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CONVERGENCE TO PROPAGATING CLINES IN THE FISHER EQUATION

BY

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1. Introduction. We are motivated by the Fisher model [1-2] of population genetics. In this model, $u(x,t)$ represents the relative density of one (say, a) of two gene alleles (a and A) at a particular gene locus in the population of a migrating diploid species. Assume three genotypes mate randomly. The function $u(x,t)$ satisfies the equation

$$\frac{\partial u}{\partial t} - \Delta u = [(f_1 + f_2)u - f_2]u(1-u) \tag{1.1}$$

where the ratio of the fitnesses of the three genotypes is

$$1 + f_1(x,t) : 1 : 1 + f_2(x,t). \tag{1.2}$$

When the fitnesses are of seasonal temporal variations only, f_j are generally considered to be periodic in t. Our study concerns with the situations where there are some propagation effects in the spatial variations of the fitnesses due to some factors (such as a spreading of "genetic defect" due to a certain desease, or some finite speed evolutions of external environmental influence). Thus the fitnesses f_j also depend on $(\nu \cdot x - ct)$ for some $\nu \in I\!R^n$. We assume that f_1 and f_2 are of the form:

$$f_j(x,t) = f_j(\nu \cdot x - ct, t) \tag{1.3}$$

which are periodic in the second variable. In that sense, the fitnesses are seasonal as well as propagating. We are interested in the population pattern under such a circumstance. Due to the form of f_j, it is nature to consider the problem in the whole $I\!R^n$ space rather than any bounded domain in x.

We consider the problem in a slightly general setting:

$$\frac{\partial u}{\partial t} = \Delta u + k \cdot \nabla u + [(f_1 + f_2)u - f_2]u(1-u), \ (x,t) \in I\!R^n \times (0,\infty). \tag{1.4a}$$

$$u(x,0) = u_0(x), \ 0 < u_0 < 1, \ \text{for } x \in I\!R^n. \tag{1.4b}$$

One might expect the solution $u(x,t) \in (0,1)$ would asymptotically converge to $0, 1$ or some limiting function of the form $\tilde{u}(\nu \cdot x - ct, t)$ as $t \to \infty$. If $u \equiv 0$, and $u \equiv 1$ are linearly unstable, then all solutions would approach some propagating clines $\tilde{u}(\nu \cdot x - ct, t)$ which is periodic in t variable. The term "propagating cline" is adopted from [9] in the expression of its propagating pattern.

In this work, we give conditions on f_j for the existence of a cline solution which is an interesting pattern in genetics. Further we show that under the same conditions, the cline is unique, and it attracts any solution of (1.4) except for 0 and 1.

The problem of the existence of time-independent clines has been considered by many authors (e.g. [5-11]) in cases where

$$f_1 = f_1(x), \quad f_2 = f_2(x).$$

The existence of stable periodic solutions for this type of equations was proved in [3, 12-13]. Our work here is influenced by a recent work of Hess and Weinberger [3] which concerns the existence of time-periodical clines at Fisher Equation and the convergence of any solutions at $t \to \infty$, and particularly by their novel use of maximum principles. Our result here is considered an extension of the methods in [3] in different equations, and however, significantly differs from [3] in the way that the domain of x here is the entire $I\!R^n$, and our cline solution is of a very particular form.

We organize this paper in the following way. In Section 2, we shall provide some technical lemmas which will be utilized later to derive the main theorems. In Section 3, we shall discuss the conditions that assure any solution of its initial value bounded away from 0 and 1 to be forever bounded away from 0 and 1. We shall then in Section 4 prove the main result about the existence and uniqueness of a propagating cline. In Section 5, we shall show that the cline is a global attractor. Some remarks will be given in Section 6.

2. Convergence conditions. We state a maximum principle which was first introduced in [3]. Our statement here differs slightly from the maximum principle in [3] in order to accommodate the $k \cdot \Delta u$ term:

Lemma 2.1. *Assume $u(x,t)$ are solutions valued in $(0,1)$ of the equation*

$$\frac{\partial u(x,t)}{\partial t} = \Delta u + k \cdot \nabla u + m(x,t,u)h(u), \ x \in I\!R^n, \ t \in (0,\infty) \tag{2.1}$$

where $k \in I\!R^n$ is a vector, $m(x,t,u)$ is nonincreasing in u and $h(u)$ is concave for $u \in (0,1)$ with $h(0) = 0$, $h(1) = 0$. If $u_1(x,t)$ and $u_2(x,t)$ are two solutions of (2.1) and $\sigma \le u_j \le 1 - \sigma$ for some $\sigma > 0$, then the nonnegative function

$$Z(x,t) \equiv \left| \int_{u_1(x,t)}^{u_2(x,t)} \frac{\partial \varsigma}{h(\varsigma)} \right|^2 \tag{2.2}$$

satisfies the maximum principle, i.e., $\max_{-\infty < x < \infty} Z(x,t)$ is nonincreasing in t; if

$$\max_x Z(x,t_1) = \max_x Z(x,t_2)$$

for some $t_1 \neq t_2$, then $Z(x,t) \equiv k$.

Proof. It can be easily verified that for any two solutions $u_1(x,t)$ and $u_2(x,t)$,

$$\begin{aligned} &\frac{\partial Z}{\partial t} - \Delta Z - k \cdot \nabla Z - h'(u_1)\left(\frac{\nabla u_2}{h(u_2)} + \frac{\nabla u_1}{h(u_1)}\right)\nabla Z = \\ &= 2[m(x,t,u_2) - m(x,t,u_1)] \int_{u_1}^{u_2} \frac{d\varsigma}{h(\varsigma)} + \\ &+ 2\left|\frac{\nabla u_2}{h(u_2)}\right|^2 [h'(u_2) - h'(u_1)] \int_{u_1}^{u_2} \frac{d\varsigma}{h(\varsigma)} - 2\left|\nabla \int_{u_1}^{u_2} \frac{d\varsigma}{h(\varsigma)}\right|^2 \le 0. \end{aligned} \tag{2.3}$$

When u_1, u_2 are uniformly bounded away from 0 and 1, $Z(x,t)$ is bounded for all x, t. Lemma 2.1 follows from the strong parabolic maximum principle in [14]. Lemma 2.1 is also true if the operator Δ is replaced by $\nabla \cdot (A\nabla)$ for $A = A' > 0$.

Lemma 2.2. *If $m(x,t,u)$ is nonincreasing in u, $m(x,t,u)$ is periodic in t of period T, and $h(u)$ is concave. Let $u(x,t)$ be a solution of (2.1) of its value bounded away from 0 and 1 uniformly in x and t. Then there exists a solution $\hat{u}$ of (2.1) which is periodic in t of period T such that $|u(x,t) - \hat{u}(x,t)| \to 0$ uniformly in x on every bounded subset of $I\!R^n$ as $t \to \infty$*

Proof. The idea of proof is essentially adopted from [3]. Take the sequence of functions $\{u(x, nT+t)\}$ to form a bounded and equicontinuous family of functions. The boundedness and equicontinuity properties of the family of functions are derived from a simple a priori bound for $u(x,t)$. For any $R>0$, $T>0$, there exist a subsequence $\{u(x, n_jT+t)\}$ which converge uniformly in $|x|<R$, $0<t<T$ to a limit function $\hat{u}(x,t)$ which is also a solution of (2.1) once the regularity is assured. Let

$$z(x,t) = \left| \int_{u(x,t)}^{u(x,T+t)} \frac{\partial\varsigma}{h(\varsigma)} \right|^2. \tag{2.4}$$

Then $z(x,t)$ satisfies the inequality (2.3). The function $\max_{x\in\mathbb{R}^n} z(x,t)$ is nonincreasing in t and its limit satisfies

$$\lim_{t\to\infty} \max_{x\in\mathbb{R}^n} z(x,t) = k^2 \geq 0.$$

Let $n_j\to\infty$, $u(x, t+n_jT)\to\hat{u}(x,t)$. Since $u(x,t)$ are uniformly bounded away from 0 and 1, so does $\hat{u}(x,t)$. We have $\lim_{n_j\to\infty} z(x, t+n_jT) = \hat{z}(x,t)$ uniformly on $|x|<R$, $t<T$ with

$$\hat{z}(x,t) = \left| \int_{\hat{u}(x,t)}^{\hat{u}(x,T+t)} \frac{\partial\varsigma}{h(\varsigma)} \right|^2. \tag{2.5}$$

Moreover, $\max_x z(x,t)\to k^2$ as $t\to\infty$; then $\max_x \hat{z}(x,t) = k^2$. The strong maximum principle then implies that $\hat{z}(x,t)\equiv k^2$ is a constant independent of x and t. We obtain

$$\int_{\hat{u}(x,t)}^{\hat{u}(x,T+t)} \frac{\partial\varsigma}{h(\varsigma)} = k. \tag{2.6}$$

Thus, we have

$$\int_{\hat{u}(x,t)}^{\hat{u}(x,nT+t)} \frac{\partial\varsigma}{h(\varsigma)} = nk. \tag{2.7}$$

Because $\hat{u}(x,t)$ is uniformly bounded away from 0 and 1 as $x\in\mathbb{R}^n$, $t\to\infty$, we conclude

$$\left| \int_{\hat{u}(x,t)}^{\hat{u}(x,nT+t)} \frac{\partial\xi}{h(\xi)} \right| \leq M. \tag{2.8}$$

Thus $k=0$ which implies that $\hat{u}(x,t)$ is periodic in t with period T since $h(\xi)>0$. Let

$$z^*(x,t) \equiv \left| \int_{\hat{u}(x,t)}^{u(x,t)} \frac{\partial\varsigma}{h(\varsigma)} \right|^2. \tag{2.9}$$

Then $z^*(x,t)$ also satisfies the maximum principle. If analogous arguments are applied to $u(x, t+n_jT)$ and $\hat{u}(x, t+n_jT) = \hat{u}(x,t)$, then $z^*(x, t+n_jT)\to 0$ as $n_j\to\infty$ from the monotonicity of $\max_x z^*(x,t)$. We get $z^*(x,t)\to 0$ uniformly in x on every bounded subset of $\mathbb{R}^n$ as $t\to\infty$. The convergence of $u(x,t)$ to $\hat{u}(x,t)$ follows from the fact that $\frac{1}{h}$ is uniform positive.

3. The existence of a propagating cline. We now analyze the Fisher Equation with propagating fitnesses (1.4). The fitnesses that we have assumed in the model, are dependent on the propagation variable $\xi_1 = \nu\cdot x - ct$ and the time t. The effect of a spreading of a genetic disease is modelled by a propagation of the fitness change with a directed finite speed. Along with that, there is a temporal (seasonal) change of the fitnesses. For further consideration, we might also consider the cases where the fitnesses are propagating along different directions (i.e., $f_j = f_j(\nu_1\cdot x - c_1t, \nu_2\cdot x - c_2t, \ldots, \nu_k\cdot x - c_kt, t)$), or maybe the propagation are seasonal as well (i.e., $f_j(\xi_1, t) = f_j(\xi_1 + X_1, t)$), or the propagation is radial ragher than directional. The results that we obtain below can be easily extended to those cases by analogous analysis. Some remarks regarding those cases will be made in Section 6.

We now consider the equation

$$\frac{\partial u}{\partial t} = \Delta u + k\cdot\nabla u + [(f_1+f_2)u - f_2]u(1-u) \tag{3.1a}$$

$$u|_{t=0} = u_0(x) \tag{3.1b}$$

where $f_j = f_j(\nu\cdot x - ct, t)$ are periodic in t of period T. The coefficients $k\in\mathbb{R}^n$, $\nu\in\mathbb{R}^n$ are constant vectors $\|\nu\|=1$. We are interested in $u(x,t)$ of values in $(0,1)$.

Lemma 3.1. *Suppose that there exists $\delta>-1$ so that $h = u(1-u)/(1+\delta u)$ is concave, and for any $(x,t)\in\mathbb{R}^n\times(0,\infty)$, $(f_1, f_2) = (f_1, f_2)(x,t)$ satisfies $(1+\delta)f_1 + f_2 \leq -|\delta||f_1+f_2|$. Assume the initial value $u_0(x)$ satisfies* *the property that $0 < \sigma \le u_0(x) \le 1-\sigma < 1$ for some $\sigma > 0$. Then either (a) $\exists (x_n, t_n)$ such that $u(x_n,t_n) \to 0$ as $n \to \infty$ or (b) $\exists (x_n,t_n)$ such that $u(x_n,t_n) \to 1$ as $n \to \infty$ or (c) there exists a solution of the form $\hat{u}(\nu\cdot x - ct, t)$ which is periodic in t of period T such that $u(x,t) \to \hat{u}(\nu\cdot x - ct, t)$ as $t \to \infty$ uniformly on every bounded set of $\xi_1 \equiv \nu\cdot x - ct, \xi_2, \xi_3, \dots, \xi_n$ where $\xi_i = \nu_j \cdot x$ are chosen so that $(\nu, \nu_2, \dots \nu_n)$ are independent.*

Proof. If we exclude the possibilities (a) and (b), then the solution $u(x,t)$ remains bounded away from 0 and 1. We take a new set of independent variables $\xi_1 = \nu\cdot x - ct, \xi_2, \dots, \xi_n$ and t. Here $\xi = (\xi_1, \dots, \xi_n)^t = B\cdot(x_1, \dots, x_n)^t - ct(1, 0, \dots, 0)^t$ with some nonsingular $B \equiv (\nu, \nu_2, \dots, \nu_n)$. Without loss of generality, we assume B to be orthonormal, i.e., $B \cdot B^t = I_n$. Then (3.1) becomes

$$\frac{\partial u}{\partial t} = \Delta_\xi u + k\cdot(B\cdot\nabla_\xi u) + c\frac{\partial u}{\partial \xi_1} + \\ +[(f_1(\xi, t) + f_2(\xi_1, t))u - f_2(\xi_1, t)]u(1-u). \tag{3.2}$$

For the convenience of the notation, we may still use $k_1\cdot\nabla$ for the first derivatives. We apply Lemma 2.2 to obtain $u(\xi_1, \xi_2, \dots, \xi_n, t) \to \hat{u}(\xi_1, \xi_2, \dots, \xi_n, t)$ for some $\hat{u}(\xi_1, \xi_2, \dots, \xi_n, t)$ which is periodic in t. In order to show that $\hat{u}$ is independent of $\xi_2, \dots, \xi_n$, we consider two functions: $\hat{u}(\xi_1, \dots, \xi_j, \dots, \xi_n, t)$ and $\hat{u}(\xi_1, \dots, \xi_j + X_j, \dots, \xi_n, t)$ for $X_j > 0$, $j \ge 2$. We note that both $\hat{u}(\xi_1, \dots, \xi_j, \dots, \xi_n, t)$ and $\hat{u}(\xi_1, \dots, \xi_j + X_j, \dots, \xi_n, t)$ are solutions of (3.2) since f_j are free of $\xi_2, \dots, \xi_n$. The corresponding function

$$\hat{Z}_{X_j}(\xi_1, \xi_2, \dots, \xi_j + X_j, \dots, \xi_n, t) \equiv \left| \int_{u(\xi_j, t)}^{u(\xi_j + X_j, t)} \frac{\partial \varsigma}{h(\varsigma)} \right|^2 \tag{3.3}$$

satisfies the maximum principle of Lemma 3.1. Follow the similar steps as showing the periodicity of $\hat{u}(\xi, t)$ in t, we can show

$$\hat{u}(\xi_1, \dots, \xi_j + X_j, \dots, \xi_n, t) = \hat{u}(\xi_1, \dots, \xi_j, \dots, \xi_n, t) \tag{3.4}$$

for any $X_j > 0$, $j \ge 2$. Thus, $\hat{u} = \hat{u}(\xi_1, t)$ is independent of ξ_j for $j \ge 2$.

Corollary 3.2. *If $f_j(\nu\cdot x - ct, t)$ is also periodic in $\xi_1 \equiv \nu\cdot x - ct$ of period X_1. Then the corresponding functions $\hat{u}(\xi_1, t) \equiv \hat{u}(\nu\cdot x - ct, t)$ are also periodic in ξ_1.*

Proof. The procedure is also similar. By comparing $\hat{u}(\xi_1 + X_1, t)$, $\hat{u}(\xi_1, t)$ through $\hat{z}(\xi, t)$, we obtain

$$\hat{u}(\xi_1 + X_1, t) = \hat{u}(\xi_1, t). \tag{3.5}$$

Thus the corollary follows.

Remark. The condition that $f_j(\nu\cdot x - ct, t)$ depends on $\nu\cdot x - ct$ periodically represents the situation where the propagating of the influencing factor is also seasonal (periodical). One might also think certain defects of generic properties are inhabited by generations. Hence the propagating change of generic fitnesses is also cyclic. Corollary 3.2 then states the corresponding propagating clines are also of the periodical propagating character.

Corollary 3.3. *If $f_j(\nu\cdot x - ct, t) \equiv f_j(\nu\cdot x - ct)$ are independent of the variable t, then the corresponding limiting solutions $\hat{u}(\nu\cdot x - ct, t)$ are of the form $\hat{u} = \hat{u}(\nu\cdot x - ct)$ which are travelling waves.*

Proof. We can follow the analogous argument to show that $\hat{u}$ is periodic of period T_1 for any $T_1 > 0$.

Lemma 3.4. *Let the same assumptions in Lemma 3.1 hold. Assume there are two solutions $\hat{u}_1$ and $\hat{u}_2$ of the form: $\hat{u}_j = \hat{u}_j(\nu\cdot x - ct, t)$ which are periodic in t of period T. Suppose that $0 < \hat{u}_j < 1$, and $\hat{u}_j$ are bounded away from 0 and 1 uniformly in x and t. Then*

(1) *Either $\hat{u}_1 - \hat{u}_2$ are bounded away from 0 uniformly in (x,t), or $\hat{u}_1 \equiv \hat{u}_2$.*

(2) *If $f_1 + f_2$ is not identically zero, then $\hat{u}_2 \equiv \hat{u}_1$.*

(3) *If $f_1 + f_2 \equiv 0$ but $f_2(\nu\cdot x - ct, t) \not\equiv f_2(t)$, then $\hat{u}_2 \equiv \hat{u}_1$.*

(4) *If there are two distinct solutions $\hat{u}_1$ and $\hat{u}_2$ which are periodic in t of period T, then $\hat{u}_j(\nu\cdot x - ct, t) \equiv \hat{u}_j(t)$, $f_1 + f_2 \equiv 0$, $f_2 \equiv f_2(t)$ and $\int_0^T f_2(t)dt = 0$. If this is the case, then all periodic solutions form a uniformly continuous, one parameter family of solutions, which satisfy the equation*

$$u_t = -f_2(t)u(1-u), \tag{3.6a}$$

$$u|_{t=0} = L, \quad 0 < L < 1. \tag{3.6b}$$

Proof. It was implied in the proof of Lemma 3.1 that there exists a constant k such that

$$\hat{Z}(x,t) = \left| \int_{u_1(x,t)}^{u_2(x,t)} \frac{d\eta}{h(\eta)} \right|^2 = k^2. \tag{3.7}$$

From this fact, we deduce that if $\hat{u}_1(x,t) = \hat{u}_2(x,t)$ at some (x,t), then $k = 0$, which then implied $\hat{u}_1(x,t) \equiv \hat{u}_2(x,t)$. If $k \neq 0$, since both $\hat{u}_j(x,t)$ are bounded away from 0 and 1, we get that $\hat{u}_2(x,t)$ are bounded away from $\hat{u}_1(x,t)$ from (3.7). This shows (1).

The assertion (2) can be derived from the maximum principle in Lemma 3.1 by substitute $\hat{Z}(x,t) = k^2$ into (2.3). Since lefthand side is identically zero, if $f_1 + f_2 \not\equiv 0$, then $u_1 \equiv u_2$.

We also observe if $f_1 + f_2 \equiv 0$ and if $u_1 \neq u_2$, then $h'(u_2) = h'(u_1)$ and $|\nabla u_2| \equiv 0$. Thus $u_2 \equiv u_2(t)$ which implies $f_2 = \dfrac{1}{h(u_2)} \cdot \dfrac{\partial u_2(t)}{\partial t} \equiv f_2(t)$. This is the conclusion of (3).

(4) is a degenerate case which has been widely studied. References can be found in [3] or [13]

4. Sufficient conditions for stable propagationg clines. We now consider the conditions upon which any solution $u(x,t)$ with any nontrivial initial condition would converge to one unique solution of the form $\hat{u} = \hat{u}(\nu \cdot x - ct, t) \subset (0,1)$ that is periodic in t of period T.

Lemma 4.1. *Let $u_0(x)$ be the initial value of a solution $u(x,t)$ of (3.2) and satisfy $1 > u_0(x) \geq \sigma > 0$. Suppose $-f_2(\xi_1,t) \geq p_1(t)$ for some T-periodic function $p_1(t)$ with $\int_0^T p_1(t)dt > 0$. Then $u(x,t)$ is bounded away from 0 in a uniform fashion in $(x,t) \in I\!R^n \times (0,\infty)$.*

Proof. We find a positive constant $u_0 > 0$ such that $u(x_0) \geq u_0$. For $0 \leq u \leq 1$, we observe that the function $\phi_1(t,u_0)$, which is the solution of

$$u_t = (p_1(t) - Mu)u(1-u) \tag{4.1a}$$

$$u|_t = u_0 \tag{4.1b}$$

where $M = \sup|f_1 + f_2|$, is a subsolution of (3.2). This is due to the fact $0 < \phi_1(t,u_0) < 1$ and the fact that

$$\begin{aligned} \frac{\partial(\phi_1)}{\partial t} &= (p_1(t) - M\phi_1)\phi_1(1-\phi_1) \leq \\ &\leq (-f_2 + (f_1+f_2)\phi_1)\phi_1(1-\phi_1) = \\ &= \Delta_\xi \phi_1 + k_1 \cdot \nabla\phi_1 + [(f_1+f_2)\phi_1 - f_2]\phi_1(1-\phi_1). \end{aligned} \tag{4.2}$$

Further, since the linearized equation of $\phi_1(t)$ at 0 has the form:

$$\frac{\partial\tilde{\phi}_1(t)}{\partial t} = p_1(t)\tilde{\phi}_1(t) \tag{4.3a}$$

$$\tilde{\phi}_1(0) = u_0 \tag{4.3b}$$

with $\int_0^T p_1(t) > 0$, we obtain $\tilde{\phi}_1(t) \to +\infty$ as $t \to \infty$. Thus, we get that $\lim_{t\to\infty} \phi_1(t) \geq \alpha_1 > 0$. Since $\phi_1(t)$ is a subsolution of (3.2), then $u(x,t) \geq \phi_1(t)$ is bounded away from 0 uniformly in (x,t).

Lemma 4.2. *Let $u(x,t)$ be a solution of (3.2) with the initial value $u(x,0) = u_0(x)$ which satisfies $0 \leq u(x) \leq 1 - \sigma < 1$. Suppose $f_1 \leq -q_1(t)$ for some T-periodic function $q_1(t)$ with $\int_0^T q_1(t)dt > 0$. Then $u(x,t)$ is bounded away from 1 in a uniform manner.*

Proof. We follow an analogous way as in Lemma 4.1 to achieve the result.

We now state our main result.

Theorem 1. *Let $u(x,t)$ be any solution of (3.1) valued in $(0,1)$. Suppose there exists $\delta > -1$ such that (f_1, f_2) satisfies $(1+\delta)f_1 + f_2 \leq \leq -|\delta||f_1 + f_2|$ for $(x,t) \in I\!R^n \times I\!R^+$. Further we assume $f_1 \leq -q_1(t)$, $f_2 \leq -p_1(t)$ for some periodic functions $p_1(t)$, $q_1(t)$ of period T with $\int_0^T p_1(t)dt > 0$, and $\int_0^T q_1(t)dt > 0$. Also assume $f_j = f_j(\nu \cdot x - ct, t)$ which is periodic in t of period T. Then*

(1) *given any u_0 which satisfies the property that $0 < \sigma \leq u_0(x) \leq 1 - \sigma < 1$, the solution $u(x,t)$ of (3.1) with the initial value $u(x,0) = u_0(x)$ satisfies the property that $\lim_{t\to\infty} |u(x,t) - \hat{u}(\nu \cdot x - ct, t)| = 0$ for some $\hat{u}(\nu \cdot x - ct, t)$ which is a solution of (3.1), and is periodic in t of period T.*

(2) *$\hat{u}(\nu \cdot x - ct, t)$ is unique, and stable. Given any $0 < \sigma \leq u_0 \leq 1 - \sigma < 1$, $u(x,t,u_0) \to \hat{u}$ in a supernorm, i.e., the basin of attraction contains at least all points in*

$$B = \{u_0(x) \mid 0 < \sigma \leq u_0(x) \leq 1 - \sigma < 1\}. \tag{4.4}$$

(3) *If $f_j \equiv f_j(\nu \cdot x - ct)$ is independent of t, then $\hat{u} \equiv \hat{u}(\nu \cdot x - ct)$, and rest of the results in (1), (2) hold, i.e., every solution of initial value uniformly bounded away from 0 and 1 will be attracted to a unique, stable travelling wave $\hat{u}(\nu \cdot x - ct)$. If $f_j(\xi_1)$ is periodic in ξ_1, then $\hat{u}(\xi)$ is also periodic in ξ_1.*

Proof. It follows from Lemmas 4.1 and 4.2 that $u(x,t)$ is bounded away from 0 and 1. Then Lemma 3.1 assures that

$$u(x,t) = u(\xi_1, \xi_2, \ldots, \xi_n, t) \longrightarrow \hat{u}(\xi_1, t) \equiv \hat{u}(\nu \cdot x - ct, t) \tag{4.5}$$

as $t \to \infty$ for some $\hat{u}(\nu \cdot x - ct, t)$, a solution of (3.1) which is periodic in t of period T. Also $\hat{u}(\nu \cdot x - ct, t)$ is bounded away from 0 and 1. Since $f_1 + f_2 \not\equiv 0$ by the condition $f_1 + f_2 \le -(p_1 + q_1)$ with $\int_0^T p_1 > 0$ and $\int_0^T q_1 > 0$, if $\hat{u}_1, \hat{u}_2$ are two solutions of the form $\hat{u}_j = \hat{u}_j(\nu \cdot x - ct, t)$ which are periodic in t, then $\hat{u}_1 \equiv \hat{u}_2$ by Lemma 3.2. From the uniqueness of the attractor, any solution $u(x,t)$ which is uniformly bounded away from 0 and 1 will approach $\hat{u}(\nu \cdot x - ct, t)$.

(3) is a special case of (1) and (2).

Remark 1. Examples of qualified systems can be handy, such as

$$u_t = \Delta u + k \cdot \nabla u + \lambda(\nu \cdot x - ct, t) \cdot (a(\nu \cdot x - ct, t) - u) u (1 - u) \tag{4.6}$$

where $\lambda \cdot a \ge p_1(t)$ with $\int_0^T p_1 > 0$ and $\lambda \cdot (1 - a) \ge q_1(t)$ with $\int_0^T q_1 > 0$. Special cases can be $\lambda \cdot a \ge \sigma_1 > 0$ and $\lambda(1 - a) \ge \sigma_2 > 0$ if both λ and a are taken to be constant.

Remark 2. In degenerate cases where $f_j = f_j(\xi_1, t)$ are periodic in ξ_1 of period X_1, the conditions for instability of $u = 0, 1$ can be strengthen into

$$\int_0^{X_1} \int_0^T f_j(\xi_1, t) dt \, d\xi_1 \le 0. \tag{4.7}$$

With periodic f_j on the variables ξ_j, we can simply consider the equation (3.2) with the periodical boundary conditions. Using the comparisons with the first eigenfunction with the semigroup generated by the linearized operator, we can derive that (4.7) implies $u = 0, 1$ are unstable. References can be found in [3].

Remark 3. When $\delta = 0$, the condition about (f_1, f_2) in Theorem 1 is simply that $f_1 + f_2 \le 0$ which is always satisfied in the case where $f_j = f_j(\xi)$.

5. The global attractor.

So far, our attention was focused on the solutions whose initial values are bounded away from 0 and 1. In this section, we show that under same conditions every solution of a initial value $u_0(x)$ with $measure\{x \mid u_0 \ne 0, 1\} > 0$ will converge to $\hat{u}(\nu \cdot x - ct, t)$. Thus $\hat{u}(\nu \cdot x - ct, t)$ is a global attractor.

Theorem 2. *Let $u(x,t)$ be any solution of (3.1) valued in $[0,1]$. Let $f_j = f_j(\nu \cdot x - ct, t)$ be periodic in t of period T. Assume there exists $\delta < -1$ such that*

$$(1 + \delta) f_1 + f_2 \le -|\delta||f_1 + f_2| \quad for \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}^+, \tag{5.1}$$

and further $f_1 \le -q_1(t)$, $f_2 \le -p_1(t)$ for periodic functions $p_1(t)$, $q_1(t)$ with period T, $\int_0^T p_1(t)dt > 0$, $\int_0^T q_1(t)dt > 0$. Then given $u_0(x)$ such that $measure\{x \mid u_0(x) \ne 0\} > 0$, $measure\{x \mid u_0(x) \ne 1\} > 0$, the solution $u(x,t)$ of (3.1) with the initial value $u_0(x)$ satisfies the property that

$$\lim_{t \to \infty} |u(x,t) - \hat{u}(\nu \cdot x - ct, t)| = 0 \tag{5.2}$$

uniformly in every bounded set of $\xi = (\xi_1, \ldots, \xi_n)$ where $\xi = x - k \cdot t$.

Proof. Without loss of generality, we show that given any $u_0(x)$ with

$$u_0(x) = \begin{cases} 1 > u_0(x) > 0 & x \in I = (a,b) \\ = 0 & x \notin [a,b] \end{cases} \tag{5.3}$$

which is of a compact support, the solution $u(x,t,u_0) \to \hat{u}(\nu \cdot x - ct, t)$. With an analogous result showing $u(x,t,v_0) \to \hat{u}(\nu \cdot x - ct, t)$ for any v_0 with

$$v_0(x) = \begin{cases} 0 < v_0(x) < 1 & x \in (c,d) \\ = 1 & x \notin [c,d], \end{cases} \tag{5.4}$$

the comparison principle would force all solutions to go to $\hat{u}(\nu \cdot x - ct, t)$. We show $u(x,t,u_0) \to \hat{u}(\nu \cdot x - ct, t)$ in the following manner. First we write the equation in the new independent variables

$$(t, \xi_1 = x_1 - k_1 t, \xi_2 = x_2 - k_2 t, \ldots, \xi_n = x_n - k_n t), k = (k_1, k_2, \ldots, k_n).$$

Then (3.1) becomes

$$\frac{\partial u}{\partial t} = \Delta_\xi u + [(f_1(\xi,t) + f_2(\xi,t)u - f_2(\xi,t)]u(1-u). \tag{5.5}$$

Let $\psi_1(x,t)$ be the solution of the equation:

$$u_t = \Delta_\xi u + (p_1(t) - Mu)u(1-u), \tag{5.6a}$$

$$u(\xi,t)|_{t=0} = u_0(\xi). \tag{5.6b}$$

Evidently, for sufficient large $M = M(f_i) > 0$, $\psi_1(x,t)$ is a subsolution of (5.5) since $\psi_1(x,t) \geq 0$ and $-f_2 > p_1(t)$, $f_1 + f_2 \geq -M$. We intend to show that there exists $\alpha > 0$, such that

$$\liminf_{t\to\infty} u(\xi,t) \geq \liminf_{t\to\infty} \psi_1(\xi,t) \geq \alpha > 0 \tag{5.7}$$

uniformly for every bounded subset of ξ in $\mathbb{R}^n$. The equation (5.6) can be rewritten as

$$\frac{\partial u}{\partial t} = \Delta_\xi u + p_1(t)u + u^2 q(u, p(t)) \tag{5.8a}$$

$$u(\xi,t)|_t = u_0(\xi) \tag{5.8b}$$

where $p_1(t)$ is periodic and $\int_0^T p_1(t) > 0$.

Let

$$P_1(t) = \int_0^t p_1(\tau)d\tau - \bar{p}_1 \tag{5.9}$$

where

$$\bar{p}_1 = \frac{1}{T}\int_0^T p_1(t)dt > 0. \tag{5.10}$$

Then P_1 is periodic of period T as well. We let

$$u = v \cdot e^{P_1(t)} \equiv v \cdot e^{(\int_0^t p_1(\tau)d\tau - \bar{p}_1)} \tag{5.11}$$

The equation (5.8) becomes

$$v_t = \Delta_\xi v + \bar{p}_1 v + v^2 h(v, P_1(t), p_1(t)), \tag{5.12a}$$

$$v(\xi,t)|_{t=0} = u_0(\xi), \tag{5.12b}$$

where $h(v, P_1, p_1)$ is uniformly bounded in t since it contains periodic functions $p_1(t)$ and $P_1(t)$.

We now consider the function $\chi_1(x,t)$ which is the solution of

$$v_t = \Delta_\xi v + \bar{p}_1 v - M_1 v^2 \equiv \Delta_\xi v + F_1(v) \tag{5.13a}$$

$$u(\xi,t)|_{t=0} = u_0(\xi). \tag{5.13b}$$

When M_1 is sufficiently large, $\chi_1(\xi,t)e^{P_1(t)}$ is a subsolution of $\psi_1(\xi,t)$ in (5.8), i.e., $\psi_1(\xi,t) \geq \chi_1(\xi,t)e^{P_1(t)}$. A famous result of A r o n s o n and W e i n b e r g e r [4] tells that if $F_1(v) > 0$ for $v < \alpha$, and $\lim_{v\to 0+} F_1(v) \cdot v^{-(1+\frac{2}{n})} > 0$ and $v(\xi,0) = u_0(\xi) > 0$ in a set of nonzero measure, then the solution $v(x,t)$ satisfies $\liminf_{t\to\infty} v(\xi,t) > \alpha$ in any bounded set of $\xi = (\xi_1,\dots,\xi_n)$ in $\mathbb{R}^n$. Thus we obtain that $\liminf_{t\to\infty} u(\xi,t) \geq \alpha$ uniformly for every boundes set of ξ in $\mathbb{R}^n$. Upper bounds can be similarly achieved provided $f_1 \leq -q_1(t)$.

We now show $u(\xi,t) \to \hat{u}(\xi,t)$ as $t\to\infty$ where $\hat{u}(\xi,t) = \hat{u}(\nu\cdot x - ct, t)$ is the unique solution of (3.1) which is periodic in t according to Lemma 4.1. We consider two functions

$$u^+(\xi,t) = \limsup_{n\in\mathbb{Z}^+}\{u(\xi, t+nT)\}, \tag{5.14a}$$

$$u^-(\xi,t) = \liminf_{n\in\mathbb{Z}^+}\{u(\xi, t+nT)\}. \tag{5.14b}$$

There exist subsequences $u(\xi, t+n_j^{\pm}T) \longrightarrow u^{\pm}(\xi,t)$ uniformly in any bounded set of $\xi \in \mathbb{R}^n$. Assured of the regularity of $u(\xi,t)$, we get that $u^{\pm}$ are solutions of (5.5). Following the same procedure, we can drive that $u^{\pm}(\xi,t) = u^{\pm}(\nu\cdot x - ct, t)$ from the fact $f_j = f_j(\nu\cdot x - ct, t)$. Also, we can show $u^{\pm}(\xi,t) \equiv u^{\pm}(\nu\cdot x - ct, t)$ are periodic in the second variable t. The argument used in Theorem 4.1 is still applicable here since $u^{\pm}(\xi,t)$ are bounded away from 0 and 1 which was achieved above by the estimate

$\psi_1(\xi,t)$. We then obtain that $u^+(\xi,t) = u^-(\xi,t) = \hat{u}$ by the uniqueness of the propagating periodic cline shown in Theorem 1. Also for every bounded set B of $\xi \in I\!R^n$

$$u(\xi, t + nT) \le u^+(\xi,t) + \varepsilon$$

for $n \ge N_1(B)$, and also $u^+(\xi, t + nT) \ge u^-(\xi,t) - \varepsilon$ for $n \ge N_2(B)$. Then since $u^+(\xi,t) = u^-(\xi,t) = \hat{u}(\xi,t)$ and $\hat{u}(\xi,t) = \hat{u}(\xi, t+T)$. We obtain that for $t > N_1 + N_2$, $|u(x,t) - \hat{u}(x,t)| \le \varepsilon$. This concludes Theorem 2.

Remark. When $k = 0$, the convergence is uniform at any bounded set of $\xi = x \in I\!R^n$.

6. Some remarks about propagating clines and convergence. We make some remarks about the existence and the convergence of propagating clines.

First if the fitnesses are evolving along the radius rather than some particular direction, i.e., $f_j = f_j(r - ct, t)$. Then the resulting propagating cline solution $\hat{u}(x,t)$ are also of the form $\hat{u} = \hat{u}(r - ct, t)$ as well. This result can be achieved by showing that $\hat{u}$ is symmetric along any spatial direction. Further, under similar conditions, every nontrivial solution will converge to $\hat{u}$ as well. These systems have realistic examples, considering some of the diseases can spread equally in every spatial direction.

Second, results can be extended to the cases where the fitnesses are of the form $f_j = f_j(\nu_1 \cdot x - c_1 t, \nu_2 \cdot x - c_2 t, \cdots, \nu_k \cdot x - c_k t, t)$ which are propagating along several directions. The resulting propagating clines are expected to be of the same form. Yet, the conditions for convergence are essentially the same.

Third, if we remove the conditions that 0, 1 are unstable, then the dynamics becomes more complicated. There could be a family of propagating clines rather than one, or there could be the cases where every solution is attracted to 0 or 1. We consider the following example:

$$\text{(6.1)} \qquad \frac{\partial u}{\partial t} = \Delta u + k \cdot \nabla u + \left(u - \frac{1}{2}\right) u(1-u) \quad \text{in } I\!R^n \times I\!R^+.$$

Suppose $k = 0$, and we consider the standing waves. We find the steady state $u = \frac{1}{2}$ is unstable and there exist a family of periodic solutions $u_\lambda(x,t)$. As the analysis of a general system of (6.1) indicated in [15] that there are some solutions of form $u(\nu \cdot x - ct, t)$ which is of the property that $\lim_{t \to -\infty} u(x,t) = 1/2$ and $\lim_{t\to\infty} u(x,t) = u_\lambda(\nu \cdot x - ct)$ where u_λ is a periodic solution. We observe also that $u = 1/2$ has at least two unstable manifolds

as any spatially uniform solution $u = u(t)$ will go to $u = 0$ or $u = 1$ respectively according to the initial conditions.

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