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Received: 4.IX.1992

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ANALELE STIINTIFICE ALE UNIVERSITATII "AL.I.CUZA" IASI
Tomul XXXVIII, s.I.a.,Matematica, 1992, f.4.

ON UNCONDITIONAL BASIS AND SEQUENCE SPACES

BY

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1.Introduction. The theory of Banach spaces has been widely studied, and exposed in several text books, by a number of authors.The purpose of this note is to approach some properties of Banach spaces with unconditional basis by means of their associated Köthe's sequence space.

Hereafter X will denote a Banach space, B_X its unit ball and X^* its topological dual.A (Schauder) basis $[x_n : n \in \mathbb{N}]$ in X is said to be unconditional if $\Sigma_{i=1}^{\infty} a_i x_i$ converges unconditionally whenever $\Sigma_{i=1}^{\infty} a_i x_i$ converges in X, and the basis $\{x_n : n \in \mathbb{N}\}$ is said to be shrinking if $\lim_{n\to\infty} \|x^*\|_n = 0$ for each $x^* \in X^*$, where $\|x^*\|_n$ is the norm of x^* when restrited to the linear span of $\{x_i : i \geq n+1\}$.

Given a basis $\{x_n : n \in \mathbb{N}\}$ in X, we shall denote by $S_{[x_n]}$ the Banach sequence space

$$S_{[x_n]} := \{a \in \omega : \sum_{i=1}^{\infty} a_i x_i \textit{ converges in } X\},$$

endowed with the norm defined by

$$|||a||| = \sup\{\|\sum_{i=1}^{n} a_i x_i\| : n \in \mathbb{N}\}.$$

Clearly, $\varphi \subset S_{[x_n]}$ and when $\{x_n : n \in \mathbb{N}\}$ is normalized, then $S_{[x_n]} \subset l^\infty$. So, when $\{x_n : n \in \mathbb{N}\}$ is a normalized unconditional basis, $S_{[x_n]}$ becomes an ideal in l^∞ with $a + b := (a_i + b_i)_{i=1}^{\infty}$ and $ab := (a_i b_i)_{i=1}^{\infty}$.

We shall denote by

$$S_{[x_n^*]} := \{b \in \omega : \sum_{i=1}^{\infty} b_i x_i^* \textit{ converges in } X^*\},$$

and

$$S_{[x_n^*]^*} := \{b \in \omega : \sum_{i=1}^{\infty} b_i x_i^* \textit{ is weak}^* \textit{ convergent in } X^*\},$$

Note that the latter contains the α -dual $S^{\times}_{[x_n]}$ of $S_{[x_n]}$, i.e. the sequence space $\{b \in \omega : \Sigma_{i=1}^{\infty}|a_i b_i| < \infty \ \forall a \in S_{[x_n]}\}$. In fact, in order to see that if $b \in S^{\times}_{[x_n]}$, then $\{\Sigma_{i=1}^{n} b_i x_i^* : n \in \mathbb{N}\}$ is weak* convergent in X^* it is enough to show that it is a weak* Cauchy sequence, since X^* is the dual of a Banach space and thus weak* sequentially complete. But this is immediate since for each $x := \Sigma_{i=1}^{\infty} a_i x_i \in X, \Sigma_{i=1}^{\infty}|a_i b_i| < \infty$.

2.Unconditional basis. A sequence space $S \subset \omega$ is said to be normal [5], if given any $a = (a_i)_{i=1}^{\infty} \in S$, then S also contains every $b = (b_i)_{i=1}^{\infty}$ with $|b_i| \leq |a_i|, i \in \mathbb{N}$. From [4,Theorem 7.2] and [1, pag.88] it follows that if $\{x_n : n \in \mathbb{N}\}$ is a basis in X, then $\{x_n : n \in \mathbb{N}\}$ is unconditional if and only if $\Sigma_{i=1}^{\infty} b_i x_i$ converges whenever $\Sigma_{i=1}^{\infty} a_i x_i$ converges in X and $|b_i| \leq |a_i|, i \in \mathbb{N}$. Bearing this in mind the following result is clear,

Proposition 1. *A basis $\{x_n : n \in \mathbb{N}\}$ in X is unconditional if and only if $S_{[x_n]}$ is a normal sequence space.*

Let us recall a sequence space λ is perfect if $\lambda^{\times\times} = \lambda$ and that every perfect space is normal and contains φ, [5]. It is worth pointing out that under the hypothesis of Proposition 1, $S_{[x_n]}$ may fail to be perfect, for instance the unit vector basis $\{e_n : n \in \mathbb{N}\}$ is unconditional in c_0 but $S_{[e_n]} = c_0$ is not perfect. And even though $S_{[x_n]}$ is perfect $\{x_n : n \in \mathbb{N}\}$ may fail to be absolute, i.e. it may be false that $\Sigma_{i=1}^{\infty} a_i x_i$ converges in X if and only if $\Sigma_{i=1}^{\infty}\|a_i x_i\|$ converges, since this is equivalent to say that X is isomorphic to l^1.

Proposition 2. *Let $\{x_n : n \in \mathbb{N}\}$ be a basis in X. If $\{x_n : n \in \mathbb{N}\}$ is unconditional then $S_{[x_n^*]} \subset S^{\times}_{[x_n]}$.*

Proof. Suppose $b \in S_{[x_n^*]}$ and $a \in S_{[x_n]}$. Then we shall set

$$x^* := \sum_{i=1}^{\infty} b_i x_i^*, x := \sum_{i=1}^{\infty} a_i x_i, \xi_i := \frac{\overline{a_i b_i}}{|a_i b_i|} \text{ for } a_i b_i \neq 0 \text{ and } \xi_i := 0 \text{ for } a_i b_i = 0.$$

Now $c = (\xi_i)_{i=1}^{\infty} \in B_{l^\infty}$ and $ac \in S_{[x_n]}$. Hence, for each $n \in \mathbb{N}$,

$$\sum_{i=1}^{n}|a_i b_i| = \sum_{i=1}^{n} a_i b_i \xi_i = \sum_{i=1}^{n} a_i \xi_i x_i(x^*) = x^*(\sum_{i=1}^{n} a_i \xi_i x_i) \leq \|x^*\| \ |||ac||| < \infty,$$

and therefore $b \in S^{\times}_{[x_n]}$.

Proposition 3. *Let $\{x_n : n \in \mathbb{N}\}$ be an unconditional basis in X. Then:*

i) $\{x_n : n \in \mathbb{N}\}$ is shrinking.

ii) $S_{[x_n^*]} = S^{\times}_{[x_n]} = S_{[x_n^*]^*}$.

Proof. i) $\to$ ii). We just have to prove $S_{[x_n^*]^*} \subset S_{[x_n^*]}$ and unconditionallity is certainly not needed for this inclusion. Therefore, assuming $x^* = weak^*\Sigma_{i=1}^{\infty} b_i x_i^*$ we ought to show that $x^* = \Sigma_{i=1}^{\infty} b_i x_i^*$ in the norm topology of X^* but this is just so when X is $\{x_n : n \in \mathbb{N}\}$ is shrinking, [2,Ch.4,Sec.3,Lemma 1].

ii) $\to$ i). As $\{x_n^* : n \in \mathbb{N}\}$ is always a weak* basis for X^*, if $x \in X^*$, there must be $b \in S_{[x_n^*]^*}$ so that $x^* = \text{weak}^*\Sigma_{i=1}^{\infty} b_i x_i^*$. But, by ii), $b \in S_{[x_n^*]}$, i.e. $\Sigma_{i=1}^{\infty} b_i x_i^*$ is convergent in the norm topology of X^* (necessarily to x^*), and consequently $\{x_n^* ; n \in \mathbb{N}\}$ is a basis in X^*. Hence $\{x_n : n \in \mathbb{N}$. is shrinking.

Corollary. *Let $\{x_n : n \in \mathbb{N}\}$ be an unconditional basis in X.If $\{x_n : n \in \mathbb{N}\}$ is shrinking, then $\{x_n^* : n \in \mathbb{N}\}$ is an unconditional basis in X^*.*

Proof. It is clear from Proposition 3 that $S_{[x_n^*]}$ is perfect as it is an α -dual sequence space. Therefore $S_{[x_n^*]}$ is normal and the conclusion follows from Proposition 1.

Remark. Given a normalized unconditional basis $\{x_n : n \in \mathbb{N}\}$ in X, if $\{x_n : n \in \mathbb{N}\}$ is shrinking then the normal topology of $S_{[x_n]}, \mathcal{T}$, is compatible with the dual pair $< S_{[x_n]}, S^*_{[x_n]} >$ and strictly coarser than the norm topology of $S_{[x_n]}$. In fact, if $\{x_n : n \in \mathbb{N}\}$ is unconditional and shrinking then the topological dual of $S_{[x_n]}$ is $S^{\times}_{[x_n]}$ because of the algebraic isomorphisms $S^*_{[x_n]} \cong X^* \cong S_{[x_n^*]} = = S^{\times}_{[x_n]}$. Therefore $\mu(S_{[x_n]}, S^{\times}_{[x_n]}) \supseteq \mathcal{T} \supseteq \sigma(S_{[x_n]}, S^{\times}_{[x_n]})$ and if $\mathcal{T}$ were the same as $\mu(S_{[x_n]}, S^{\times}_{[x_n]})$, for each $\sigma(S^*_{[x_n]}, S_{[x_n]})$ -bounded subset A of $S^*_{[x_n]}$, there would be some $v \in S^{\times}_{[x_n]} = S^*_{[x_n]}$ so that $\sup\{|u_n|, u \in A\} \leq v_n$. Taking $A = \{e_n : n \in \in \mathbb{N}\} \subset S^*_{[x_n]} = S_{[x_n^*]}$, then A is norm bounded in $S_{[x_n^*]}$ and, therefore, $\sigma(S^*_{[x_n]}, S_{[x_n]})$ -bounded. Taking $v \in S_{[x_n^*]}$ such that $\sup\{|\delta_{nm}|, m \in \mathbb{N}\} \leq v_n$ for each $n \in \mathbb{N}$, then $v_n \geq 1$ for each $n \in \mathbb{N}$ and $\Sigma_{n=1}^{\infty} v_n x_n^*$ converges in X^*. Therefore $\lim_{n\to\infty} v_n\|x_n^*\| = 0$ but $\{x_n : n \in \mathbb{N}\}$ is normalized and $x_n^*(x_n) = 1$, so $\|x_n^*\| \geq 1$ for each $n \in \mathbb{N}$. Hence $\lim_{n\to\infty} v_n = 0$. Contradiction.

From this remark it is clear that the normal topologies of c_0 and $l^p, 1 < p \leq \infty$, are strictly coarser than their respective norm topologies and that $\{e_n : n \in \mathbb{N}\}$ is a normalized unconditional basis in l^1 which is not shrinking since the norm and normal topologies coincide on l^1.

Proposition 4. *Let X be a Banach space with an unconditional shrinking basis $\{x_n : n \in \mathbb{N}\}$. Then:*

i) X is reflexive if and only if X is weak sequentially complete.

ii) A subset M of X^* is relatively compact if and only if $\lim_{n\to\infty}\sup\{\|\Sigma_{i=n}^{\infty} x^*(x_i)x_i^*\|, x^* \in M\} = 0$.

Proof. i) $S_{[x_n]}$ is weak sequentially complete if and only if $S_{[x_n]}$ is perfect, [5,Sec.30,5(3)]. But if $S_{[x_n]}$ is perfect, as it is $\mu(S_{[x_n]}, S^*_{[x_n]})$ -barrelled and $S^{\times}_{[x_n]} = S_{[x_n^*]}$ is $\beta(S^*_{[x_n]}, S_{[x_n]})$ -sequentially separable since X^* is separable, it follows from [5,Sec.30,7(4)] that $S_{[x_n]}$ is reflexive.

ii) Let $x \in B_X$ and choose $\xi_i \in \mathbf{K}, |\xi_i| = 1$, so that $\xi_i x_i^*(x)x_i(x^*) = |x_i^*(x)x_i(x^*)|$. Then

$$\|\sum_{i=n}^{\infty} x_i(x^*)x_i^*\| \geq \|x\| \ \|\sum_{i=n}^{\infty} x_i(x^*)x_i^*\| \geq \|x\| \ \|\sum_{i=n}^{\infty} \xi_i x_i(x^*)x_i^*\|$$

since $\{x_n^* : n \in \mathbf{N}\}$ is an unconditional basis of X^*. Therefore,

$$\|\sum_{i=n}^{\infty} x_i(x^*)x_i^*\| \geq x(\sum_{i=n}^{\infty} \xi_i x_i(x^*)x_i^*) = \sum_{i=n}^{\infty} \xi_i x_i(x^*)x_i^*(x) = \sum_{i=n}^{\infty} |x_i^*(x)x_i(x^*)|,$$

so $\lim_{n\to\infty} \sup\{\sum_{i=n}^{\infty} |x_i^*(x)x_i(x^*)|, x \in B_X, x^* \in M\} = 0$ and the result follows from [7,Ch.2,Sec.1,4(20)].

Finally, note that l^1 is perfect but not reflexive and, therefore, no unconditional basis of l^1 may be shrinking. On the other hand if $X = c_0$ or $X = l^p, 1 < p \leq \leq \infty$, then $\{e_n : n \in \mathbf{N}\}$ is an unconditional shrinking basis of X. Hence Proposition 4 guarantees that the bounded subsets K of $l^q, q \geq 1$, are relatively compact if and only if

$$\lim_{n\to\infty} \sup\{(\sum_{i=n}^{\infty} |k_i|^q)^{1/q}, k \in K\} = 0.$$

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Received:23.X.1992

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Tomul XXXVIII, s.I.a, Matematica, 1992, f.4.

INVARIANTS IN REPRESENTATION THEORY

BY

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1.The Character table. The origin of representation theory was to get information on a finite group from its characters.This was also the main goal of F r o b e n i u s , who at the suggestion of D e d e k i n d in 1896 studied the group determinant with the aim of detecting properties of the underlying group in the irreducible factors of the group determinant, their degree and multiplicity of these factors. Eventually he derived the group characters from the irreducible factors of the group determinant. This original aspect was also stressed by R i c h a r d B r a u e r in his Harvard Lecture on 'Modern Mathematics' [1], where he askes in 1963:

What in addition to the character table determines a finite group?

Remark 1. There are several invariants of a finite group G arising in representation theory:

1. The character table $CT(G)$.
2. The spectral table $SP(G)$; i.e. the character table with the p-power map; i.e. the map $K_g \to K_{g^p}$ for K_g the conjugacy class of $g \in G$ and p a rational prime.
3. The 2-characters $CT^2(G)$ -disscused by G. F r o b e n i u s in 1896 [6] -which give the obstruction to the characters being homomorphisms; i.e. $\chi^2(g,h) = \chi(g\cdot h) - \chi(g)\cdot\chi(h)$.
4. The 3-characters - more generally the k-characters $CT^k(G)$ - which Frobenius derived from the group determinant [6]; the k -characters occur as coefficients of some monomials in the group determinant. (We shall discuss these construction below.)
5. The group determinant [6]; i.e. $\mathcal{D}_G := det(X_{g\cdot h^{-1}})$.
6. The table of marks - the Burnside ring $\Omega(G)$ - which tries to describe the group via its permutation representations [2].
7. The rational group algebra QG.