$$(II) \qquad H = L_{x_0-\alpha} + R_{y_0+\alpha}.$$

Since $X$ is a primitive $C^*$-algebra, we can represent it as a $C^*$-subalgebra of $BL(K)$, for a suitable Hilbert space $K$. Again, since $X$ has a unit element, there are no multipliers outside $X$. So, there exists a hermitian operator $T$ on $BL(K)$ such that $Ta = Ha$ for every $a$ in $X$.

Therefore, $T = L_{x_0} + R_{y_0}$. Now, applying to Theorem 3 in [5], we obtain

$$\|T\| = \|x_0 - \alpha\| + \|y_0 - \alpha\|, \qquad \alpha \in M(x_0, y_0).$$

Our next goal is to prove that $\|H\| = \|T\|$. The inequality $\|T\| \geq \|H\|$ comes free. To establish the reverse inequality, let $\varepsilon > 0$. Then there is an operator $U \in BL(K)$ with $\|U\| \leq 1$ such that

$$(III) \qquad \|T\| < \frac{\varepsilon}{2} + \|x_0 U + U y_0\|$$

In addition, there is a $\xi \in K$ with $\|\xi\| \leq 1$ such that

$$\|x_0 U + U y_0\| < \frac{\varepsilon}{2} + \|(x_0 U + U y_0)\xi\|.$$

Now, by Kadison's irreducibility theorem [4], there exists $U_0 \in X$ with $\|U_0\| \leq 1$ such that $U\xi = U_0\xi$ and $Uy_0\xi = U_0 y_0 \xi$.

Using these results in (III),

$$\|T\| < \varepsilon + \|x_0 U_0 \xi + U_0 y_0 \xi\| \leq \varepsilon + \|H\|.$$



The arbitrariness of $\varepsilon$ implies that $\|T\| \leq \|H\|$. As a result, we obtain

$$\|H\| = \|T\| = \|x_0 - \alpha\| + \|y_0 + \alpha\|,$$

for every $\alpha \in M(x_0, y_0)$. Replacing $x_0 - \alpha$ and $y_0 + \alpha$ by $x$ and $y$ respectively, the desired result (ii) follows. Q.E.D.

The theorem that follows gives an application of the above result.

**Theorem 3.2.** *Let $X$ be a structure $C^*$-algebra. Then for every hermitian operator $H$ on $X$, there exist hermitian elements $x, y$ in $X$ and an element $z$ in the centre of $X$ such that $H = L_{x-z} + R_{y+z}$, and*

$$\|H\| = \sup\{\|x - z + P\| + \|y + z + P\| : P \in \mathrm{Prim}(X)\}.$$

**Proof.** Lemma 2.2 ensures that there are hermitians $x, y$ in $X$ such that $H = L_x + R_y$. If $z$ is in the centre of $X$, then $L_{x-z} + R_{y+z} = L_x + R_y$.

Let $P$ be a primitive ideal in $X$. Then the quotient space $X/P$ is a primitive $C^*$-algebra. Also, as $X$ has the unit element and every derivation on $X$ is inner, so $X/P$ possesses the same merits.

We now define a map $H_P$ from $X/P$ into itself by $H_P(a+P) = Ha + P$, $a \in X$. This map is well-defined because the image of $P$ under $H$ is contained in $P$. Next, we show that $\|H_P\| \leq \|H\|$. Choose $t \in X$ with $\|t\| \leq 1$. Then $\|x + P\| \leq \|t\| \leq 1$. Now $\|H_P\| \leq \|H_P(t+P)\| = \|Ht + P\| \leq \|Ht\| \leq$ $\leq \|H\|$. Hence $\|\exp i\alpha H_P\| \leq \exp i\alpha H\| = 1$, for every $\alpha \in \mathbb{R}$. This yields $\|\exp i\alpha H_P\| = 1$, showing that $H_P$ is a hermitian operator on the primitive $C^*$-algebra $X/P$. Again $x + P$ is a hermitian element in $X/P$ and so there are reals $a_P$, $b_P$ such that $V(X/P, x+P) = [a_P, b_P]$. Set $\alpha_P = \frac{a_P + b_P}{2}$. Lemma 2.3 shows that $P \to a_P$ and $P \to b_P$ are continuous and hence the map $f : P \to \alpha_P$ from $\mathrm{Prim}(X)$ to the real line is continuous. Moreover, by invoking Theorem 3.1 we obtain that

$$\|H_P\| = \|x - \alpha_P + P\| + \|y + \alpha_P + P\|.$$

Now, by Dauns-Hofmann's theorem, for the unit element $I \in X$ there exists a unique element $z$ of $X$ such that

$$z + P = f(P)\,(I + P) = \alpha_P\,(I + P), \quad \textit{for all } P \in \mathrm{Prim}(X).$$

Since a $C^*$-algebra is semi-simple, $z$ is in the centre of $X$. The above result is a key step in this theorem, because the scalar $\alpha_P$ depends upon $P$ but the element $z$ is independent of $P$. Using this result,

$$\|H\| = \|x - z + P\| + \|y + z + P\|, \quad P \in \mathrm{Prim}(X).$$

As $\|H\| = \sup\{\|H_P\| : P \in \mathrm{Prim}(X)\}$, (also see [3]), we can conclude from above that $H = L_{x-z} + R_{y+z}$ and

$$\|H\| = \sup\{\|x - z + P\| + \|y + z + P\| : \ P \in \mathrm{Prim}(X)\}$$

which are as wanted.

**Remark 3.3.** Because of Remark 2.4, we guess that the above theorem may not be be true for a non-structure $C^*$-algebra.

**A counter example 3.4.** It would seem reasonable to try to extend Theorem 3.1 on more general $C^*$-algebra. One possible generalization of these results leads to the following question: If $X$ is a unital $C^*$-algebra with all derivations inner and if $H$ is a Hermitian operator on $X$, are there Hermitians $x, y \in X$ and $z$ is in the centre of $X$ such that

$$H = L_x + R_y$$

and

$$\|H\| = \|x - z\| + \|y + z\|$$

Although the first result is true (Lemma 2.2) the second part is not true in general. A counter example is given below.

Let $K$ be a Hilbert space of dimension greater than or equal to 2, and let $E = BL(K)$ and $X = E \oplus E$. We can choose $x = x^*$ and $y = y^*$ in $E$ with $\|x\| = 2$, $\pm 2 \in \mathrm{Sp}(x)$, $\|y\| = 1$ and $\pm 1 \in \mathrm{Sp}(y)$. Thus the numerical ranges of $x$ and $y$ are $V(x) = [-2, 2]$ and $V(y) = [-1, 1]$ respectively. Put $H = L_{x \oplus y} + R_{y \oplus x}$. Then $H$ is a Hermitian operator on $X$ and $\|H\| = 3$. The latter is immediately seen to be true by writing $H = (L_x + R_y) \oplus (L_y + R_x)$, and observing that $\|L_x + R_y\| = 3$ and $\|Lyx + R_x\| = 3$.

Now if $z$ is in the centre of $X$, then $z = \lambda 1 \oplus \mu 1$ for some $\lambda, \mu \in C$, (the complex field), because the centre of $E$ is $C \cdot 1$. If $\lambda, \mu \in \mathbb{R}$, and if $z = \lambda 1 \oplus \mu 1$, then

$$\begin{aligned} &\|x \oplus y - z\| + \|y \oplus x + z\| = \\ &= \|(x - \lambda) \oplus (y - \mu)\| + \|(y + \lambda) \oplus (x + \mu)\| = \\ &= \max\{2 + |\lambda|, 1 + |\mu|\} + \max\{1 + |\lambda|, 2 + |\mu|\} \end{aligned}$$

(because $\|x - \lambda\| = 2 + |\lambda|$ and $\|y + \mu\| = 1 + |\mu|$, etc.). Hence $\|x \oplus y - z\| + {}+\|y \oplus x + z\| \geq 4$. Consequently, $\inf\{\|x \oplus y - z\| + \|y \oplus xy - z\| : \ z \in$ centre of $X\} \geq 4 > 3 = \|H\|$. Therefore, $\|H\| = \|x - z\| + \|y + z\|$ is impossible in general.



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