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ON THE EQUATION $y' + a(x,y)f(by + g(y)) = 0$

BY

ADRIAN CORDUNEANU

We consider the differential equation

$$y' + a(x,y)f(by + g(y)) = 0, \ x \geq 0 \tag{1}$$

where the function $a = a(x,y)$ is defined for $x \geq 0, y \in \mathbf{R}$, the functions $f = f(r)$ and $g = g(y)$ are defined on $\mathbf{R}$, b is a positive constant. We denote by $\mathbf{R}$ the set of all real numbers. The aim of this Note is to study the behaviour of the solutions of equation (1) for $x \to \infty$, assuming in what follows that the conditions which ensure the local existence and the uniqueness of the solution, satisfying the initial data $y(x_0) = y_0$ with $x_0 \geq 0, y_0 \in \mathbf{R}$, are fulfilled. We make the following hypotheses:

(H_1) The function $g = g(y)$ satisfies the Lipschitz condition

$$|g(y) - g(z)| \leq L|y - z|, \ \forall y, z \in \mathbf{R}; \ 0 \leq L < b. \tag{2}$$

(H_2) The continuous function $f = f(r)$ satisfies the condition

$$f(r)/r \geq b_1 > 0 \ for \ r \neq 0; \ b_1 = \text{const}. \tag{3}$$

(H_3) There exists a continuous function $m = m(x)$ such that

$$a(x,y) \geq m(x) \geq 0 \ for \ x \geq 0, \ y \in \mathbf{R}; \ \int_0^\infty m(x)dx = \infty. \tag{4}$$

We remark that the continuous function $h(y) = by + g(y)$ is a bijection on $\mathbf{R}$ and hence there exists a single value y^* in $\mathbf{R}$ such that $h(y^*) = 0$. Because $f(0) = 0$ and $f(r) \neq 0$ for $r \neq 0$, it easily follows that $y = y^*$ is the unique constant solution of equation (1).

Proposition 1. *Under the hypothesis $(H_1),(H_2)$ and (H_3) for the solution of (1), we have:*

(i) *if $y(0) > y^*$, then $y = y(x)$ decreases to y^* for $x \to \infty$ and*

$$0 < y(x) - y^* \le (y(0) - y^*) \cdot \exp\left[-b_1(b-L)\int_0^x m(t)dt\right],\ x \ge 0; \tag{5}$$

(ii) *if $y(0) < y^*$, then $y = y(x)$ increases to y^* for $x \to \infty$ and*

$$0 < y^* - y(x) \le (y^* - y(0) \cdot \exp\left[-b_1(b-L)\int_0^x m(t)dt\right],\ x \ge 0. \tag{6}$$

Proof. Firstly, we remark that $f(r) > 0$ for $r > 0$ and $f(r) < 0$ for $r < 0$. If we assume that $y(0) > y^*$, the graph of the solution $y = y(x)$ remains above the line $y = y^*(y(x) > y^*)$ and since we have $h(y) > 0$ for $y > y^*$, we obtain from (1) $y'(x) \le 0$ for this solution; consequently, $y = y(x)$ decreases and it is defined for all $x \ge 0$. From

$$\int_{y(0)}^{y(x)} \frac{dt}{f(bt+g(t))} \le -\int_0^x a(t,y(t))dt \le -\int_0^x m(t)dt,\ x \ge 0 \tag{7}$$

and from the inequality

$$\begin{aligned} f(bt+g(t)) &\ge b_1(bt+g(t)) = b_1[bt+g(t)-(by^*+g(y^*))] \ge \\ &\ge b_1(b-L)(t-y^*),\ t > y^* \end{aligned} \tag{8}$$

we obtain

$$\int_0^x m(t)dt \le \frac{1}{b_1(b-L)} \ln\frac{y(0)-y^*}{y(x)-y^*},\ x \ge 0 \tag{9}$$

which implies that inequality (5) is true. If we assume $y(0) < y^*$, the graph of the corresponding solution $y = y(x)$ remains below the line $y = y^*(y(x) < y^*)$, and because we have $h(y) < 0$ for $y < y^*$, we deduce from (1) that $y'(x) \ge 0$ for this solution, hence $y = y(x)$ increases and it is defined for all $x \ge 0$. Using again (7) and the inequality

$$f(bt+g(t)) \le b_1(bt+g(t)) \le b_1(b-L)(t-y^*) < 0,\ t < y^* \tag{10}$$

we now obtain

$$\int_0^x m(t)dt \le \frac{1}{b_1(b-L)} \ln\frac{y^*-y(0)}{y^*-y(x)},\ x \ge 0 \tag{11}$$

which implies inequality (6). Interesting particular cases are obtained, taking $a(x,y) = a(x) \ge 0$, $a(x) = 1$ or $f(r) = r$. The geometrical meaning of the condition (3) is that the graph of the function $y = f(x)$ lies under the line $y = b_1 x$ for $x < 0$ and above it for $x > 0$.

Let us now replace the hypotheses (H_2) and (H_3) by other ones, namely

(H'_2) The continuous function $f = f(r)$ satisfies the condition

$$0 < f(r)/r \le b_2 \ \textit{for}\ r \ne 0;\ b_2 = \text{const}. \tag{12}$$

(H'_3) There exists a continuous function $M = M(x)$ such that

$$0 \le a(x,y) \le M(x)\ \textit{for}\ x \ge 0,\ y \in \mathbf{R};\ \int_0^\infty M(x)dx < \infty. \tag{13}$$

Proposition 2. *Under the hypothesis $(H_1),(H'_2)$ and (H'_3) for the solution of equation (1), we have*

(i) *if $y(0) > y^*$, then $y = y(x)$ decreases to $y(\infty) > y^*$ for $x \to \infty$ and*

$$0 \le y(x) - y(\infty) \le K\int_x^\infty M(t)dt,\ x \ge 0, \tag{14}$$

(ii) *if $y(0) < y^*$, then $y = y(x)$ increases to $y(\infty) < y^*$ for $x \to \infty$ and*

$$0 \le y(\infty) - y(x) \le K\int_x^\infty M(t)dt,\ x \ge 0; \tag{15}$$

in both cases, the constant $K = b_2(b+L)|y(0)-y^*| > 0$ depends on the choosen solution.

Proof. (i) The fact that $y = y(x)$ decreases for $x \ge 0$ may be justified as in Proposition 1. Obviously, it follows $y(\infty) \ge y^*$. Since, for $t > y^*$, we have

$$0 < f(bt+g(t)) \le b_2(bt+g(t)) \le b_2(b+L)(t-y^*), \tag{16}$$

the assumption that $y(\infty) = y^*$ leads us to a contradiction, following from the inequality

$$\int_{y(x)}^{y(0)} \frac{dt}{f(bt+g(t)} = \int_0^x a(t,y(t))dt \le \int_0^x M(t)dt,\ x \le 0 \tag{17}$$

if we remark that the left hand side becomes infinite for $x \to \infty$, while the right hand side remains finite. This contradiction shows that $y(\infty) > y^*$. Integrating equations (1) on $[x, \infty)$, we find

$$y(x) - y(\infty) \leq \int_x^\infty M(t) f(by(t) + g(y(t)))dt, \ x \geq 0. \tag{18}$$

If we take into account (16) and use the inequality

$$|y(x) - y^*| \leq |y(0) - y^*|, \ x \geq 0 \tag{19}$$

it easily follows the desired inequality (14).

(ii) The fact that $y = y(x)$ increases for $x \geq 0$ may be proved as in Proposition 1; it also follows that $y(\infty) \leq y^*$. Since, for $t < y^*$, we have

$$0 > f(bt + g(t)) \geq b_2(bt + g(t)) \geq b_2(b + L)(t - y^*), \tag{20}$$

the assumption that $y(\infty) = y^*$ leads us to a contradiction, letting $x \to \infty$ in the same relationship (17). Integrating again equation (1) over $[x, \infty)$ and taking into account (19) and (20), we conclude that inequality (15) also holds.

Remark 1. If we assume in equation (1) that $a(x, y) = a(x) \geq 0$ and $f(r) = r$, all the required hypotheses are fulfilled with $b_1 = b_2 = 1$ and $m(x) = M(x) = a(x)$ for $x \geq 0$, hence for the differential equation with separable variables

$$y' + a(x)(by + g(y)) = 0, \ x \geq 0 \tag{21}$$

hold the conclusions of Proposition 1 or of Proposition 2, depending on whether

$$\int_0^\infty a(x)dx = \infty \ or \ \int_0^\infty a(x)dx < \infty, \tag{22}$$

respectively.

Remark 2. The assumptions that $g = g(y)$ and $f = f(r)$ are defined on the whole real axis may be weakened, requiring only that $g = g(y)$ is defined on an interval containing the root y^* of the function $h(y) = by + g(y)$ and that $f = f(r)$ is defined in an open set containing the range of the function $h = h(y)$. Of course, the other restrictions imposed in the preceding Propositions remain valid.

We now state a result on the existence of periodic solutions for the equation

$$y' + a(x)f(by + g(y)) = 0, \ x \geq 0 \tag{23}$$

introducing a new hypothesis, namely

(H_4) *The function $a = a(x)$ is continuous* and T-periodic for $x \geq 0$.

Proposition 3. *Under the hypotheses $(H_1), (H'_2)$ and (H_4), for the equation* (23) *we have:*

(i) *if $\int_0^T a(x)dx = 0$, all its solution are T-periodic;*

(ii) *if $\int_0^T a(x)dx \neq 0$, there exists no T-periodic solution, except $y(x) \equiv y^*$.*

For the proof, we denote by $H = H(y)$ an antiderivative of the function $1/f(h(y))$, defined for $y > y^*$. We can take $H(y)$ as given by

$$H(y) = \int_\alpha^y \frac{dt}{f(bt + g(t))}, \ y > y^* \tag{24}$$

where $\alpha > y^*$ is a fixed number. Using (16), it follows that $H(y) \to \infty$ for $y \to \infty$ and $H(y) \to -\infty$ for $y \to y^*$. We also remark that $H = H(y)$ is strictly increasing on (y^*, ∞) and that every solution of (23), with $y(0) > y^*$, is given by

$$H(y) = c - \int_0^x a(t)dt, \ x \geq 0; \ c = H(y(0)), \tag{25}$$

which is equivalent to

$$y(x) = H^{-1}(c - \int_0^x a(t)dt), \ x \geq 0. \tag{26}$$

The inverse function H^{-1} is strictly increasing on the real axis and takes its values in the interval (y^*, ∞). In case (i), $y(0) = y(T)$ and hence every solution with $y(0) > y^*$ is $T-periodic$ for $x \geq 0$. In case (ii), $y(0) \neq y(T)$ and hence we have not $T-periodic$ solution with $y(0) > y^*$. Similar arguments may be used for the solutions with $y(0) < y^*$, employing (20) and an antiderivative of the same function $1/f(h(y))$, defined for $y < y^*$.

We conclude this Note with the remark that similar propositions may be formulated concerning the differential equation

$$y' = a(x, y)f(by + g(y)), \ x \geq 0 \tag{27}$$

keeping the same hypotheses and notations as above. For example, we may state the following

Proposition 4. *Under the hypotheses* $(H_1), (H'_2)$ *and* (H'_3), *for the solution of equation* (27), *we have*

(i) *if* $y(0) > y^*$, *then* $y = y(x)$ *increases to* $y(\infty)$ = *finite for* $x \to \infty$ *and*

$$0 \le y(\infty) - y(x) \le \tilde{K} \int_x^\infty M(t)dt, \ x \ge 0; \tag{28}$$

(ii) *if* $y(0) < y^*$, *then* $y = y(x)$ *decreases to* $y(\infty)$ = *finite for* $x \to \infty$ *and*

$$0 \le y(x) - y(\infty) \le \tilde{K} \int_x^\infty M(t)dt, \ x \ge 0 \tag{29}$$

where the constant $\tilde{K}$ *depends of the chosen solution.*

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Department of Mathematics
Polytechnic Institute
R-66oo, Iasi
ROMANIA

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GENERALIZED RECURRENT RIEMANNIAN MANIFOLD

BY

MILEVA PRVANOVIĆ

1.Introduction. Let (M, g) be a connected n-dimensional, $n > 3$, Riemannian smooth manifold where g is the Riemannian metric of M. We denote by ∇, R and S the Levi-Civita connection, the Riemann-Cristoffel curvature tensor and the Ricci tensor of (M, g) respectively.

Recently M.C. Chaki [3] considered the non-flat manifolds whose curvature tensor, in the local coordinates, satisfies

$$\begin{aligned} \nabla_s R_{ijkl} = 2A_s R_{ijkl} + A_i R_{sjkl} + \\ + A_j R_{iskl} + A_k R_{ijsl} + A_l R_{ijks}, \end{aligned} \tag{1.1}$$

where A_i are the components of a non-zero covariant vector field. Such a manifold was called by Chaki pseudo-symmetric and A_i is called its associated vector. But in many papers the name pseudo-symmetric manifold is used for semi-Riemannian manifold satisfying the condition

$$R \cdot R = \mathcal{L}Q(g, R)$$

for some function $\mathcal{L}$ on M (see for ex. [4],[5],[6],[7]). In the local coordinates this condition has the form

$$\begin{aligned} &\nabla_r \nabla_s R_{ijkl} - \nabla_s \nabla_r R_{ijkl} = \\ &= \mathcal{L}(g_{ri}R_{sjkl} + g_{rj}R_{iskl} + g_{rk}R_{ijsl} + g_{rl}R_{ijks} - \\ &- g_{si}R_{rjkl} - g_{sj}R_{irkl} - g_{sk}R_{ijrl} - g_{sl}R_{ijkr}). \end{aligned}$$

On the other hand, if the manifold, besides (1.1) satisfies

$$A_s R_{ijkl} + A_k R_{ijls} + A_l R_{ijsk} = 0,$$