keeping the same hypotheses and notations as above. For example, we may state the following

Proposition 4. *Under the hypotheses $(H_1), (H'_2)$ and (H'_3), for the solution of equation* (27), *we have*

(i) *if $y(0) > y^*$, then $y = y(x)$ increases to $y(\infty)$ = finite for $x \to \infty$ and*

$$0 \le y(\infty) - y(x) \le \tilde{K} \int_x^\infty M(t)dt, \; x \ge 0; \tag{28}$$

(ii) *if $y(0) < y^*$, then $y = y(x)$ decreases to $y(\infty)$ = finite for $x \to \infty$ and*

$$0 \le y(x) - y(\infty) \le \tilde{K} \int_x^\infty M(t)dt, \; x \ge 0 \tag{29}$$

where the constant $\tilde{K}$ depends of the chosen solution.

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GENERALIZED RECURRENT RIEMANNIAN MANIFOLD

BY

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1.Introduction. Let (M, g) be a connected n-dimensional, $n > 3$, Riemannian smooth manifold where g is the Riemannian metric of M. We denote by ∇, R and S the Levi-Civita connection, the Riemann-Cristoffel curvature tensor and the Ricci tensor of (M, g) respectively.

Recently M.C. C h a k i [3] considered the non-flat manifolds whose curvature tensor, in the local coordinates, satisfies

$$\begin{aligned} \nabla_s R_{ijkl} = 2A_s R_{ijkl} + A_i R_{sjkl} + \\ + A_j R_{iskl} + A_k R_{ijsl} + A_l R_{ijks}, \end{aligned} \tag{1.1}$$

where A_i are the components of a non-zero covariant vector field. Such a manifold was called by Chaki pseudo-symmetric and A_i is called its associated vector. But in many papers the name pseudo-symmetric manifold is used for semi-Riemannian manifold satisfying the condition

$$R \cdot R = \mathcal{L}Q(g, R)$$

for some function $\mathcal{L}$ on M (see for ex. [4],[5],[6],[7]). In the local coordinates this condition has the form

$$\begin{aligned} &\nabla_r \nabla_s R_{ijkl} - \nabla_s \nabla_r R_{ijkl} = \\ &= \mathcal{L}(g_{ri} R_{sjkl} + g_{rj} R_{iskl} + g_{rk} R_{ijsl} + g_{rl} R_{ijks} - \\ &- g_{si} R_{rjkl} - g_{sj} R_{irkl} - g_{sk} R_{ijrl} - g_{sl} R_{ijkr}). \end{aligned}$$

On the other hand, if the manifold, besides (1.1) satisfies

$$A_s R_{ijkl} + A_k R_{ijls} + A_l R_{ijsk} = 0,$$

the condition (1.1) reduces to

$$\nabla_s R_{ijkl} = 4A_s R_{ijkl},$$

i.e. (M, g) is a recurrent Riemannian manifold. It is easily to see that conversely, every recurrent Riemannian manifold also satisfies the condition of the form (1.1).

Thus, for the manifold satisfying (1.1) but which is not a recurrent one and is not of constant curvature, we shall use the name *generalized recurrent Riemannian manifold*, in short GR-manifold.

Among others, in [3] it is shown that, under certain condition, the curvature tensor of a simply connected conformally flat GR-manifold can be expressed in the form

$$\begin{aligned} R_{ijkh} &= \sigma^2(g_{ih}g_{jk} - g_{ik}g_{jh}) + \\ &+ (\sigma\bar{\sigma} - \sigma^2)(g_{ih}T_jT_k + g_{jk}T_iT_h - g_{ik}T_jT_h - g_{jh}T_iT_k), \end{aligned} \tag{1.2}$$

where σ and $\bar{\sigma}$ are some functions and vector field T is codirectional with A.

The Riemannian manifold whose curvature tensor has the form (1.2) is said to be the space of quasi-constant curvature ([1],[2]).

In this paper we shall prove some further properties of generalized recurrent Riemannian manifold. In Sec.2, we prove that every GR-manifold satisfies

$$R \cdot R = Q(\bar{A}, R),$$

where $\bar{A}$ is a symmetric (0,2) tensor field, i.e. satisfies a relation of the pseudo-symmetric type. In Sec.3 we prove that if the associated vector is concircular, a GR-manifold is a space of quasi-constant curvature. The question under wich condition the space of quasi-constant curvature would be a GR-manifold, will be answered in Sec.4.

2.The relation of pseudo-symmetric type. We shall prove

Theorem. *The curvature tensor of a GR-manifold satisfies*

$$\begin{aligned} &\nabla_r\nabla_s R_{ijkl} - \nabla_s\nabla_r R_{ijkl} = \\ &= A_{ri}R_{sjkl} + A_{rj}R_{iskl} + A_{rk}R_{ijsl} + A_{rl}R_{ijks} - \\ &- A_{si}R_{rjkl} - A_{sj}R_{irkl} - A_{sk}R_{ijrl} - A_{sl}R_{ijkr}, \end{aligned} \tag{2.1}$$

where

$$A_{ri} = \nabla_r A_i - A_r A_i \tag{2.2}$$

is symmetric tensor field.

Proof. Applying the operator ∇_r on (1.1) and antisymmetrizing in r and s, we obtain

$$\begin{aligned} &\nabla_r\nabla_s R_{ijkl} - \nabla_s\nabla_r R_{ijkl} = 2(A_{rs} - A_{sr})R_{ijkl} + A_{ri}R_{sjkl} + \\ &+ A_{rj}R_{iskl} + A_{rk}R_{ijsl} + A_{rl}R_{ijks} - \\ &- A_{si}R_{rjkl} - A_{sj}R_{irkl} - A_{sk}R_{ijrl} - A_{sl}R_{ijkr}. \end{aligned} \tag{2.3}$$

By the cyclic permutation in pair of indices i, j; k, l; s, r and the use of known identity ([9], p.153, Lemma 1)

$$\begin{aligned} &\nabla_r\nabla_s R_{ijkl} - \nabla_s\nabla_r R_{ijkl} + \nabla_j\nabla_i R_{klrs} - \nabla_i\nabla_j R_{klsr} + \\ &+ \nabla_l\nabla_k R_{srij} - \nabla_k\nabla_l R_{srij} = 0, \end{aligned}$$

we obtain

$$\begin{aligned} &2(A_{rs} - A_{sr})R_{ijkl} + 2(A_{ji} - A_{ij})R_{klsr} + 2(A_{lk} - A_{kl})R_{srij} + \\ &+ A_{ri}R_{sjkl} + A_{rj}R_{iskl} + A_{rk}R_{ijsl} + A_{rl}R_{ijks} - \\ &- A_{si}R_{rjkl} - A_{sj}R_{irkl} - A_{sk}R_{ijrl} - A_{sl}R_{ijkr} + \\ &+ A_{jk}R_{ilsr} + A_{jl}R_{kisr} + A_{js}R_{klir} + A_{jr}R_{klsi} - \\ &- A_{ik}R_{jlsr} - A_{il}R_{kjsr} - A_{is}R_{kljr} - A_{ir}R_{klsj} + \\ &+ A_{ls}R_{krij} + A_{lr}R_{skij} + A_{li}R_{srkj} + A_{lj}R_{srik} - \\ &- A_{ks}R_{lrij} - A_{kr}R_{slij} - A_{ki}R_{srlj} - A_{kj}R_{sril} = 0. \end{aligned} \tag{2.4}$$

Now, we shall show that (2.4) implies that either all the $A_{rs} - A_{sr}$ are zero, or all the R_{ijkl} are zero.

Suppose that one of $A_{rs} - A_{sr}$, say $A_{12} - A_{21} \neq 0$. Then (2.4) with $i, k, s = 1$ and $j, l, r = 2$ gives

$$12(A_{21} - A_{12})R_{1212} = 0,$$

and therefore $R_{1212} = 0$.

Putting $i = s = 1$, $j = l = r = 2$ in (2.4), we now have

$$8(A_{21} - A_{12})R_{12k2} = 0,$$

and therefore $R_{12k2} = 0$.

Similarly, putting $i = l = s = 1$ and $j = r = 2$ in (2.4), we obtain $R_{12k2} = 0$. Now (2.4) with $i = s = 1$ and $j = r = 2$ gives $R_{12kl} = 0$.

Putting $i = s = 1$ and $r = 2$ in (2.4), we find

$$\text{(2.5)} \quad \begin{aligned} &3(A_{21} - A_{12})R_{1jkl} + (A_{2k} - A_{k2})R_{1j1l} + (A_{2l} - A_{l2})R_{1jk1} + \\ &+ (A_{1k} - A_{k1})R_{1jl2} + (A_{l1} - A_{1l})R_{1jk2} = 0 \end{aligned}$$

which, for $k = 1$ and $k = 2$ gives respectively

$$R_{1j1l} = 0, \qquad R_{1j2l} = 0.$$

Thus (2.5) reduces to $R_{1jkl} = 0$. Now (2.4) with $i = r = 2$, $s = 1$ gives

$$\text{(2.6)} \quad 3(A_{21} - A_{12})R_{2jkl} + (A_{k1} - A_{1k})R_{2j2l} + (A_{l1} - A_{1l})R_{2jk2} = 0,$$

from which, for $k = 2$, we get $R_{2j2l} = 0$. Therefore (2.6) reduces to $R_{2jkl} = 0$. Finally, putting $s = 1$, $r = 2$ in (2.4) and using $R_{1jkl} = 0$, we get $R_{ijkl} = 0$.

We see that if tensor $A_{rs} - A_{sr}$ has at least one component different from zero, then all R_{ijkl} must be zero. But this contradicts the assumption that (M, g) is non-flat. Thus, all $A_{rs} - A_{sr}$ must be zero and (2.3) reduces to (2.1). This completes the proof of the theorem.

Remark. The condition $A_{ij} - A_{ji} = 0$ can be expressed in the form

$$\text{(2.7)} \quad \nabla_i A_j - \nabla_j A_i = 0.$$

3.GR-manifold whose associated vector field is concircular. Vector field A is said to be concircular if

$$\text{(3.1)} \quad \nabla_j A_i = \alpha g_{ij} + \gamma A_i A_j,$$

where α and γ are some functions. Then

$$\text{(3.2)} \quad A_{ji} = \nabla_j A_i - A_j A_i = \alpha g_{ij} + \beta A_i A_j,$$

where

$$\text{(3.3)} \quad \beta = \gamma - 1.$$

Also we have

$$\text{(3.4)} \quad \nabla_k \nabla_j A_i - \nabla_j \nabla_k A_i = g_{ij} W_k - g_{ik} W_j + \gamma_k A_i A_j - \gamma_j A_i A_k,$$

where we have put

$$W_k = \alpha_k = \alpha\gamma A_k, \quad \alpha_k = \frac{\partial \alpha}{\partial x^k}, \quad \gamma_k = \frac{\partial \gamma}{\partial x^k}$$

In view of the Ricci identity, (3.4) can be written in the form

$$\text{(3.5)} \quad A_t R^t{}_{ikj} = W_k g_{ij} - W_j g_{ik} + \gamma_k A_i A_j - \gamma_j A_i A_k.$$

Transvecting (3.5) with A^i, we find

$$(W_k + \gamma_k A_t A^t) A_j = (W_j + \gamma_j A_t A^t) A_k,$$

from which we get

$$W_k + \gamma_k A_t A^t = \pi A_k,$$

where π is some function. Thus, assuming $A_t A^t \neq 0$, this relation can be written in the form

$$\text{(3.6)} \quad \gamma_k = \beta A_k + q W_k,$$

where p is some function and

$$q = -\frac{1}{A_t A^t}.$$

Substituting (3.6) into (3.5), we find

$$\text{(3.7)} \quad A_t R^t{}_{ikj} = W_k (g_{ij} + q A_i A_j) - W_j (g_{ik} + q A_i A_k),$$

from which we get

$$\text{(3.8)} \quad A_t S^t{}_j = (2 - n) W_j + Q A_j, \quad Q = q W_a A^a.$$

Some remarks concerning the case $A_t A^t = 0$ will be given in Sec.5.

Substituting (3.2) into (2.1), and using the Ricci identity, we get

$$\begin{aligned} &- R^a{}_{jkl} R_{srai} + R^a{}_{ikl} R_{sraj} - R^a{}_{eij} R_{srak} + R^a{}_{kij} R_{srae} = \\ &= (\alpha g_{ri} + \beta A_r A_i) R_{sjkl} - (\alpha g_{rj} + \beta A_r A_j) R_{sikl} + \\ &+ (\alpha g_{rk} + \beta A_r A_k) R_{slij} - (\alpha g_{rl} + \beta A_r A_l) R_{skij} - \\ &- (\alpha g_{si} + \beta A_s A_i) R_{rjkl} - (\alpha g_{sj} + \beta A_s A_j) R_{irkl} - \\ &- (\alpha g_{sk} + \beta A_s A_k) R_{ijrl} - (\alpha g_{sl} + \beta A_s A_l) R_{ijkr} \end{aligned}$$

from which, transvecting with A^s and using (3.7), we obtain

$$
\begin{aligned}
&-W_aR^a{}_{jkl}(g_{ri}+qA_rA_i)+W_aR^a{}_{ikl}(g_{rj}+qA_rA_j)-\\
&-W_aR^a{}_{lij}(g_{rk}+qA_rA_k)+W_aR^a{}_{kij}(g_{rl}+qA_rA_l)+\\
&+2q[A_rW_iW_k(g_{jl}+qA_jA_l)-A_rW_iW_l(g_{jk}+qA_jA_k)-\\
&-A_rW_jW_k(g_{il}+qA_iA_l)+A_rW_jW_l(g_{ik}+qA_iA_k)]+\\
&+W_iR_{rjkl}+W_jR_{irkl}+W_kR_{ijrl}+W_lR_{ijkr}+\\
&+(\alpha+\beta A_aA^a)[A_iR_{rjkl}+A_jR_{irkl}+A_kR_{ijrl}+A_lR_{ijkr}]=\\
&=W_k[\alpha(g_{ri}g_{jl}-g_{rj}g_{il})+\alpha q(g_{ri}A_jA_l-g_{rj}A_iA_l)+\\
&+\beta(g_{jl}A_rA_i-g_{il}A_rA_j)]+W_l[\alpha(g_{rj}g_{ik}-g_{ri}g_{jk})\\
&+\alpha q(g_{rj}A_iA_k-g_{ri}A_jA_k)+\beta(g_{ik}A_rA_j-g_{jk}A_rA_i)]+\\
&+W_i[\alpha(g_{rk}g_{lj}-g_{rl}g_{kj})+\alpha q(g_{rk}A_lA_j-g_{rl}A_kA_j)+\\
&+\beta(g_{lj}A_rA_k-g_{kj}A_rA_l)]+W_j[\alpha(g_{rl}g_{ki}-g_{rk}g_{li})\\
&+\alpha q(g_{rl}A_kA_i-g_{rk}A_lA_i)+\beta(g_{ki}A_rA_l-g_{li}A_rA_k)].
\end{aligned}
\tag{3.9}
$$

Transvecting (3.9) with g^{ir} and using (3.7), we find

$$
\begin{aligned}
&(1-n-qA_aA^a)W_tR^t{}_{jkl}-W_kS_{jl}+W_lS_{jk}+\\
&+(\alpha+\beta A_aA^a)(A_tR^t{}_{jkl}-A_kS_{jl}+A_lS_{jk})+\\
&+(qW_aW^a+\beta W_aW^a)(g_{kj}A_l-g_{lj}A_k)+\\
&+(2qW_aA^a-\alpha n-\beta A_aA^a)(W_kg_{jl}-W_lg_{jk})+\\
&+q(3+2qA_aA^a)(W_lA_k-W_kA_l)W_j+(n\alpha q-\beta)(W_lA_k-W_kA_l)A_j=0.
\end{aligned}
$$

Transvecting with A^j and using (3.7) and (3.8), we get

$$\left[(n-6)\frac{W_aA^a}{A_tA^t}+(n-4)\beta A_tA^t-(n+2)\alpha\right](W_lA_k-W_kA_l)=0.$$

Thus, if

$$(n-6)\frac{W_aA^a}{A_tA^t}+(n-4)\beta A_tA^t-(n+2)\alpha\neq 0, \tag{3.10}$$

we have

$$W_lA_k-W_kA_l=0,$$

from which

$$W_k=\Psi A_k,$$

where Ψ is some function. Substituting this into (3.7) and then into (3.9), we find

$$A_iT_{rjkl}+A_jT_{irkl}+A_kT_{ijrl}+A_lT_{ijkr}=0 \tag{3.11}$$

where

$$
\begin{aligned}
T_{rjkl}=R_{rjkl}&+(\Psi+\alpha)(g_{rl}g_{jk}-g_{rk}g_{jl})+\\
&+\beta(g_{rl}A_kA_j+g_{kj}A_rA_l-g_{jl}A_rA_k-g_{rk}A_jA_l).
\end{aligned}
$$

Now, the relation (3.11), in view of [8], Lemma 4, yields $A_i=0$ or $T_{rijk}=0$ i.e. $T_{rijk}=0$ because of the assumption $A_i\neq 0$. Thus

Theorem. *Let the associated vector field of a generalized recurrent Riemannian manifold satisfies $A_tA^t\neq 0$, (3.1) and (3.10). Then the curvature tensor of the manifold satisfies*

$$
\begin{aligned}
R_{rjkl}&=(\Psi+\alpha)(g_{rk}g_{jl}-g_{rl}g_{jk})+\\
&\beta(g_{jl}A_rA_k+g_{rk}A_jA_l-g_{rl}A_kA_j-g_{kj}A_lA_r),
\end{aligned}
\tag{3.12}
$$

i.e. this manifold is a space of quasi-constant curvature.

In the next section we consider the question: under which condition the space of quasi-constant curvature is a generalized recurrent Riemannian manifold?

4.A manifold of quasi-constant curvature. Let us consider a manifold of quasi-constant curvature, i.e. a Riemannian manifold with positive definite metric whose curvature tensor is given by

$$
\begin{aligned}
R_{ijkl}&=a(g_{il}g_{jk}-g_{ik}g_{jl})+\\
&+b(g_{il}v_jv_k+g_{jk}v_iv_l-g_{ik}v_jv_l-g_{jl}v_iv_k),
\end{aligned}
\tag{4.1}
$$

where a,b are differentiable functions. Without the loss of the generality, we can suppose that $v^i=g^{it}v_t$ is a unit vector field. Then

$$(\nabla_iv_j)v^j=0. \tag{4.2}$$

Also, we suppose $a\neq 0$, $b\neq 0$.

T. A d d a t t i and Y. W a n g ([1]) proved the following properties of a space of quasi-constant curvature.

Theorem A. *If $a\neq$const., then $a_i=\frac{\partial a}{\partial x^i}=\tau v_i$ where $\tau=a_tv^t$.*

Theorem B. *Vector field v_i is gradient if and only if $b_i = \frac{\partial b}{\partial x^i} = \eta v_i$ where $\eta = b_t v^t$. Moreover, then*

$$\nabla_j v_i = \rho(g_{ij} - v_i v_j), \quad \textit{where} \quad \rho = \frac{\tau}{2b}. \tag{4.3}$$

From (4.1), in view of theorem A, we have

$$\begin{aligned}\nabla_r R_{ijkl} = {} & \tau v_r (g_{il} g_{jk} - g_{ik} g_{jl}) + \\ & + b_r (g_{il} v_j v_k + g_{jk} v_i v_l - g_{ik} v_j v_l - g_{jl} v_i v_k) + \\ & + b[g_{il}(v_k \nabla_r v_j + v_j \nabla_r v_k) + g_{jk}(v_i \nabla_r v_l + v_l \nabla_r v_i) - \\ & - g_{ik}(v_j \nabla_r v_l + v_l \nabla_r v_j) - g_{jl}(v_i \nabla_r v_k + v_k \nabla_r v_i)].\end{aligned}$$

Now let us suppose that the space of quasi-constant curvature is generalized recurrent. In view of (1.1),(4.1),(4.2) and (4.4), we have

$$\begin{aligned} & 2A_r[a(g_{il}g_{jk} - g_{ik}g_{jl}) + b(g_{il}v_jv_k + g_{jk}v_iv_l - g_{ik}v_jv_l - g_{jl}v_iv_k)] + \\ & + A_i[a(g_{rl}g_{jk} - g_{rk}g_{jl}) + b(g_{rl}v_jv_k + g_{jk}v_rv_l - g_{rk}v_rv_l - g_{jl}v_rv_k)] + \\ & + A_j[a(g_{il}g_{rk} - g_{ik}g_{rl}) + b(g_{il}v_rv_k + g_{rk}v_iv_l - g_{ik}v_rv_l - g_{rl}v_iv_k)] + \\ & + A_k[a(g_{il}g_{jr} - g_{ir}g_{jl}) + b(g_{il}v_jv_r + g_{jr}v_iv_l - g_{ir}v_jv_l - g_{jl}v_iv_r)] + \\ & + A_l[a(g_{ir}g_{jk} - g_{ik}g_{jr}) + b(g_{ir}v_jv_k + g_{jk}v_iv_r - g_{ik}v_jv_r - g_{jr}v_iv_k)] = \\ & = \tau v_k(g_{il}g_{jk} - g_{ik}g_{jl}) + b_r(g_{il}v_jv_k + g_{jk}v_iv_l - g_{ik}v_jv_l - g_{jl}v_iv_k) + \\ & + b[g_{jl}(v_k\nabla_rv_j + v_j\nabla_rv_k) + g_{jk}(v_l\nabla_lv_i + v_i\nabla_rv_l) - \\ & - g_{ik}(v_j\nabla_rv_l + v_l\nabla_rv_j) - g_{jl}(v_i\nabla_rv_k + v_k\nabla_rv_i)].\end{aligned}$$

Transvecting with g^{il}, we find

$$\begin{aligned} & 2A_r\{[(n-1)a + b]g_{jk} + (n-2)bv_jv_k\} + \\ & + a(A_rg_{jk} - A_jg_{rk}) + b(A_rv_jv_k + A_tv^tg_{jk}v_r - \\ & A_tv^tg_{rk}v_j - A_jv_rv_k) + a(A_rg_{jk} - A_kg_{jr}) + b(A_rv_jv_k + A_tv^tg_{jk}v_r - \\ & - A_kv_jv_r - A_tv^tg_{jr}v_k) + A_j\{[(n-1)a + b]g_{rk} + (n-2)bv_rv_k\} + \\ & + A_k\{[(n-1)a + b]g_{rj} + (n-2)bv_rv_j\} = \\ & = (n-1)\tau v_rg_{jk} + b_r[(n-2)v_jv_k + g_{jk}] + \\ & + (n-2)b[v_k\nabla_rv_j + v_j\nabla_rv_k].\end{aligned} \tag{4.5}$$

Transvecting (4.5) with g^{jk}, we obtain

$$[(2n-1)(n+2)a + 4nb]A_r = [n(n-1)\tau - 4(n-2)bA_tv^t]v_r + 2(n-1)b_r. \tag{4.6}$$

Transvecting (4.5) with v^jv^k, we get

$$2(a+b)[nA_r + (n-2)A_tV^tv_r] = (n-1)(\tau v_r + b_r). \tag{4.7}$$

Thus, if $a + b = 0$, we have $b_r = -\tau v_r$.

If $a + b \neq 0$, eliminating A_r from (4.6) and (4.7), we find

$$\frac{(n^2-1)a}{a+b} b_r = \left[2(n-1)(n+2)aA_tv^t - \frac{n-1}{a+b}(a - nb)\tau\right] v_r.$$

Thus, in every case, $b_r = \eta v_r$ and, according to Theorem B, v_i is a gradient vector field. Now, in view of second part of same theorem, the relation (4.4) becomes

$$\begin{aligned}\nabla_r R_{ijkl} = {} & \tau v_r(g_{il}g_{jk} - g_{ik}g_{jl}) + \\ & \frac{\tau}{2}[(g_{jk}g_{rl} - g_{jl}g_{rk})v_i + (g_{il}g_{rk} - g_{ik}g_{rl})v_j + \\ & + (g_{il}g_{rj} - g_{jl}g_{ri})v_k + (g_{jk}g_{ri} - g_{ik}g_{rj})v_l] + \\ & + (\eta - \tau)(g_{il}v_rv_jv_k + g_{jk}v_rv_lv_i - g_{ik}v_rv_lv_j - g_{jk}v_rv_kv_i).\end{aligned}$$

Supposing $\eta \neq \tau$, we can rewrite this relation in the form

$$\begin{aligned}\nabla_r R_{ijkl} = {} & \tau v_r(g_{il}g_{jk} - g_{ik}g_{jl}) + \\ & \frac{\tau}{2}[(g_{jk}g_{rl} - g_{jl}g_{rk})v_i + (g_{il}g_{rk} - g_{ik}g_{rl})v_j + \\ & + (g_{il}g_{rj} - g_{jl}g_{ri})v_k + (g_{jk}g_{ri} - g_{ik}g_{rj})v_l] + \\ & + \frac{\eta - \tau}{4}[2v_r(g_{il}v_kv_j + g_{jk}v_lv_i - g_{ik}v_lv_j - g_{jl}v_kv_i) + \\ & + v_i(g_{rl}v_jv_k + g_{jk}v_rv_l - g_{rk}v_jv_l - g_{jl}v_rv_k) + \\ & + v_j(g_{il}v_rv_k + g_{rk}v_iv_l - g_{ik}v_rv_l - g_{rl}v_iv_k) + \\ & + v_k(g_{il}v_jv_r + g_{jr}v_iv_l - g_{ir}v_jv_l - g_{jl}v_iv_r) + \\ & + v_l(g_{ir}v_jv_k + g_{jk}v_iv_r - g_{ik}v_jv_r - g_{jr}v_iv_k)],\end{aligned}$$

or in the form

$$\begin{aligned}\nabla_r R_{ijkl} = {} & 2A_r\underline{R}_{ijkl} + A_iR_{rjkl} + \\ & A_j\underline{R}_{irkl} + A_k\underline{R}_{ijrl} + A_l\underline{R}_{ijkr},\end{aligned} \tag{4.8}$$

where we have put

$$A_i = \frac{\tau}{2a}v_i, \qquad b_1 = \frac{\eta - \tau}{2\tau}a$$

$$R_{ijkl} = a(g_{il}g_{jk} - g_{ik}g_{jl}) + b(g_{il}v_jv_k + g_{jk}v_iv_l - g_{ik}v_jv_l - g_{jl}v_iv_k).$$

It is easy to see that $\underline{R}_{ijkl} = R_{ijkl}$ if and only if $\frac{\eta-\tau}{2\tau}a = b$. In this case (4.8) becomes (1.1).

Summing up, we can state

Theorem. *A manifold of quasi-constant curvature satisfying $a \neq$ const., is a generalized recurrent Riemannian manifold if the vector field v_i is gradient and $\frac{a}{2} + b = \frac{\eta}{2\tau}$ ($\tau = a_tv^t$, $\eta = b_tv^t$). In the case $\eta = \tau$, its curvature tensor satisfies*

$$\nabla_rR_{ijkl} = \frac{\tau}{2}[2v_r(g_{il}g_{jk} - g_{ik}g_{jl}) + v_i(g_{jk}g_{rl} - g_{jl}g_{rk}) + v_j(g_{il}g_{rk} - g_{ik}g_{rl}) + v_k(g_{il}g_{rj} - g_{jl}g_{ri}) + v_l(g_{jk}g_{ri} - g_{ik}g_{rj})].$$

5.Remarks.

Remark 1. If $\beta = 0$ (i.e. if $\gamma = 1$), the condition (3.12) reduces to

$$R_{rjkl} = (\Psi + \alpha)(g_{rk}g_{jl} - g_{rl}g_{jk}),$$

i.e. the manifold is a space of constant curvature. Thus

The associated vector field of *generalized recurrent Riemannian manifold* cannot satisfy the condition

$$\nabla_jA_i = \alpha g_{ij} + A_iA_j.$$

Remark 2. If in (3.1) $\alpha = 0$ i.e. if the the *associated vector* field of *generalized recurrent Riemannian manifold satisfies*

(5.1) $$\nabla_jA_i = \gamma A_iA_j,$$

then $A_tA^t = 0$.

In fact, if $\alpha = 0$, then $W_k = 0$ too, and (3.5) reduces to

(5.2) $$A_tR^t{}_{ikj} = (\gamma_kA_j - \gamma_jA_k)A_i.$$

Transvecting (5.2) with A^i, we obtain

$$A_tA^t(\gamma_kA_j - \gamma_jA_k) = 0.$$

Thus, if we suppose $A_tA^t \neq 0$ then $\gamma_kA_j - \gamma_jA_k = 0$ and (5.2) reduces to

(5.3) $$A_tR^t{}_{ikj} = 0$$

Differentiating (5.3), we have

$$(\nabla_sA_t)R^t{}_{ikj} + A^t\nabla_sR_{tikj} = 0,$$

from which, substituting (5.1) and (1.1) and using (5.3), we get

$$A_tA^tR_{sikj} = 0.$$

But this contradicts assumptions $A_tA^t \neq 0$, $R_{sikj} \neq 0$. Thus

There does not exist a generalized recurrent Riemannian manifold of a positive definite metric whose associated vector field satisfies (5.1).

Now, let us suppose $A_tA^t = 0$. Then $(\nabla_iA_t)A^t = 0$ and transvecting (3.1) with A^i we get $\alpha A_j = 0$ i.e. $\alpha = 0$. On the other hand, in view of $\alpha = 0$, $W_i = 0$ and $A_tA^t = 0$, the relation (3.9) is satisfied identically and the way described in Sec.3 does not give the result.

Remark 3. If the associated vector field of generalized recurrent Riemannian manifold satisfies (3.1) (including $\alpha = 0$), the gradient of the scalar curvature is codirectional with associated vector field.

In fact, trasvecting (1.1) with $g^{il}g^{jk}$ we get

(5.4) $$\nabla_sK = 2A_sK + 4A_tS^t_\sigma,$$

where K denotes the scalar curvature of the manifold. If $\alpha = 0$, transvecting (5.2) with g^{ik} we have

$$A_tS^t{}_j = (\gamma_aA^a)A_j,$$

so that (5.4) reduces to

$$\nabla_sK = (2K + 4\gamma_aA^a)A_s.$$

If $\alpha \neq 0$, we have $W_j = \Psi A_j$ because of which (3.8) becomes

$$A_tS^t{}_j = [(2-n)\Psi + Q]A_j$$

and (5.4) reduces to

$$\nabla_sK = [2K + 4(2-n)\Psi + 4Q]A_s.$$

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ON UNCONDITIONAL BASIS AND SEQUENCE SPACES

BY

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1.Introduction. The theory of Banach spaces has been widely studied, and exposed in several text books, by a number of authors.The purpose of this note is to approach some properties of Banach spaces with unconditional basis by means of their associated Köthe's sequence space.

Hereafter X will denote a Banach space, B_X its unit ball and X^* its topological dual.A (Schauder) basis $[x_n : n \in \mathrm{N}]$ in X is said to be unconditional if $\Sigma_{i=1}^{\infty} a_i x_i$ converges unconditionally whenever $\Sigma_{i=1}^{\infty} a_i x_i$ converges in X, and the basis $\{x_n : n \in \mathrm{N}\}$ is said to be shrinking if $\lim_{n\to\infty} \|x^*\|_n = 0$ for each $x^* \in X^*$, where $\|x^*\|_n$ is the norm of x^* when restrited to the linear span of $\{x_i : i \geq n+1\}$.

Given a basis $\{x_n : n \in \mathrm{N}\}$ in X, we shall denote by $S_{[x_n]}$ the Banach sequence space

$$S_{[x_n]} := \{a \in \omega : \sum_{i=1}^{\infty} a_i x_i \textit{ converges in } X\},$$

endowed with the norm defined by

$$|||a||| = \sup\{\|\sum_{i=1}^{n} a_i x_i\| : n \in \mathrm{N}\}.$$

Clearly, $\varphi \subset S_{[x_n]}$ and when $\{x_n : n \in \mathrm{N}\}$ is normalized, then $S_{[x_n]} \subset l^\infty$. So, when $\{x_n : n \in \mathrm{N}\}$ is a normalized unconditional basis, $S_{[x_n]}$ becomes an ideal in l^∞ with $a + b := (a_i + b_i)_{i=1}^{\infty}$ and $ab := (a_i b_i)_{i=1}^{\infty}$.

We shall denote by

$$S_{[x_n^*]} := \{b \in \omega : \sum_{i=1}^{\infty} b_i x_i^* \textit{ converges in } X^*\},$$

and

$$S_{[x_n^*]^*} := \{b \in \omega : \sum_{i=1}^{\infty} b_i x_i^* \textit{ is weak}^* \textit{ convergent in } X^*\},$$