since $\{x_n^* : n \in \mathbf{N}\}$ is an unconditional basis of X^*. Therefore,

$$\|\sum_{i=n}^{\infty} x_i(x^*)x_i^*\| \geq x(\sum_{i=n}^{\infty} \xi_i x_i(x^*)x_i^*) = \sum_{i=n}^{\infty} \xi_i x_i(x^*)x_i^*(x) = \sum_{i=n}^{\infty} |x_i^*(x)x_i(x^*)|,$$

so $\lim_{n\to\infty} \sup\{\sum_{i=n}^{\infty} |x_i^*(x)x_i(x^*)|, x \in B_X, x^* \in M\} = 0$ and the result follows from [7,Ch.2,Sec.1,4(20)].

Finally, note that l^1 is perfect but not reflexive and, therefore, no unconditional basis of l^1 may be shrinking. On the other hand if $X = c_0$ or $X = l^p, 1 < p \leq \leq \infty$, then $\{e_n : n \in \mathbf{N}\}$ is an unconditional shrinking basis of X. Hence Proposition 4 guarantees that the bounded subsets K of $l^q, q \geq 1$, are relatively compact if and only if

$$\lim_{n\to\infty} \sup\{(\sum_{i=n}^{\infty} |k_i|^q)^{1/q}, k \in K\} = 0.$$

REFERENCES

1. B e a u z a m y, B. - *Introduction to Banach spaces and their geometry,* Math. Studies 68. North Holland 1982.
2. D a y, M.M. - *Normed linear spaces,* Springer Verlag 1973.
3. D i e s t e l, J. - *Sequences and series in Banach spaces,* Springer Verlag 1984.
4. J a m e s, R.C. - *Bases in Banach spaces,* Amer. Math. Monthly 89 (1982), 625 - 640.
5. K ö t h e, G. - *Topological vector spaces,* Springer Verlag 1969.
6. M a r t i, J. - *Introduction to the theory of bases,* Springer Verlag 1969.
7. V a l d i v i a, M. - *Topics in locally convex spaces,* Math. Studies 67, North Holland 1982

Received:23.X.1992

Departamento de Matemática Aplicada
Universidad Politécnica de Valencia
Apartado 22012
46071 Valencia
SPAIN

ANALELE STIINTIFICE ALE UNIVERSITATII "AL.I.CUZA" IASI
Tomul XXXVIII, s.I.a, Matematica, 1992, f.4.

INVARIANTS IN REPRESENTATION THEORY

BY

K.W. ROGGENKAMP

1.The Character table. The origin of representation theory was to get information on a finite group from its characters.This was also the main goal of F r o b e n i u s , who at the suggestion of D e d e k i n d in 1896 studied the group determinant with the aim of detecting properties of the underlying group in the irreducible factors of the group determinant, their degree and multiplicity of these factors. Eventually he derived the group characters from the irreducible factors of the group determinant. This original aspect was also stressed by R i c h a r d B r a u e r in his Harvard Lecture on 'Modern Mathematics' [1], where he askes in 1963:

What in addition to the character table determines a finite group?

Remark 1. There are several invariants of a finite group G arising in representation theory:

1. The character table $CT(G)$.
2. The spectral table $SP(G)$; i.e. the character table with the p-power map; i.e. the map $K_g \to K_{g^p}$ for K_g the conjugacy class of $g \in G$ and p a rational prime.
3. The 2-characters $CT^2(G)$ -disscused by G. F r o b e n i u s in 1896 [6] -which give the obstruction to the characters being homomorphisms; i.e. $\chi^2(g,h) = \chi(g \cdot h) - \chi(g) \cdot \chi(h)$.
4. The 3-characters - more generally the k-characters $CT^k(G)$ - which Frobenius derived from the group determinant [6]; the k -characters occur as coefficients of some monomials in the group determinant. (We shall discuss these construction below.)
5. The group determinant [6]; i.e. $\mathcal{D}_G := det(X_{g\cdot h^{-1}})$.
6. The table of marks - the Burnside ring $\Omega(G)$ - which tries to describe the group via its permutation representations [2].
7. The rational group algebra QG.

8. The group algebra FG for all fields.
9. The group algebra over all p-adic rings of integers.
10. The integral group ring ZG.
11. The integral cohomology ring $H^*(G, Z)$.
12. The *mod p* cohomology ring $H^*(G, F)$ for a field F of characteristic $p > 0$.

I shall elaborate here on some aspects of each of the problems related with the above topics.

Remark 2.

1.**Character table.** The quaternion group of order 8 and the dihedral group of order 8 have isomorphic character tables - this was known already to B u r n s i d e, F r o b e n i u s and S c h u r, and was presumably the reason that the question "What properties of a finite group are reflected by the character table" was adressed only relatively late in this century.

On the other hand, the character table reflects quite a lot of properties of the underlying group:

The character table of a finite group G determines:

(a) the length of the conjugacy classes,

(b) the lattice of normal subgroups of G,

(c) the character table of quotient groups,

(d) a chief series of G; a result that was proved by W. K i m m e r l e and in colaboration with R. L y o n s, R. S a n d l i n g and D. T e a g u e [10],[12] a chief series is a normal series of maximal lenght; i.e. the chief factors are simple G -modules.

(e) whether or not for a set π of primes, G has abelian Hall π -subgroups.[1] If the Hall π -subgroups are abelian, then they are determined up to isomorphism. This was proved by K i m m e r l e and S a n d l i n g ([10],[16]) and answers another question of R. B r a u e r [1].

2.**Brauer pairs.** Dade has constructed in 1964,([5]) the first Brauer pair; i.e. non isomorphic groups with isomorphic spectral tables - answering another question of R. B r a u e r [1]. A further series with this property, which we shall discuss in detail later, are certain subgroups of the groups of semi-linear transformations of finite fields; i.e., subgroups of

$$\mathbf{F}_{p^n} \times \mathbf{F}^*{}_{p^n} \times \phi_n \tag{1}$$

where $\mathbf{F}^*{}_{p^n}$ is the group of units in $\mathbf{F}_{p^n}$ and ϕ_n is the Frobenius automorphism. This was observed by G. C l i f f and S u r i n d e r K. S e h g a l [3].

[1] A Hall π subgroup of G is a subgroup H of G such that $|G : H|$ has no prime factor in π.

Group determinant and higher characters. Before we come to the result of the group determinant and higher characters, let us recall the definitions:

Let $G = \{g_1, ..., g_n\}$ be a finite group, and let $\{X_{g1} = X_1, ..., X_{gn} = X_n\}$ be independent indeterminants over the field K. Let us denote by $K(X)$ the field of rational functions over K in these indeterminates.

Dedekind has defined the GROUP DETERMINANT (cf.above) as

$$\mathcal{D}_G = \mathcal{D}_G(X) := det(X_{g_i \cdot g_j^{-1}}) \in K(X). \tag{2}$$

Then $\mathcal{D}_G$ is a homogeneous polynomial of degree $|G|$ in the variables X_g. The group determinant is independent of the numbering of the group elements, as follows from the definition of the determinant.

In modern terminology: Let

$$\mathcal{H}_G = \sum_{g \in G} X_g \cdot g \in K(X)G$$

be the 'generic' element. Then $\mathcal{D}_G$ is the determinant of the element $\mathcal{H}_G$ under the regular representation ρ.

In earlier times $K = \mathbf{C}$ was the field of complex numbers. Let us thus assume that $K = \mathbf{C}$.

In $K(X)$ the group determinant decomposes uniquely - after normalization - into irreducible factors (homogeneous polynominals):

$$\mathcal{D}_G = \Pi_{1 \leq i \leq h} \Phi_i^{m_i}.$$

In modern language: We know that $\mathbf{C}G$ is semisimple, and so

$$\mathbf{C}(X)G = \oplus_{1 \leq i \leq h} M_i^{m_i}$$

decomposes into a direct sum of non isomorphic irreducible modules M_i where M_i has multiplicity $m_i, 1 \leq i \leq h$. Accordingly, the regular representation $\rho = \oplus_{1 \leq i \leq h} \rho_i^{m_i}$ decomposes into the representations ρ_i on M_i, and so

$$\mathcal{D}_G = \Pi_{1 \leq i \leq h} det(\rho_i(\mathcal{H}_G))^{m_i}$$

and one has found a decomposition of $\mathcal{D}_G$. It turns out that this decomposition is even a decomposition into irreducible factors.

F r o b e n i u s in 1896 though did not know about semi-simple algebras, and some of his goals - he eventually reached all of them - were to obtain group theoretical interpretations of the following invariants:

1. The number of the distinct irreducible factors in the decomposition of the group determinant. We know that this is the number of conjugacy classes.

2. The degree of these factors.

3. The multiplicity with which the different irreducible factors occur in the group determinant. We know that the multiplicity coincides with the degree.

Frobenius associated with the group determinant certain functions - the characters - from the finite group G to $\mathbf{C}$ is as follows: If Φ is an irreducible factor of $\mathcal{D}_G$, then he made the following definitions:

1. The CHARACTER [2] $\chi\Phi$ associated to Φ is defined as

- $\chi\Phi(1) = f$, where f is the degree of (Φ).
- For $1 \neq g \in G$ the value $\chi\Phi(g)$ is defined as the coefficient of $X_1^{f-1} \cdot X_g$ in Φ.

IN FROBENIUS OPINION, THE MOST IMPORTANT PROPERTY OF THESE 'CHARACTERS' IS THAT THEY ARE CONSTANT ON CONJUGACY CLASSES; i.e. $\chi\Phi(g \cdot h) = \chi\Phi(h \cdot g)$. He writes on April 17, 1896 to Dedekind:"My feeling that the equation $\chi(a \cdot b) = \chi(b \cdot a)$ provided the key did not deceive me. I still have a long way to go but I am certain I have chosen the right path." [3]

2. More generally, for a natural number k, the k -character associated to the irreducible factor Φ is defined as follows: For $k \leq f$, where f is the degree of Φ, the k -character χ_Φ^k has value on a k - tuple of group elemants $(\gamma_1, \gamma_2, ..., \gamma_k)$, defined as follows:

- the coefficient of $X_1^{f-k} \cdot X_{\gamma_1} \cdot ... \cdot X_{\gamma_k}$ in Φ, provided none of the $\gamma_i = 1$.
- If $\gamma_i = 1$ for some i, then

$$\chi_\Phi^k(\gamma_1, \gamma_2, ..., \gamma_{i-1}, 1, \gamma_{i+1}, ..., \gamma_k) =$$
$$\chi_\Phi^{k-1}(\gamma_1, \gamma_2, ..., \gamma_{i-1}, \gamma_{i+1}, ..., \gamma_k).$$

[2] Frobenius had asked Dedekind, whether he knew of a good name for what Frobenius had discovered, and he asked whether this should not be termed 'character' since for abelian groups it reduced by Dedekinds formula to the characters introduced by Gauß.

[3] later at the suggestion of Dedekind, Frobenius tried to link his characters with the then by Schur developed theory of representations, and found out that his characters are the traces of the irreducible representations. At first - even for the orthogonality relations, derived by Frobenius - representations did not enter into the picture.

It is then clear that for an irreducible factor Φ of the group determinant one has

$$(3) \qquad \Phi = (1/f!) \cdot \left(\sum_{\{\gamma_1, \gamma_2, ..., \gamma_f\}} \chi_\Phi^f(\gamma_1, \gamma_2, ..., \gamma_f) \cdot X_{\gamma_1} \cdot X_{\gamma_2} \cdot ... \cdot X_{\gamma_f} \right)$$

Moreover, already Frobenius noticed that the k -characters can be derived inductively from the ordinary characters via the following formula:

$$\chi_\Phi^k(\gamma_1, \gamma_2, ..., \gamma_i, ..., \gamma_k) = \chi_\Phi(\gamma_1) \cdot \chi_\Phi^{k-1}(\gamma_2, ..., \gamma_k) -$$
$$-\chi_\Phi^{k-1}(\gamma_1 \cdot \gamma_2, ..., \gamma_i, ..., \gamma_k) - ... - \chi_\Phi^{k-1}(\gamma_1, ..., \gamma_{k-1} \cdot \gamma_k)$$

This formula is closed related to the differential of the standard resolution for groups:

$$(\gamma_1, ..., \gamma_k)d_k = (\gamma_2, ..., \gamma_k) +$$
$$\Sigma_{j=1,...,k}(-1)^j(\gamma_1, ..., \gamma_j \cdot \gamma_{j+1}, ..., \gamma_k) + (-1)^k(\gamma_1, ..., \gamma_{k-1} \cdot \gamma_k)$$

(Cf. also the singular homology in topology).
THE k -CHARACTERS CAN BE DERIVED FROM THE ORDINARY CHARACTERS, IF ONE KNOWS IN ADDITION TO THE CHARACTER TABLE $CT(G) := (\chi_i(K_j))_{1 \leq i,j \leq h}$ – K_j ARE THE CONJUGACY CLASSES - THE VALUES OF $\chi_i(\gamma_1 \cdot ... \cdot \gamma_k)$ FOR k BOUNDED BY THE MAXIMUM OF THE DEGREES OF THE IRREDUCIBLE REPRESENTATIONS; i.e. one has to know the map

$$\psi_k : G \times G \times ... \times G \to \cup K_j,$$
$$(\gamma_1, ..., \gamma_k) \to K_{\gamma_1 \cdot ... \cdot \gamma_k}.$$

This apparently is not as strong as the knowledge of the multiplication table for G. However, as we shall see later, this is a very powerful condition.

1.(**2-characters**) It is noted by K.W. Johnson and Surinder K. Sehgal [9] in 1991 that the groups from Equation 1 also have the same 2-characters. Their 3-characters though are different.

2.(**3-characters**) It was shown independently by H.J.Hoehnke in colaboration with K.W.Johnson , [8] and with different techniques by Kimmerle - Roggenkamp [15] [4] that the character table and the 3-characters determine a finite group up to isomorphism.

[4] It was reported to us, that other people had made the same observation.

THIS IS ONE ANSWER TO R. BRAUER'S QUESTION:'What is needed in addition to the character table, to determine the finite groop G?'

3.(k **-characters**) The k -characters where introduced by Frobenius in 1986 [6] as coefficients of the irreducible factors of the group determinant, and so they are determined by the group determinant. On the other hand, the group obviously determines the group determinant, and so the above implies that the ordinary characters and the 3-characters determine the group determinant.

4.(**Group determinant**) On the other hand, in 1991 E. Formanek and D. Sibley have shown -using invariant theory- that the group determinant determines the group up to isomorphism [7], so that the above result incorporates this one. R. Mansfield [17], has given an easy direct proof of this result by Formanek and Sibley.

Table of Marks. Let us recall the definition of the Burnside ring and the Burnside matrix: For the finite group G let $\Omega'(G)$ be the category of finite left G -sets with G -equivariant maps. Every finite G -set can be written uniquely as a disjoint union of transitive G -sets; i.e. G -sets isomorphic to $G//H$ - the set of left cosets of the subgroup H in G. Two such transitive G -sets are isomorphic if and only if the corresponding subgroups are conjugate in G .The isomorphism classes of finite G -sets form the Burnside ring $\Omega(G)$; multiplication is given by the cartesian product and the sum is the disjoint union. A $\mathbf{Z}$ -basis is given by the isomorphism classes of transitive G -sets. The Burnside matrix has the rows and columns indexed by the conjugacy classes of subgroups of G, and the entry corresponding to the classes of subgroups $((U),(V))$ is given by $Hom_{\Omega(G)}(G//U, G//V) = \{G \in G|^gU \subset V\}$.

Burnside matrices 1. It is easy to construct two non isomorphic groups which have isomorphic Burnside matrices - i.e. table of marks; this means essentialy that the groups have the same lattice of subgroups. The easiest examples are the two non isomorphic groups

$$(4)\qquad \begin{aligned} G &:=< a,b,c \mid a^7, b^{13}, c'[a,b],\ {}^ca = a^2,\ {}^cb = b^3 > \ \textit{and} \\ H &:=< a,b,c \mid a^7, b^{13}, c'[a,b],\ {}^ca = a^4,\ {}^cb = b^3 > . \end{aligned}$$

It is not so easy though - and was long an open problem - to construct a Brauer pair with the same table of marks. A detailed analysis of the Brauer pair constructed by Cliff and Surinder Sehgal [3] (cf. above) - we give an explicit description below - shows however that they also have the same table of marks, as was noted by Kimmerle and the author [15]. These groups

1. have isomorphic spectral tables,
2. have isomorphic Burnside matrices and
3. have isomorphic 2-characters.

A detailed analysis of the above examples gave theoretical arguments, which answer these questions in the following way [15]:

Proposition 1. *1. Let G_1 and G_2 be isomorphic groups, which act linearly on an elementary abelian p-group V such that*

(a) the groups G_i act fixed point freely on V; i.e. $Stab_{G_i}(v) = 1$ for every $0 \neq v \in V$, where $Stab_*(*)$ is the stabilizer.

(b) For every $v \in V$ we have for the orbits

$$\mathcal{O}_{G_1}(v) = \mathcal{O}_{G_2}(v);$$

i.e., G_1 and G_2 have the same orbits on V.
Then the semi-direct products $H_i = V \times G_i$ have isomorphic spectral tables. [5]

2. Under the above assumption, there is a unique bijection for each $v \in V$

$$\tau_v : G_1 \to G_2 \ \textit{defined by} \ {}^gv = {}^{\tau_v(g_1)}v,$$

for $g_1 \in G_1$.

Moreover, these maps τ_v determine maps

$$\rho_v : G_1 \times G_1 \to G_1 \ \textit{defined by} \ {}^{\tau_v(g)\cdot h}v = {}^{\rho_v(g,h)\cdot h}v,$$

where $g, h \in G_1$. We assume now that in addition to the above

(a) all minimal subgroups of V are conjugate under G_1 and G_2,

(b) there exists $v_0 \in V$ such that for every subgroup $H \leq G_1$ we have

$$\rho_{v_0} \downarrow_{H \times H} : H \times H \to H,$$

i.e., ρ_{v_0} somehow preserves the subgroup structure of G_1.
Then the above semi direct products H_1 and H_2 have isomorphic Burnside matrices.

3. Assume that $\rho : G \to H$ is an isomorphism of finite groups and that M is a module for both G and H. If

(a) there exits a ρ -equivariant bijection - not neccesarily a group homomorphism -

$$\sigma : M \to M \ \textit{with} \ \sigma(g \cdot m) = \rho(g) \cdot \sigma(m); g \in G, m \in M,$$

(b) $M\backslash\{0\}$ consists of a single orbit for both G and H,

[5] *The examples of Cliff and Sehgal [3] are of this type; as a matter of fact, there $V\backslash\{0\}$ consists of exactly one orbit under G_i.*

then the semi direct products $M \times G$ and $M \times H$ have isomorphic 2- characters.

Example 1. Let us briefly DECRIBE THE GROUPS: For a prime p we denote by $\mathbf{F}_{p^n}$ the field with p^n elements. ϕ_n is the Frobenius automorphism, and

$$\Omega(p,n) = \mathbf{F}_{p^n} \times \mathbf{F}_{p^n}^{\times} \times < \phi_n > .$$

Then the groups which we have constructed [6] are certain subgroups of $\Omega(p,n)$, which are described in detail in [15]. The smallest example is for $p = 7$ and $n = 3$. Then for these numbers

$$\mathbf{F}_{7^3}^{\times} = < a > \cdot < b > \cdot < c >$$

is the product of a cyclic groups of order 19,2,9 resp. and

$$H_1 := \mathbf{F}_{7^3} \times < a, b, c \cdot \phi_3 >$$

and

$$H_2 := \mathbf{F}_{7^3} \times < a, b, c \cdot_3^2 >$$

are not isomorphic groups, which have

1. isomorphic spectral tables,
2. isomorphic 2-characters and
3. isomorphic Burnside matrices.

2.The Isomorphism problem and Cohomology. We now turn to the various aspects centering around the isomorphism problem.

Note 1. 1. Since abelian groups are determinated by their primary parts, it is easily seen that an abelian group is determined by its rational group algebra. But a much stronger statement is valid: Let $A = \Pi A_p$ be the p-primary decomposition of A. Then $A = lim.proj_p A_p$ is also the projective limit of its p-primary parts. The group ring $\mathbf{Q}A$ though is not the projective limit of the group rings of the p-primary components. Let $\Gamma_{\mathbf{Q}A} := lim.proj_p \mathbf{Q}A_p$ - an example is given below - then $\Gamma_{\mathbf{Q}A}$ is a proper epimorphic image of $\mathbf{Q}A$. The above discussion actually shows:

2. Let $\Gamma_{\mathbf{Q}A} \simeq \Gamma_{\mathbf{Q}B}$ for an abelian group A, then $A \simeq B$; i.e. the structure of the abelian group is captured already in a small part of the rational group algebra.

3. E.C. Dade has constructed in 1971, [4] two non isomorphic groups G and H of order $p^3 \cdot q^6$ which have isomorphic group rings over every field

[6] These groups are very similar to those considered by Cliff and Sehgal [3]

and even over the p-adic integers $\hat{\mathbf{Z}}_p$ for every prime p. These groups though have non isomorphic integral groups rings, since they are metabelian. One can construct smaller groups than Dade's with these properties [19].

4. Under the aspect of Dade's examples the most far reaching result thus far [20] is the following: Let G be a finite group with a normal p-subgroup N, such that $C_G(N)$, the centralizer of N is a p-group. If $\mathbf{Z}G \simeq \mathbf{Z}H$, then G and H are p-adically conjugate; in particular, G and H are conjugate in $\mathbf{Q}G$.

5. In particular, this implies that for a solvable group G, the various quotients $G/O_{p'}(G)$ are uniquely determined by the integral group ring. The group itself is the projective limit of the projective system generated by the various $G/O_{pt}(G)$; but as mentioned above for abelian groups, the group ring $\mathbf{Z}G$ is not the projective limit of the group rings $\mathbf{Z}G/O_{p'}(G)$.

6. Important consequences of this result are

(a) Let E be a finite group given via the exact sequence

$$0 \to N \to E \to G \to 0,$$

where N is an abelian group and $G = \Pi_{1 \leq i \leq n} G_i$, where $O_{p_i'}(G_i) = 1$ and the various G_i have relatively prime order, then the isomorphism problem for $\mathbf{Z}G$ has a positive answer. (This was proved by L.L.Scott in collaboration with the author in case G was nilpotent and in the general situation by A.Zimmermann [21] following suggestions of the author.)

(b) Assume that $[G, G]$ is nilpotent, then the isomorphism problem for $\mathbf{Z}G$ has a positive answer.[11]

7. Let G and H be non isomorphic groups such that for every rational prime p, the goups $G/O_{pt}(G)$ and $H/O_{pt}(H)$ are isomorphic, then the cohomology rings

$$H^*(G,R) := \sum_{i=0,1\dots} H^i(G,R) \text{ and } \sum_{i=0,1,\dots} H^i(H,R)$$

are isomorphic for all fields R and all complete Dedekind domains of characteristic zero with finite residue field. Since $H^*(G,\mathbf{Z})$ is determined by the p -adic cohomology rings, this shows at the same time that

$$H^*(G,\mathbf{Z}) \simeq H^*(H,\mathbf{Z}).$$

Examples of such groups are the groups from Equation 4.

Example 2. Let us describe the ring $\Gamma_{\mathbf{Q}G}$: Let $G = C_p \times C_q$ be the direct product of a cyclic group of order p and one of order q, where p and q are different prime numbers. Then

$$\mathbf{Q}G \doteq \mathbf{Q} \oplus \mathbf{Q}(\zeta_p) \oplus \mathbf{Q}(\zeta_q) \oplus \mathbf{Q}(\zeta_{p \cdot q}],$$

where ζ_n denotes a primitive n -th root of unity. Then

$$\Gamma_{\mathbf{Q}G} = \mathbf{Q} \oplus \mathbf{Q}(\zeta_p) \oplus \mathbf{Q}(\zeta_q).$$

3.The isomorphism problem and variations on the Zassenhaus conjecture for group bases. We have listed above some classes of groups for which the isomorphism problem has a positive answer. We shall briefly recall some properties, which are detected by the integral group ring:

Proposition 2. *Assume that $ZG = ZH$ as augmented algebras* [7] *Then*

1. the class sums of G and those of H coincide; i.e. there is a bijection between the conjugacy classes of G and H such that correponding class sums coincide in $\mathbf{Z}G$,

2. G and H have isomorphic lattices of normal subgroups.

3. The group ring determines nilpotent Hall subgroups up to isomorphism, as well as hamiltonian Hall subgroups (W. Kimmerle [10] and R. Sandling [16]),

4. G and H have isomorphic spectral tables - hence abelian Hall subgroups are determined.

Conjecture 1.

1. One of the fascinating open problems in representation theory of finite groups is the isomorphism problem, which asks

$$ZG \simeq ZH \Longrightarrow G \simeq H?.$$

2. This is closely related to the **Zassenhaus conjecture:**

$$\mathbf{Z}G = \mathbf{Z}H \Longrightarrow G = a \cdot K \cdot a^{-1} \text{ for some } a \in \mathbf{Q}H.$$

3. The Zassenhaus conjecture is equivalent to the following statement: Assume that $\mathbf{Z}G = \mathbf{Z}H$ as augmented algebras. Then the class sum correspondence says that there is a **bijection**[8] $\beta : G \to H$ such that in $\mathbf{Z}G$ we have

$$K_g = H_{\beta(g)} \text{ for every } g \in G.$$

The Zassenhaus conjecture now is true if and only if β can be choosen, to be a group **isomorphism.**

4. Yet another way of phrasing the Zassenhaus conjecture is as follows:

(a) The isomorphism problem has a positive answer for G.

(b) For every augmented automorphism α of $\mathbf{Z}G$ there is a group automorphism ρ of G such that $\alpha \cdot \rho^{-1}$ is a **central** automorphism.

5. The above rephrasing of the Zassenhaus conjecture gives rise to an interesting modification, which has mainly been considered by W. Kimmerle:

(a) **Zassenhaus conjecture for p-power classes:** Assume that $ZG = ZH$ as augmented algebras, then there exists an isomorphism $\beta : G \to H$ such that $K_g = K_{\beta(g)}$ for $g \in G$ a p -power element for some prime p. Clearly this variation also implies that the isomorphism problem has a positive answer.

Let me summarize the known results on the Zassenhaus conjecture and its variation.

Proposition 3. *1. To the Zassenhaus conjecture:*

(a) The class of groups, for which the Zassenhaus conjecture holds is closed under direct products - the same statement holds for its variation.

(b) The Zassenhaus conjecture is true for groups which have a normal p-subgroup containing its centralizer - and hence for products of those. In particular, it is true for nilpotent groups.

(c) The Zassenhaus conjecture is true symmetric groups, as was noted by G. Peterson [18] (cf. also W. Kimmerle [11]).

(d) If G is solvable, and $\mathbf{Z}G = \mathbf{Z}H$ as augmented algebras, then the Sylow p-subgroups of G and H are conjugate in $\mathbf{Q}G$[13].

(e) There is a metabelian group and an automorphism

$$\alpha : \mathbf{Z}G \to \mathbf{Z}G,$$

which is a counterexample to the Zassenhaus conjecture.

(f) Zassenhaus has further - after the above counterexample was known -conjectured that the Zassenhaus conjecture is true for abelian Sylow tower groups [9] *There is not yet known a complete proof. However, in a first step M. Hertweck - advised by W. Kimmerle - has shown that for metabelian groups with abelian Sylow subgroups, central automorphisms are inner, thus verifying the variation of this conjecture for certain nilpotent extensions of these groups.*

2. To the variation of the Zassenhaus conjecture.

(a) The variation surely holds for a class of groups, provided the Zassenhaus conjecture holds for this class.

[7] This means that ZG and ZH have the same augmentation map $ZG \to Z$.

[8] Note that not every bijection between groups is an isomorphism.

[9] These groups are defined inductively: G has a normal abelian Sylow p-subgroup P for some prime p and G/P is an abelian Sylow tower group.

(b) The variation of the Zassenhaus conjecture holds for G provided [G,G] is nilpotent. The counterexample to the Zassenhaus conjecture mentioned above is of this form.

Remark 3. *Let me speculate about the isomorphism problem: It was known to L.L. Scott and the author that a counterexample to the isomorphism problem would imply the existence of a counterexample to the isomorphism problem would imply the existence of a counterexample to the Zassenhaus conjecture. Therefore it was neccesary, to construct a counterexample to the Zassenhaus conjecture. Once we had done this, we thought that a counterexample to the isomorphism problem should now be relatively close. This was not the case though. However, we did not have a counterexample to the variation of the Zassenhaus conjecture, and one can show that a counterexample to the isomorphism problem would imply the existance of a counterexample to the variation of the Zassenhaus conjecture.*

REFERENCES

1. Brauer, R. – *Representations of finite groups.Lectures on modern mathematics*, Vol.I, 133 - 175, Wiley New York, 1963, reproduced in Vol.II of: Richard Brauer: collected papers. MIT press, Cambridge MA, 1980.
2. Curtis, C.W., Reiner, I. – *Methods of Representation Theory*, Vol.2, John Wiley Interscience (1987).
3. Cliff, G., Surinder, K.Sehgal – *On groups having the same character tables*, Com. Algebra 9 (1981), 627 - 640.
4. Dade, E. – *Deux groupes finis ayant la même algèbre de groupe sur tout corps*, Mathematische Zeitschrift 119 (1971), 345 - 348.
5. Dade, E. – *Answer to a question of R. Brauer* , J. of Algebra, 1 (1964), 1-4.
6. Frobenius, G. – *Über die Primfaktoren der Gruppendeterminante*, Sizungsbrichte der Akad. d. Wiss. Berlin 1896, 1343 - 1382.
7. Formanek, E., Sibley, D. – *The group determinant determines the group*, to appear Proc. AMS 1991.
8. Hoehnke, H.-J., Johnson, K. – *The 1-, 2-, 3- characters are sufficient to determine a group*, Preprint 1991.
9. Johnson, K., Surinder, K.Sehgal – *The 2 -character table does not determine a group*, MS 1991.
10. Kimmerle, W. – *Beiträge zur ganzzahligen Darstellungstheorie endlicher Gruppen*, Bayreuther Math. Schr. Heft 36 (1991).
11. Kimmerle, W. – *Variations of the Zassenhaus conjecture in Group Rings and Class Groups*, Ed. Roggenkamp, Taylor, DMV Seminar 18, Birkhäuser 1992, 91 - 103.

12. Kimmerle, W., Lyons, R., Sandling, D. Teague – *Composition factors from the group ring and Artin's theorem on orders of simple groups*, Proceedings LMS (3) 60 (1990), 89 - 122.
13. Kimmerle, W., Roggenkamp, K., – *A Sylowlike Theorem for Integral Group Rings of Finite Solvable Groups*, appears in Arch. d.Math.
14. Kimmerle, W., Roggenkamp, K., – *Group determinants and 3-characters*, MS Stuttgart 1991.
15. Kimmerle, W., Roggenkamp, K., – *Non isomorphic groups with isomorphic spectral tables and Burnside matrices*, MS Stuttgart 1992.
16. Kimmerle, W., Sandling, S. – *Group and group ring theoretic determination of certain Sylow and Hall subgroups and the resolution of a question of R. Brauer*, Preprint 1991.
17. Mansfield, R. – *A group determinant determines its group*, Preprint 1991.
18. Peterson, G. – *Automorphisms of the integral group rings of S_n*, Proceedings AMS Vol. 59, No.1, (1976), 14 - 18.
19. Roggenkamp, K.W. – *Group Rings and Class Groups*, Ed. Roggenkamp, Taylor, DMV Seminar 18, Birkhäuser 1992, 91 - 103.
20. Scott, L.L. – *Recent progress on the isomorphic problem*, Proc. of Symposia in Pure Math., Vol.47, 1987, 259 - 273.
21. Zimmermann, A. – *Das Isomorphieproblem ganzzahliger Gruppenringe für Gruppen mit abelschen Normalteiler und Quotienten, der eine Vermutung von Hans Zassenhaus erfüllt*, Diplomarbeit, Univerität Stuttgart, (1990).

Received:5.XI.1992

Mathematisches Institut
Universität Stuttgart
Deutschland