Remarque. L'application v du diagramme 1 est monotone.

En effet, si $(i,x) \leq (j,y)$, alors il existe $k \in I$ un majorant commun de i et j tel que $f_{ki}(x) \leq f_{kj}(y)$. Compte tenu du fait que u_k est monotone et des relations 1 on a successivement

$$v(i,x) = v\sigma_i(x) = u_i(x) = u_k f_{ki}(x) \leq u_k f_{kj}(y) = c_j(y) = v\sigma_j(y) = v(j)$$

et donc $v(i,x) \leq v(j,y)$. Puisque "$\sim$" est l'équivalence associée à la relation de préordre "$\leq$" considérée sur S $(S/\sim, \leq)$ et ordonné, aussi, il existe une unique application monotone u telle que $uq = v$.

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SOME KINDS OF APPROXIMATE EFFICIENT POINTS AND ITS APPLICATIONS TO THE VECTORIAL OPTIMIZATION THEORY WITH MULTIFUNCTIONS

BY

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0. The approximate extremum points and the approximate subdifferential of a convex function have been proved to be a useful tool in real and vectorial analysis from the theoretical viewpoint as well as for practical purposes.

The vectorial optimization problems have been studied specially in finite dimensional spaces having in view to establish necessary, sufficient or necessary and sufficient conditions for the existence of the solutions and to obtain duality results for vectorial optimization programs with multivalued objective mappings. Thus, for example, T. T a n i n o and Y. S a w a r a g i in [8],[9] constructed and developed a duality theory for multiobjective optimization problems and H. K a w a s a k i [5] extended their results using the concepts of "subdiferential" and "conjugate" for multifunctions from a linear space to $R^n_{\leq}$. For optimization problems with the objective maps taking values in order complete vector lattices then are some notable results in the monographies [2], [5].

In general vectorial ordered spaces, the optimization problems are strongly related with the efficient points of sets. There are some existence results for this in [2], [6].

In [2] the efficient points are presented in connection with the study of the vectorial subdifferential for multifunctions.

This paper intends to present the natural extensions of the ***approximate efficient points*** and the ***approximate vectorial subdifferential*** for multifunctions with their general properties.

1. In what follows $(Z, \leq)$ will denote a real vector space ordered by a convex cone Z_+. We suppose that Z_+ is pointed, i.e.

$$Z_+ \cap (-Z_+) = \{0\}.$$

We add to Z a *smallest element* denoted by $-\infty$ and a *largest element* denoted by $+\infty$; we consider $\bar{Z} = Z \cup \{+\infty\} \cup \{-\infty\}$.

The operations on Z can be extended to $\bar{Z}$ using the following conventions:

$$(\pm\infty) + x = x + (\pm\infty) = \pm\infty \quad \forall x \in Z$$

$$(\pm\infty) + (\pm\infty) = \pm\infty$$

$$\lambda(\pm\infty) = (\mathrm{sgn}\lambda)(\pm\infty) \quad \forall \lambda \in R$$

We denote $a < b$ if $b - a \in Z_+$ and $a \neq b$. Let A be a nonempty set of $\bar{Z}$ such that $A \neq \{+\infty\}$ and $p \in \bar{Z}$, $\varepsilon \in Z$.

Definition 1.1. i) *We shall say that $p \in \bar{Z}$ is an ε infimum for S if there is no $a \in A$ such that $a < p - \varepsilon$. The set of ε infimum points for A is denoted by* ${}^{\varepsilon}\mathrm{Inf}\, A$.

ii) *We shall say that $p \in \bar{Z}$ is an ε minimum for A if p is an ε infimum for A and $p \in A$. The set of ε-minimum points for A is denoted by* ${}^{\varepsilon}\mathrm{Min}\, A$.

iii) *We shall say that $p \in \bar{Z}$ is an ε proximal infimum for A if p is an ε-infimum for A and for all $p \in Z$ with $p' > p$ there is $a' \in A$ such that $a' < p'$. The set of ε proximal infimum points for A is denoted by* ${}^{\varepsilon}\mathrm{Inf}_1\, A$.

In a similar manner, for a nonempty set $A \subseteq \bar{Z}, A \neq \{+\infty\}, p \in \bar{Z}$, $\varepsilon \in Z$ we define the sets ${}^{\varepsilon}\mathrm{Sup}\, A, {}^{\varepsilon}\mathrm{Max}\, A, {}^{\varepsilon}\mathrm{Sup}_1\, A$.

For $\varepsilon = 0$ we get the notions from [2], denoted by $\mathrm{Inf}\, A, \mathrm{Inf}_1\, A, \mathrm{Min}\, A$, $\mathrm{Sup}\, A, \mathrm{Sup}_1\, A, \mathrm{Max}\, A$. If $A = \emptyset$ or $A = \{+\infty\}$ we consider ${}^{\varepsilon}\mathrm{Inf}\, A = \bar{Z}$ and if $A = \emptyset$ or $A = \{-\infty\}$ we consider ${}^{\varepsilon}\mathrm{Sup}\, A = \bar{Z}$, $\forall \varepsilon \in Z$.

Remarks.

a) If $\varepsilon \in -Z_+$ then ${}^{\varepsilon}\mathrm{Min}\, A = \emptyset$.

b) ${}^{\varepsilon}\mathrm{Inf}\, A = -{}^{\varepsilon}\mathrm{Sup}(-A)$

${}^{\varepsilon}\mathrm{Inf}_1\, A = -{}^{\varepsilon}\mathrm{Sup}_1(-A)$

${}^{\varepsilon}\mathrm{Min}\, A = -{}^{\varepsilon}\mathrm{Max}(-A)$

c) ${}^{\varepsilon}\mathrm{Inf}\, A = \mathrm{Inf}(A + \varepsilon) = \mathrm{Inf}\, A + \varepsilon$

d) ${}^{\varepsilon}\mathrm{Inf}_1\, A = {}^{\varepsilon}\mathrm{Inf}_1(A + Z_+)$

${}^{\varepsilon}\mathrm{Sup}_1\, A = {}^{\varepsilon}\mathrm{Sup}_1(A - Z_+)$

e) If $B, C \subset \bar{Z}$, $B \neq \emptyset$, $B \subseteq C$ then we have

${}^{\varepsilon}\mathrm{Inf}\, C \subset {}^{\varepsilon}\mathrm{Inf}\, B$ and ${}^{\varepsilon}\mathrm{Sup}\, C \subset {}^{\varepsilon}\mathrm{Sup}\, B$.

f) $\varepsilon_1 \leq \varepsilon_2 \implies {}^{\varepsilon_1}\mathrm{Inf}\, A \subset {}^{\varepsilon_2}\mathrm{Inf}\, A$.

Proposition 1.1. i) *If A is a nonempty set in $\bar{Z}$ such that $A \neq \{+\infty\}$ then*

$${}^{\varepsilon}\mathrm{Min}\, A = A \cap {}^{\varepsilon}\mathrm{Inf}\, A = A \cap {}^{\varepsilon}\mathrm{Inf}_1\, A$$

ii) *If A is a nonempty set in $\bar{Z}$ such that $A \neq \{-\infty\}$ then*

$${}^{\varepsilon}\mathrm{Max}\, A = A \cap {}^{\varepsilon}\mathrm{Sup}\, A = A \cap {}^{\varepsilon}\mathrm{Sup}_1\, A$$

Proof. i) It is evident that $A \cap^{\varepsilon} \mathrm{Inf}_1\, A \subseteq A \cap^{\varepsilon} \mathrm{Inf}\, A$. Now let $a \in A \cap^{\varepsilon} \mathrm{Inf}\, A$ and $p' > a$. We get that there is $a' \in A, a' = a < p'$ so $A \cap^{\varepsilon} \mathrm{Inf}\, A \subseteq A \cap^{\varepsilon} \mathrm{Inf}_1\, A$ and i) is proved. ii) It is similar.

Remark 1.1. It is obvious that for $A \subset Z$;

$$\mathrm{Min}\, A \subset {}^{\varepsilon}\mathrm{Min}\, A, \quad \forall \varepsilon \in Z_+$$

The converse inclusion does not holds.

For example let $Z = R^2$ with its usual positive cone and let, $A = \{(x_1, x_2), x_1^2 + x_2^2 < 1\}$.

We have $\mathrm{Min}\, A = \emptyset$ but ${}^{\varepsilon}\mathrm{Min}\, A = [(A - \varepsilon) \cap (A + \mathbb{R}_+^2)^c] + \varepsilon$.

Proposition 1.2. *Let A be a nonempty set from $\bar{Z}$. If* $\mathrm{Inf}\, A \neq \emptyset$ *then* ${}^{\varepsilon}\mathrm{Min}\, A \neq \emptyset$, $\forall \varepsilon \in Z_+ \setminus \{0\}$.

Proof. Let $a \in \mathrm{Inf}\, A$ and $\varepsilon \in Z_+ \setminus \{0\}$. Thus there is $a \in A$ such that $a + \varepsilon > a$. Suppose that $a \notin^{\varepsilon} \mathrm{Min}\, A$, so there is $b \in A$ such that $b < a - \varepsilon$. This implies that $b < a$, contradicting that $a \in \mathrm{Inf}_1\, A \subset \mathrm{Inf}\, A$.

Remark 1.2. For $\varepsilon = 0$ the proposition 1.2 is not true. See for that the example from Remark 1.1.

Definition 1.2. i) *Let A be a nonempty set, $A \subset \bar{Z}, p \in \bar{Z}$ and $\varepsilon \in Z_+$. We shall say that p is an ε -ideal minimum for A if $p \in A$ and $p - \varepsilon \leq a, \forall a \in A$. The set of ε -ideal minimum points for A while be denoted by* ${}^{\varepsilon}\mathrm{IMin}\, A$.

ii) *If (Z, τ) is a locally convex space, let $A \neq \emptyset, A \subset Z, p \in Z, \varepsilon \in Z_+$. We shall say that p is an ε - properly minimum for A if there is K a convex cones, $K \neq Z$ with $Z_+ \setminus \{0\} \subset K^0$ such that p is an ε minimum point for A with respect to K. The set of all ε - properly minimum points for A will be denoted by* ${}^{\varepsilon}\mathrm{PrMin}\, A$.

iii) *If (Z, τ) is a locally convex space and $\overset{\circ}{Z}_+ \neq \emptyset$, let $A \neq \emptyset, A \subset Z$, $p \in Z$, $\varepsilon \in Z_+$. We shall say that p is an ε-weak minimum point for A if p is an ε-minimum point for A with respect to $\{0\} \cup \overset{\circ}{Z}_+$. The set of all ε-weak minimum points for A will be denoted by* ${}^{\varepsilon}\mathrm{WMin}\, A$.

For $\varepsilon = 0$ we get the notion from [6] denoted by $\mathrm{IMin}\,A, \mathrm{PrMin}\,A, \mathrm{WMin}\,A$.

Also, some basic properties, established in [6] may be extended.

Proposition 1.3. *For all nonempty sets $A \subset Z$, we have the following inclusions:*

$$^{\varepsilon}\mathrm{PrMin}\,A \subset {}^{\varepsilon}\mathrm{Min}\,A \subset {}^{\varepsilon}\mathrm{WMin}\,A$$

Proof. We are proving the inclusion

$$^{\varepsilon}\mathrm{PrMin}\,A \subset {}^{\varepsilon}\mathrm{Min}\,A.$$

Let $y \in {}^{\varepsilon}\mathrm{PrMin}\,A$ and posit to the contrary that $y \notin {}^{\varepsilon}\mathrm{Min}\,A$ so there is $a \in A$ such that $a < y - \varepsilon$. Thus $y - \varepsilon \in a + Z_+\setminus\{0\} \subset a + \mathring{K} \subset a + K\setminus\{0\}$ which implies that $y \notin {}^{\varepsilon}\mathrm{Min}(A|K)$. Further, to prove ${}^{\varepsilon}\mathrm{Min}\,A \subset {}^{\varepsilon}\mathrm{WMin}\,A$ let $x \in {}^{\varepsilon}\mathrm{Min}\,A$ and $Q = \{0\} \cup \mathring{Z}_+$. Suppose that $x \in {}^{\varepsilon}\mathrm{WMin}\,A$ and thus there is $a \in A$ such that $x - \varepsilon - a \in Q\setminus\{0\} = \mathring{Z}_+$. In this case $x - \varepsilon \in a + \mathring{Z}_+ \subset a + Z_+\setminus\{0\}$ which say that $x \notin {}^{\varepsilon}\mathrm{Min}\,A$. This contradiction proves the inclusion.

The following propositions are immediate.

Proposition 1.4. *Let $A, B \subset Z$ such that $B \supset A$. Then*

i) $({}^{\varepsilon}\mathrm{IMin}\,B) \cap A \subseteq {}^{\varepsilon}\mathrm{IMin}\,A$
ii) $({}^{\varepsilon}\mathrm{Min}\,B) \cap A \subset {}^{\varepsilon}\mathrm{Min}\,A$
iii) $({}^{\varepsilon}\mathrm{PrMin}\,B) \cap A \subset {}^{\varepsilon}\mathrm{PrMin}\,A$
iv) $({}^{\varepsilon}\mathrm{WMin}\,B) \cap A \subset {}^{\varepsilon}\mathrm{WMin}\,A$

Proposition 1.5. *Suppose that there exists a convex pointed cone K containing Z_+. Then*

i) ${}^{\varepsilon}\mathrm{IMin}(A \mid K) \supseteq {}^{\varepsilon}\mathrm{IMin}(A \mid Z_+)$
ii) ${}^{\varepsilon}\mathrm{PrMin}(A \mid K) \subseteq {}^{\varepsilon}\mathrm{PrMin}(A \mid Z_+)$
iii) ${}^{\varepsilon}\mathrm{Min}(A \mid K) \subseteq {}^{\varepsilon}\mathrm{Min}(A \mid Z_+)$
iv) ${}^{\varepsilon}\mathrm{WMin}(A \mid K) \subseteq {}^{\varepsilon}\mathrm{WMin}(A \mid Z_+)$

2. In this paragraph we present an existence result for ε minimum points (proposition 2.1) and some scalarization properties which are the extensions of the similar results from [2].

Definition 2.1. *Let $A \subset Z$, $A \neq \emptyset$, $\varepsilon \in Z_+$, $x \in A$. The set $A \cap (x - Z_+ + \varepsilon) = A_x^{\varepsilon}$ is called ε-section of A at x.*

Proposition 2.1. *Let $A \subset Z, A \neq \emptyset, \varepsilon \in Z_+$. The set ${}^{\varepsilon}\mathrm{Min}\,A$ is nonempty if and only if A has a nonempty ε-section with ε-minimum points set being nonempty. In fact we have that $\bigcup_{x \in A} {}^{\varepsilon}\mathrm{Min}\,A_x^{\varepsilon} = {}^{\varepsilon}\mathrm{Min}\,A$.*

Proof. First we prove that for every $x \in A$ we have ${}^{\varepsilon}\mathrm{Min}\,A_x^{\varepsilon} \subset {}^{\varepsilon}\mathrm{Min}\,A$. Suppose to the contrary that there is $y \in {}^{\varepsilon}\mathrm{Min}\,A_x^{\varepsilon} \setminus {}^{\varepsilon}\mathrm{Min}\,A$ so there is $a_1 \in A$ such that $y - \varepsilon - a_1 \in Z_+\setminus\{0\}$. Thus $a_1 \in y - \varepsilon - Z_+$, $y \in A_x^{\varepsilon}$ which implies $a_1 \in x - Z_+ \subset x - Z_+ + \varepsilon$ and so $a_1 \in A_x^{\varepsilon}$.

Since $y \in {}^{\varepsilon}\mathrm{Min}\,A_x^{\varepsilon}$ then $y - \varepsilon - a \notin Z_+\setminus\{0\}$. This contradiction shows that

$$\bigcup_{x \in A} {}^{\varepsilon}\mathrm{Min}\,A_x^{\varepsilon} \subseteq {}^{\varepsilon}\mathrm{Min}\,A$$

Now let $x \in {}^{\varepsilon}\mathrm{Min}\,A$ and we want to prove that $x \in {}^{\varepsilon}\mathrm{Min}\,A_x^{\varepsilon}$. Since $A_x^{\varepsilon} \subset A$ then there is no $a \in A_x^{\varepsilon}$ with $x - \varepsilon > a$ so $x \in {}^{\varepsilon}\mathrm{Inf}\,A_x^{\varepsilon} \cap A_x^{\varepsilon} = {}^{\varepsilon}\mathrm{Min}\,A_x^{\varepsilon}$. This shows that ${}^{\varepsilon}\mathrm{Min}\,A \subseteq \bigcup_{x \in A} {}^{\varepsilon}\mathrm{Min}\,A_x^{\varepsilon}$ and finally ${}^{\varepsilon}\mathrm{Min}\,A = \bigcup_{x \in A} \mathrm{Min}\,A_x^{\varepsilon}$ as claimed.

Definition 2.2. *A set $A \subset Z, A \neq \emptyset$ is Z_+-convex if for all $n \in N^*\setminus\{1\}$ and for all $a_1, a_2, \ldots, a_n \in A; \lambda_1, \lambda_2, \ldots, \lambda_n \geq 0$ with $\lambda_1 + \lambda_2 + \ldots + \lambda_n = 1$ there is $a \in A$ such that $\lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_n a_n - a \in Z_+$.*

It is obvious that a set A is Z_+ convex if and only if $A + Z_+$ is convex.

Proposition 2.2. *If A is a Z_+ convex set then ${}^{\varepsilon}\mathrm{Min}\,A = ({}^{\varepsilon}\mathrm{Min}(conv\ A)) \cap A$.*

Proof. Let $b \in {}^{\varepsilon}\mathrm{Min}\,A$ and posit to the contrary that $b \notin {}^{\varepsilon}\mathrm{Min}(conv\ A)$, i.e. there is $x \in conv\ A$ with $x < b - \varepsilon$, so there is $a_1, \ldots, a_n \in A$, $\lambda_1 \ldots \lambda_n \in R_+$ $\sum_{i=1}^{n} \lambda_i = 1$ such that $x = \sum_{i=1}^{n} \lambda_i a_i$ and $\sum_{i=1}^{n} \lambda_i a_i < b - \varepsilon$. Since A is a Z_+ convex set, there is $a \in A$ such that

$$a \leq \sum_{i=1}^{n} \lambda_i a_i < b - \varepsilon$$

so $a < b - \varepsilon$ which is contradicting the fact that $b \in {}^{\varepsilon}\mathrm{Min}\,A$. Then we have ${}^{\varepsilon}\mathrm{Min}\,A \subseteq {}^{\varepsilon}\mathrm{Min}(conv\ A)$. Now let $b \in ({}^{\varepsilon}\mathrm{Min}\,conv\ A) \cap A$, then there is no $x \in conv\ A$ such that $x < b - \varepsilon$, so there is no $x \in A$ with $x < b - \varepsilon$. This implies that $b \in {}^{\varepsilon}\mathrm{Min}\,A$ and thus $({}^{\varepsilon}\mathrm{Min}\,conv\ A) \cap A \subseteq {}^{\varepsilon}\mathrm{Min}\,A$. Finally we get the conclusion by the proposition.

Definition 2.3. *Let (Y, τ) be a locally convex space and $y_0 \in Y, \varepsilon \in Z_+$. We say that $y_0 \in Y$ is a locally ε minimal point for a multifunction $f : Y \to Z$ if $y_0 \in D(f)$ and there is U_{y_0} a neighbourhood for y_0, $U_{y_0} \subset D(f)$ such that $f(y_0) \cap {}^{\varepsilon}\mathrm{Min}\,f(U_{y_0}) \neq \emptyset$. (Here $D(f) = \{y \in Y \mid f(y) \neq \emptyset\}$.)*

The set of locally ε minimal points is denoted by ${}^{\varepsilon}\mathrm{Min\,loc}\,f$.

This concept extends to multifunctions the notion of local ε minimal point for functions. On the other hand, we remark that for $\varepsilon = 0$ this notions have been given in [2].

Remark. Let $\varphi : Y \to R$ be a function and we consider the multifunction

$$\bar{\varphi} : Y \to \bar{R}, \bar{\varphi}(y) = \{t \in (-\infty, +\infty], \varphi(y) \le t\}$$

Of course $D(\bar{\varphi}) = Y$ and let $\varepsilon > 0$. We can show that $y_0 \in Y$ *is a locally ε minimal point for φ if and only if y_0 is a locally ε minimal point for $\bar{\varphi}$.*

For that, $y_0 \in {}^{\varepsilon}\mathrm{Min\ loc}\ \varphi$ if and only if there is U_{y_0} such that $\varphi(y_0) \cap {}^{\varepsilon}\mathrm{Min}\,\bar{\varphi}(U_{y_0}) \neq \emptyset$. This means that $y_0 \in {}^{\varepsilon}\mathrm{Minloc}\,\varphi$ if and only if there is $b \in \bar{\varphi}(y_0)$ such that there is no $z \in \bar{\varphi}(U_{y_0})$ with $z < b - \varepsilon$.

Finally we get $y_0 \in {}^{\varepsilon}\mathrm{Minloc}\,\bar{\varphi}$ if and only if there is $b \in (-\infty, +\infty]$ with $\varphi(y_0) \le b$ such that for all $y \in U_{y_0}$, and all $z \in (-\infty, +\infty]$ with $\varphi(y) \le z$ we have $z \ge b - \varepsilon$ and thus the statement is proved.

The following definitions was given in [2].

Definition 2.4. *Let A be a nonempty set of a locally convex space Z. We say that A is strictly Z_+-convex if for all $n \in N, n \ge 2, a_1, a_2, ..., a_n \in A$, $\lambda_1, ..., \lambda_n \ge 0$ with $\sum_{i=1}^{n} \lambda_i = 1$ there is $a \in A$ such that*

$$x^*(a) > \lambda_1 x^*(a_1) + ... + \lambda_n x^*(a_n), \ \forall x^* \in Z_+ \backslash \{0\}$$

(Here Z_+^0 means the polar cone for Z_+, i.e.

$$Z_+^0 = \{x^* \in Z^* \mid x^*(z) \le 0\ \forall z \in Z_+\})$$

Proposition 2.3. *If (Z, τ) is a locally convex space ordered by a convex, closed cone Z_+ which has a nonempty interior and $f : Y \to \bar{Z}$ is a multifunction such that $f(y) \neq \emptyset, \{+\infty\}$ for all $y \in Y$ and $f(y)$ is a Z_+-convex set then*

i) ${}^{\varepsilon}\mathrm{Min}\, f(y) \subset \bigcup_{x^* \in Z_+^0 \backslash \{0\}} \{u \in f(y), x^*(u) \ge x^*(t) + x^*(\varepsilon),\ \forall t \in f(y)\}$

ii) *If $f(y)$ is a strictly Z_+-convex set then the converse inclusion is valid that is*

$${}^{\varepsilon}\mathrm{Min}\, f(y) = \bigcup_{x^* \in Z_+^0 \backslash \{0\}} \{u \in f(y), x^*(u) \ge x^*(t) + x^*(\varepsilon),\ \forall t \in f(y)\}$$

Proof. i) If ${}^{\varepsilon}\mathrm{Min}\, f(y) = \emptyset$ then it is trivial that

$${}^{\varepsilon}\mathrm{Min}\, f(y) \subseteq \bigcup_{x^* \in Z_+^0 \backslash \{0\}} \{u \in f(y), x^*(u) \ge x^*(t) + x^*(\varepsilon),\ \forall t \in f(y)\}$$

Now we are supposing that ${}^{\varepsilon}\mathrm{Min}\, f(y) \neq \emptyset$ and let $z_1 \in {}^{\varepsilon}\mathrm{Min}\, f(y)$. We can see that

$$f(y) + \mathrm{int}\ Z_+ - z_1 + \varepsilon \subset \mathrm{int}(f(y) + Z_+ - z_1 + \varepsilon)$$

(where int A denote the interior of a subset A of Z).

Since $z_1 \in {}^{\varepsilon}\mathrm{Min}\, f(y)$ we get

$$\mathrm{int}(f(y) + Z_+ - z_1 + \varepsilon) \cap (-Z_+) = \emptyset$$

So by theorem 1.14 given in [3] it follows that there exists a continuous and linear functional x^* on $Z, x^* \neq 0$ with the property

$$x^*(t + z - z_1 + \varepsilon) \le x^*(-z_2),\ \forall t \in f(y), z_1, z_2 \in Z_+$$

So, $x^*(z_1) \ge x^*(t) + x^*(\varepsilon),\ \forall t \in f(y), z_1 \in Z_+$. On the other hand if t is fix and $z_2 = 0$, we have for all $\lambda > 0, x^*(t + \lambda z - z_1 + \varepsilon) \le 0$ so $x^*(z) \le 0,\ \forall z \in Z_+$ which means that $x^* \in Z_+^0$ and i) is proved

ii) Let $f(y)$ be strictly Z_+ convex and $z_1 \in f(y)$, $x^* \in Z_+^0$ with $x^*(z_1) \ge x^*(t) + x^*(\varepsilon),\ \forall t \in f(y)$. We suppose that $z_1 \notin {}^{\varepsilon}\mathrm{Min}\, f(y)$ so there is $z_2 \in f(y)$ with $z_1 - \varepsilon > t_2$ which implies that

$$x^*(z_1) \le x^*(z_2) + x^*(\varepsilon).$$

Thus we obtain $x^*(z_1) = x^*(z_2) + x^*(\varepsilon)$.

For $\lambda = 0$ we get

$$x^*(z_3) + x^*(\varepsilon) > x^*(z_1)$$

which is contradicting the hypotesis, so

$$\bigcap_{x^* \in Z_+^0 \backslash \{0\}} \{u \in f(y), x^*(u) \ge x^*(t) + x^*(\varepsilon),\ \forall u \in f(y)\} \subseteq {}^{\varepsilon}\mathrm{Min}\, f(y)$$

and using i) we obtain ii).

In fact we have

Proposition 2.3.' *Let (X, τ) be a locally convex space ordered by a convex, closed cone Z_+ which has a nonempty interior and $A \subset Z$, a Z_+-convex set. Then*

i) ${}^{\varepsilon}\mathrm{Min}\, A \subseteq \bigcup_{x^* \in Z_+^0 \setminus \{0\}} \{a \in A, x^*(a) \geq x^*(t) + x^*(\varepsilon), \forall t \in A\}$

ii) *If A is a strictly Z_+ convex set then we have*

$${}^{\varepsilon}\mathrm{Min}\, A = \bigcup_{x^* \in Z_+^0 \setminus \{0\}} \{a \in A, x^*(a) \geq x^*(t) + x^*(\varepsilon), \forall t \in A\}$$

3. In the following, for $A \subset \overline{Z}$ and $\delta \in Z$ we shall say that A has the property

(P^{δ}) if ${}^{\delta}\mathrm{Inf}_1\, A \neq \emptyset, \{-\infty\}$ and $A \subset {}^{\delta}\, \mathrm{Inf}_1\, A + Z_+$

(P'^{δ}) if ${}^{\delta}\mathrm{Sup}_1\, A \neq \emptyset, \{+\infty\}$ and $A \subset {}^{\delta}\, \mathrm{Sup}_1\, A - Z_+$

(P_1^{δ}) if ${}^{\delta}\mathrm{Inf}_1\, A \neq \emptyset$, and $\mathrm{Inf}\, A \subseteq ({}^{\delta}\mathrm{Inf}_1\, A - Z_+) \cup \{-\infty\}$

$(P_1'^{\delta})$ if ${}^{\delta}\mathrm{Sup}_1\, A \neq \emptyset$, and $\mathrm{Sup}\, A \subseteq ({}^{\delta}\mathrm{Sup}_1\, A + Z_+) \cup \{+\infty\}$

When $\delta = 0$ we denote simply P, P', P_1, P_1'. In the end of this paragraph we shall present some sufficient conditions for a set A to have these properties.

We begin with some auxiliary properties.

Proposition 3.1. *Let $A \subset \overline{Z}$ be a nonempty set such that $A \neq \{-\infty\}$ and $\varepsilon, \delta \in Z$.*

We have: i) ${}^{\varepsilon}\mathrm{Sup}\, {}^{\delta}\mathrm{Sup}_1\, A \supset {}^{\varepsilon}\mathrm{Sup}_1\, A$

ii) *If A has no property (P'^{δ}) then* ${}^{\varepsilon}\mathrm{Sup}_1\, {}^{\delta}\mathrm{Sup}_1\, A = {}^{\varepsilon}\mathrm{Sup}_1\, A$.

Proof. i) If ${}^{\varepsilon}\mathrm{Sup}_1\, A = \emptyset$ or $\{+\infty\}$ the inclusion is evident. Now suppose ${}^{\varepsilon}\mathrm{Sup}_1\, A \neq \emptyset_1\{+\infty\}$ and let $\alpha \in {}^{\varepsilon}\mathrm{Sup}_1\, A$. Posit to the contrary that $\alpha \notin {}^{\varepsilon}\mathrm{Sup}\, {}^{\varepsilon}\mathrm{Sup}_1\, A$ so there is $\beta \in {}^{\delta}\mathrm{Sup}_1\, A$ such that $\beta > \alpha + \varepsilon$. Since $\beta \in {}^{\delta}\mathrm{Sup}_1\, A$ will exist $a \in A$ with $a > \alpha + \varepsilon$ which is in contradiction with $\alpha \in {}^{\varepsilon}\mathrm{Sup}_1\, A$.

ii) We have from i) that ${}^{\varepsilon}\mathrm{Sup}_1\, A \subset {}^{\varepsilon}\mathrm{Sup}\, {}^{\delta}\mathrm{Sup}_1\, A$. For $\alpha \in {}^{\varepsilon}\mathrm{Sup}_1\, A$ let $\alpha' < \alpha$ and thus there is $a \in A$, $a > \alpha'$. Since A has property (P'^{δ}) there is $\beta \in {}^{\delta}\mathrm{Sup}_1\, A$, $a < \beta$ which implies that $\alpha' < \beta$ so ${}^{\varepsilon}\mathrm{Sup}_1\, A \subseteq {}^{\varepsilon}\mathrm{Sup}_1\, {}^{\delta}\mathrm{Sup}_1\, A$. Now if $\alpha \in {}^{\varepsilon}\mathrm{Sup}_1\, {}^{\delta}\mathrm{Sup}_1\, A$, suppose that $\alpha \notin {}^{\varepsilon}\mathrm{Sup}\, A$. In this case, there is $a \in A$, $a > \alpha + \varepsilon$. Since A has property (P'^{δ}) there is $\beta \in {}^{\delta}\mathrm{Sup}_1\, A$, $\beta > a$ so $\beta > \alpha + \varepsilon$, contradicting $\alpha \in {}^{\varepsilon}\mathrm{Sup}_1\, {}^{\delta}\mathrm{Sup}_1\, A$. Let $\alpha' < \alpha$, then there is $\beta \in {}^{\delta}\mathrm{Sup}_1\, A$, $\beta > \alpha$, and so there is $a \in A$ with $a > \alpha'$. This shows that ${}^{\varepsilon}\mathrm{Sup}_1\, {}^{\delta}\, \mathrm{Sup}_1\, A \subseteq {}^{\varepsilon}\mathrm{Sup}_1\, A$ and finally we get the equality.

Corolary 3.1. *Let $A \subset \overline{Z}, A \neq \emptyset, \{-\infty\}$. We have*

i) $\mathrm{Sup}_1\, A \subset \mathrm{Sup}\, \mathrm{Sup}_1\, A$

ii) *If A has property (P') then* $\mathrm{Sup}_1\, \mathrm{Sup}_1\, A = \mathrm{Sup}_1\, A$.

In the same way we have.

Proposition 3.1.' *Let A be a nonempty set, $A \subseteq \overline{Z}, A \neq \{\infty\}$, $\varepsilon, \delta \in Z$. We have*

i) ${}^{\varepsilon}\mathrm{Inf}\, {}^{\delta}\mathrm{Inf}_1\, A \supset {}^{\varepsilon}\mathrm{Inf}_1\, A$

ii) *If A has property (P^{δ}) then* ${}^{\varepsilon}\mathrm{Inf}_1\, {}^{\delta}\mathrm{Inf}_1\, A = {}^{\varepsilon}\mathrm{Inf}_1\, A$

Corolar 3.1.' *Let $A \subset \overline{Z}, A \neq \emptyset, \{-\infty\}$. We have*

i) $\mathrm{Inf}_1\, A \subseteq \mathrm{Inf}\, \mathrm{Inf}_1\, A$

ii) *If A has property (P') then* $\mathrm{Inf}_1\, \mathrm{Inf}_1\, A = \mathrm{Inf}_1\, A$.

Proposition 3.2. *Let $A \subset \overline{Z}, A \neq \emptyset, \{\infty\}$ and $\mathrm{Inf}\, A \neq \{-\infty\}$. We have*

i) $\mathrm{Sup}_1\, \mathrm{Inf}\, A \subset \mathrm{Inf}\, A$

ii) *If A has property (P_1) then* $\mathrm{Sup}_1\, \mathrm{Inf}\, A = \mathrm{Inf}_1\, A$

Proof. First we observ the $\mathrm{Sup}_1\, \mathrm{Inf}\, A \neq \{+\infty\}$ because for $a \in A$ there is no $\alpha \in \mathrm{Inf}\, A$ with $\alpha > a$.

The conclusion is trivial if $\mathrm{Sup}_1\, \mathrm{Inf}\, A = \emptyset$. Now suppose $\mathrm{Sup}_1\, \mathrm{Inf}\, A \neq \emptyset$ and let $\alpha \in \mathrm{Sup}_1\, \mathrm{Inf}\, A$. If $\alpha \notin \mathrm{Inf}\, A$ then there is $a \in A, a < \alpha$. Since $\alpha \in \mathrm{Sup}_1\, \mathrm{Inf}\, A$ there exists $\beta \in \mathrm{Inf}\, A$ with $\beta > a$ contradicting the fact that $\beta \in \mathrm{Inf}\, A$.

ii) Now let A has the property (P_1).

Using i) we get $\mathrm{Sup}_1\, \mathrm{Inf}\, A \subseteq (\mathrm{Inf}_1\, A - Z_+) \cup \{-\infty\}$.

But $-\infty \notin \mathrm{Sup}_1\, \mathrm{Inf}\, A$ because $\mathrm{Inf}\, A \neq \{-\infty\}$ then for all $\alpha \in \mathrm{Sup}_1\, \mathrm{Inf}\, A$ there is $\alpha' \in \mathrm{Inf}_1\, A \subseteq \mathrm{Inf}\, A$ such that $\alpha \leq \alpha'$ so $\alpha = \alpha'$ and $\mathrm{Sup}_1\, \mathrm{Inf}\, A \subseteq \mathrm{Inf}_1\, A$.

If $\mathrm{Inf}_1\, A = \emptyset$, the converse inclusion does hold trivial, so suppose $\mathrm{Inf}_1\, A \neq \emptyset$. Let $\alpha \in \mathrm{Inf}_1\, A$ and suppose $\alpha \notin \mathrm{Sup}\, \mathrm{Inf}\, A$. In this case there is $\beta \in \mathrm{Inf}\, A$ with $\beta > \alpha$ and $a \in A, a < \beta$, false.

Now if $\alpha' < \alpha$ there is $\beta = \alpha \in \mathrm{Inf}\, A, \beta > \alpha'$. Thus we have $\mathrm{Inf}_1\, A \subseteq \mathrm{Sup}_1\, \mathrm{Inf}\, A$ and finaly we get the equality.

Similar, we get

Proposition 3.2' *Let $A \subset \overline{Z}, A \neq \emptyset, \{-\infty\}$ and $\mathrm{Sup}\, A \neq \{+\infty\}$. We have:*

i) $\mathrm{Inf}_1\, \mathrm{Sup}\, A \subset \mathrm{Sup}\, A$.

ii) *If A has property (P_1') then* $\mathrm{Inf}_1\, \mathrm{Sup}\, A = \mathrm{Sup}_1\, A$.

Proposition 3.3 i) *Let $A \subset \overline{Z}, A \neq \emptyset, \{+\infty\}$. We have* ${}^{\varepsilon}\mathrm{Inf}_1\, A \subset {}^{\varepsilon}\mathrm{Sup}_1\, {}^{\varepsilon}\mathrm{Inf}_1\, A \subset {}^{\varepsilon}\mathrm{Inf}\, A$

ii) *Let $A \subset Z, A \neq \emptyset, \{-\infty\}$.*

We have ${}^{\varepsilon}\mathrm{Sup}_1\, A \subset {}^{\varepsilon}\mathrm{Inf}_1\, {}^{\varepsilon}\mathrm{Sup}_1\, A \subset {}^{\varepsilon}\mathrm{Sup}\, A$.

Proposition 3.4. i) *Let* $A \subset \overline{Z}$, $A \neq \emptyset, \{+\infty\}$ *and* $\mathrm{Inf}\, A \neq \{-\infty\}$. *If* A *has property* (P_1) *then*

$$\mathrm{Sup}\,\mathrm{Inf}\, A = C(\mathrm{Inf}\, A) \cup \mathrm{Inf}_1\, A$$

ii) *Let* $A \subset \overline{Z}, A \neq \emptyset, \{-\infty\}$ *and* $\mathrm{Sup}\, A \neq \{+\infty\}$. *If* A *has property* (P_1') *then*

$$\mathrm{Inf}\,\mathrm{Sup}\, A = C(\mathrm{Sup}\, A) \cup \mathrm{Sup}_1\, A.$$

Proof. From proposition 3.2 we have $\mathrm{Inf}_1\, A = \mathrm{Sup}_1\,\mathrm{Inf}\, A$, so that $\mathrm{Inf}_1\, A \subset \mathrm{Sup}\,\mathrm{Inf}\, A$.

If $\alpha \in C(\mathrm{Inf}\, A)$ there is no $\beta \in \mathrm{Inf}\, A$ with $\beta > \alpha$ so $\alpha \in \mathrm{Sup}\,\mathrm{Inf}\, A$ and $C(\mathrm{Inf}\, A) \cup \mathrm{Inf}_1\, A \subset \mathrm{Sup}\,\mathrm{Inf}\, A$. Now let $\alpha \in \mathrm{Sup}\,\mathrm{Inf}\, A$. If $\alpha \in \mathrm{Inf}\, A$ then $\alpha \in \mathrm{Max}\,\mathrm{Inf}\, A \subseteq \mathrm{Sup}_1\,\mathrm{Inf}\, A = \mathrm{Inf}_1\, A$. So $\alpha \in \mathrm{Inf}_1\, A \cup C(\mathrm{Inf}\, A)$ and i) is proved. For statement ii) the proof is similar.

Proposition 3.5. *Let* $A, B \subset \overline{Z}$, $A, B \neq \emptyset, \{-\infty\}$ *with* $\dot{A} \subset B - Z_+$ *and* B *has property* (P'^{δ}).

In these hypothesis:

$${}^{\delta}\mathrm{Sup}_1\, A \subset \bigcap_{\varepsilon \in Z_+ \setminus \{0\}} ({}^{\delta}\mathrm{Sup}_1\, B - Z_+ + \varepsilon), \forall \delta \in Z$$

and ${}^{\delta}\mathrm{Sup}\, B \subset {}^{\delta}\mathrm{Sup}\, A$.

Proof. It is evident that if $A \subset B \dot{-} Z_+$ then

$${}^{\delta}\mathrm{Sup}\, B \subset {}^{\delta}\mathrm{Sup}\, A, \quad \forall \delta \in Z.$$

Now if ${}^{\delta}\mathrm{Sup}_1\, A = \emptyset$ the conclusion is trivial. Let ${}^{\delta}\mathrm{Sup}_1\, A \neq \emptyset$ and $\alpha \in {}^{\delta}\mathrm{Sup}_1\, A$.

If $\alpha = +\infty$ then for all $z \in Z$ there is $a \in B$ and $b_a \in B$ such that $z - \delta < a < b_a$ and thus ${}^{\delta}\mathrm{Sup}_1\, B = \{+\infty\}$. But B has propety (P'^{δ}) so ${}^{\delta}\mathrm{Sup}_1\, A \neq \{+\infty\}$.

Let $\alpha \in {}^{\delta}\mathrm{Sup}_1\, A$ and $\varepsilon \in Z_+$. Thus there is $a \in A$ and $b_a \in B$ such that $b_a > a > \alpha - \varepsilon$. Since B has property (P'^{δ}) there is $\beta \in {}^{\delta}\mathrm{Sup}_1\, B$ such that $\beta > b_a > a > \alpha - \varepsilon$ and thus $\alpha \in {}^{\delta}\mathrm{Sup}_1\, B - Z_+ + \varepsilon$ which implies the conclusion.

In the same way we get.

Proposition 3.5.' *Let* $A, B \subset \overline{Z}$, $A, B \neq \emptyset, \{-\infty\}$ *with* $A \subset B + Z_+$ *and* B *has property* (P^{δ}).

In these hypothesis:

$${}^{\delta}\mathrm{Inf}_1\, A \subset \bigcap_{\varepsilon \in Z_+ \setminus \{0\}} ({}^{\delta}\mathrm{Inf}_1\, B + Z_+ - \varepsilon), \quad \forall \delta \in Z$$

and ${}^{\delta}\mathrm{Inf}\, B \subset {}^{\delta}\mathrm{Inf}\, A$.

Now we are interesed to get some topological proprieties for the efficient points set. Throughout the following we'll consider (Z, τ) a separated, locally convex space, ordered by $Z_+ = \{0\} \cup \overset{\circ}{Z}_+$ which is normal.

(We denote by $\bar{A}, \overset{\circ}{A}$ the closure and the interior of A, respectively. A space (Z, τ) is called normal if for all $V \in V_\tau(0), V = (V - Z_+) \cap (V + Z_+)$ see [4].)

We observe that the assumption about the space and cone implies that $U(a) = \{(a - \varepsilon, a + \varepsilon), \varepsilon \in Z_+ \setminus \{0\}\}$ forms a foundamental system of neighbourhoods for $a \in Z$. (Here $(a, b) = (a + Z_+ \setminus \{0\}) \cap (b - Z_+ \setminus \{0\}) \subset V$, and $V \subset \overline{Z}$ is a neighbourhood for $\pm\infty$ if there is $a \in Z$ such that $\{\pm\infty\} \cup (a \pm Z_+) \subset V$.)

Proposition 3.6. i) *Let* $A \subset \overline{Z}$, $A \neq \emptyset, \{+\infty\}, \varepsilon \in Z$. *Then*

${}^{\varepsilon}\mathrm{Inf}\, A = {}^{\varepsilon}\mathrm{Inf}\, \bar{A} = \overline{{}^{\varepsilon}\mathrm{Inf}\, A}$

${}^{\varepsilon}\mathrm{Inf}_1\, A = {}^{\varepsilon}\mathrm{Inf}_1\, \bar{A} = \overline{{}^{\varepsilon}\mathrm{Inf}_1\, A}$

ii) *Let* $A \subset \overline{Z}$, $A \neq \emptyset, \{-\infty\}, \varepsilon \in z$. *Then*

${}^{\varepsilon}\mathrm{Sup}\, A = {}^{\varepsilon}\mathrm{Sup}\, \bar{A} = \overline{{}^{\varepsilon}\mathrm{Sup}\, A}$

${}^{\varepsilon}\mathrm{Sup}_1\, A = {}^{\varepsilon}\mathrm{Sup}_1\, \bar{A} = \overline{{}^{\varepsilon}\mathrm{Sup}_1\, A}$

Proof. i) Since $A \subset \bar{A}$ we get from proposition 3.5′, ${}^{\varepsilon}\mathrm{Inf}\, \bar{A} \subset {}^{\varepsilon}\mathrm{Inf}\, A$. If ${}^{\varepsilon}\mathrm{Inf}\, A = \{-\infty\}$ we have ${}^{\varepsilon}\mathrm{Inf}\, A \subset {}^{\varepsilon}\mathrm{Inf}\, \bar{A}$ and thus the equality does holds.

Now suppose ${}^{\varepsilon}\mathrm{Inf}\, A \neq \{-\infty\}$.

Let $\alpha \in {}^{\varepsilon}\mathrm{Inf}\, A$ and posit to the contrary that $\alpha \notin {}^{\varepsilon}\mathrm{Inf}\, \bar{A}$. In this case there is $b \in \bar{A}$ with $b < \alpha - \varepsilon$. Let $V = (2b - \alpha + \varepsilon, \alpha - \varepsilon) \in U(b)$ and since $b \in \bar{A}$, we get $V \cap A \neq \emptyset$. We found $a \in A, a < \alpha - \varepsilon$, false. So we have ${}^{\varepsilon}\mathrm{Inf}\, A = {}^{\varepsilon}\mathrm{Inf}\, \bar{A}$.

Now if ${}^{\varepsilon}\mathrm{Inf}_1\, A = \emptyset$ we have ${}^{\varepsilon}\mathrm{Inf}_1\, \bar{A} \subseteq {}^{\varepsilon}\mathrm{Inf}_1\, A$. Thus suppose ${}^{\varepsilon}\mathrm{Inf}_1\, \bar{A} \neq \emptyset$ and let $\alpha \in {}^{\varepsilon}\mathrm{Inf}_1 \bar{A}$ and $p' > \alpha$. (Of course $\alpha \neq +\infty$ because $A \neq \{+\infty\}$).

This implies that there is $b \in \bar{A}$, $b < p'$. If $b = -\infty$ then ${}^{\varepsilon}\mathrm{Inf}_1\, \bar{A} = \{-\infty\}$ and ${}^{\varepsilon}\mathrm{Inf}\, A = \{-\infty\}$ so ${}^{\varepsilon}\mathrm{Inf}_1\, A = \{\infty\}$ and we have the equality. Now for $b \neq -\infty$ since $V = (2b - p', p')$ is a neighbourhood for b we get $V \cap A \neq \emptyset$, thus there is $a \in A$, $a < p'$ so $\alpha \in {}^{\varepsilon}\mathrm{Inf}_1\, A$ and ${}^{\varepsilon}\mathrm{Inf}_1\, \bar{A} \subseteq {}^{\varepsilon}\mathrm{Inf}_1\, A$. If ${}^{\varepsilon}\mathrm{Inf}_1\, A = \emptyset$ the equality does hold. Now suppose ${}^{\varepsilon}\mathrm{Inf}_1\, A \neq \emptyset$ and let $\alpha \in {}^{\varepsilon}\mathrm{Inf}_1\, A$; we already know that $\alpha \in {}^{\varepsilon}\mathrm{Inf}\, \bar{A}$ and let $\alpha' > \alpha$. In this case we get $a \in A \subset \bar{A}, a < \alpha'$ so $\alpha \in {}^{\varepsilon}\mathrm{Inf}_1\, \bar{A}$ and ${}^{\varepsilon}\mathrm{Inf}_1\, A = {}^{\varepsilon}\mathrm{Inf}_1\, \bar{A}$.

Always we have ${}^{\varepsilon}\mathrm{Inf}\, A \subseteq \overline{{}^{\varepsilon}\mathrm{Inf}\, A}$ and ${}^{\varepsilon}\mathrm{Inf}_1\, A \subseteq \overline{{}^{\varepsilon}\mathrm{Inf}_1\, A}$. If $\overline{{}^{\varepsilon}\mathrm{Inf}\, A} = \{-\infty\}$ then ${}^{\varepsilon}\mathrm{Inf}\, A = \{-\infty\} = \overline{{}^{\varepsilon}\mathrm{Inf}\, A}$.

Suppose $\overline{{}^{\varepsilon}\mathrm{Inf}\, A} \neq \{-\infty\}$ and let $\alpha \in \overline{{}^{\varepsilon}\mathrm{Inf}\, A}, \alpha \neq -\infty$. Posit to the contrary that $\alpha \notin {}^{\varepsilon}\mathrm{Inf}\, A$. This implies that there exists $a \in A$, $a < \alpha - \varepsilon$ and thus $(a + \varepsilon, \alpha + \varepsilon) \in V(\alpha)$. We shall get $\beta \in {}^{\varepsilon}\mathrm{Inf}\, A$, $\beta > a + \varepsilon$ and this contradiction prove that ${}^{\varepsilon}\mathrm{Inf}\, A = \overline{{}^{\varepsilon}\mathrm{Inf}\, A}$.

Now, if $\overline{{}^{\varepsilon}\mathrm{Inf}_1\, A} \neq \emptyset$ or $\{-\infty\}$, equality does hold, so let $\overline{{}^{\varepsilon}\mathrm{Inf}_1\, A} \neq \emptyset, \{-\infty\}$ and $\alpha \in \overline{{}^{\varepsilon}\mathrm{Inf}_1\, A}$.

Of course $\alpha \neq -\infty$ because $\overline{{}^{\varepsilon}\mathrm{Inf}_1\, A} = \{-\infty\}$. We get already $\alpha \in {}^{\varepsilon}\mathrm{Inf}\, A$ and let $\alpha' > \alpha$. Thus $(\alpha' - 2\alpha, \alpha') \in U(\alpha)$ and there is $\beta \in {}^{\varepsilon}\mathrm{Inf}_1\, A$, $\beta < \alpha'$. Now we get $a \in A$, $a < \alpha'$ which implies that $\overline{{}^{\varepsilon}\mathrm{Inf}_1\, A} \subseteq {}^{\varepsilon}\mathrm{Inf}_1\, A$ and finaly ${}^{\varepsilon}\mathrm{Inf}_1\, A = \overline{{}^{\varepsilon}\mathrm{Inf}_1\, A}$

ii) The proof is similar.

Proposition 3.7. *Let $A \subset \overline{Z}$; $A \neq \emptyset, \{-\infty\}, \{+\infty\}$. A has property (P') if and only if for all $a \in A$ there is $d > a$ such that $\mathrm{Min}\, B_d \neq \emptyset$ where $B_d = [a, d] \cap \mathrm{Sup}\, A$.*

Proof. " $\Rightarrow$ " If $A \neq \emptyset, \{-\infty\}, \{+\infty\}$ and A has property (P') then $\mathrm{Sup}_1\, A \neq \emptyset, \{+\infty\}$ and $A \subset \mathrm{Sup}_1\, A - Z_+$ so for all $a \in A$ there is $d' \in \mathrm{Sup}_1\, A$ such that $a \leq d' < d' + \varepsilon, \forall \varepsilon \in Z_+\backslash\{0\}$. For $d = d' + \varepsilon$ the set $B_d = [a, d] \cap \mathrm{Sup}\, A$ contains d'. We posit to the contrary that there is $\alpha \in \mathrm{Sup}\, A, \alpha < d'$. In this case there is $b \in A, b > \alpha$ which is contradicting that $\alpha \in \mathrm{Sup}\, A$. So $d \in \mathrm{Inf}\, B_d \cap B_d = \mathrm{Min}\, B_d$.

" $\Leftarrow$ " Let $a \in A$ and $d > a$ such that $B_d = \{\alpha \in [a, d], \alpha \in \mathrm{Sup}\, A\}$ has $\mathrm{Min}\, B_d \neq \emptyset$. In this case $B_d \neq \emptyset$. Let $\alpha \in \mathrm{Min}\, B_d$, so $\alpha \in \mathrm{Sup}\, A$ and let $\alpha' < \alpha$. If $a = \alpha$ then $\alpha \in \mathrm{Max}\, A \subset \mathrm{Sup}_1\, A$ and thus $a \in \mathrm{Sup}_1\, A - Z_+$. If $a \neq \alpha$, then $a < \alpha$ and let $V_1 = (a, 2\alpha - a)$ and $V_2 = (\alpha', 2\alpha - \alpha')$ be two neighbourhoods for α. We can get $\mu \in Z_+\backslash\{0\}$ such that $(\alpha - \mu, \alpha + \mu) \in V_1 \cap V_2$. In this conditions we have $\alpha > \alpha - \mu > a$ and $\alpha - \mu > \alpha'$ so $\alpha - \mu \in [a, \alpha]$ and $\alpha - \mu \notin \mathrm{Sup}\, A$ because $\alpha \in \mathrm{Min}\, B_d$ and $\alpha - \mu < \alpha$.

Then there is $b \in A$ such that $b > \alpha - \mu > \alpha$ so $\alpha \in \mathrm{Sup}_1\, A$ and $a < \alpha$, so $a \in \mathrm{Sup}_1\, A - Z_+$. In the same way we get:

Proposition 3.7.' *Let $A \subset \overline{Z}$, $A \neq \emptyset, \{+\infty\}, \{-\infty\}$. A has property (P) if and only if for all $a \in A$ there is $d < a$ such that $B_d = [d, a] \cap \mathrm{Inf}\, A$ has $\mathrm{Max}\, B_d \neq \emptyset$.*

Theorem 3.1. *Let (Z, τ) be a separated locally convex space ordered by a convex pointed, normal cone Z_+ such that $Z_+ = \{0\} \cup \overset{\circ}{Z}_+$. If one of the following conditions is satisfied:*

a) *$\overset{\circ}{Z}_+$ is locally compact;*

b) *$\overset{\circ}{Z}_+$ is supernormal and Z is quasicomplet;*

c) *Z_+ is Daniell;*

d) *Z_+ is well based by a complete set;*

then

i) *for all set $A \subset Z$, $A \neq \emptyset, \{-\infty\}, \{+\infty\}$ with $\mathrm{Inf}\, A \neq \{-\infty\}$, A has property (P) and*

$$\mathrm{Inf}\, A = (\mathrm{Inf}_1\, A - Z_+) \cup \{-\infty\}$$

ii) *for all sets $A \subset Z$, $A \neq \emptyset, \{-\infty\}, \{+\infty\}$ with $\mathrm{Sup}\, A \neq \{+\infty\}$, A has property (P') and*

$$\mathrm{Sup}\, A = (\mathrm{Sup}_1\, A + Z_+) \cup \{+\infty\}$$

Proof. ii) Since $\mathrm{Sup}\, A \neq \{+\infty\}$ we have that $+\infty \notin A$. Let $a \in A$ and $c \in \mathrm{Sup}\, A$, $c \neq +\infty$. Since $Z_+ \neq \emptyset$, from [4] there is $d \in Z$, $d > a$, $d > c$. The set $B = (a + Z_+) \cap \mathrm{Sup}\, A$ is a nonempty , closed set and $a \leq b$ for all $b \in B$. Thus by c) and Lemma 3.5. from [6] we get $\mathrm{Min}(B|Z_+) \neq \emptyset$. We observe that $B_d = B \cap (d - Z_+)$ is nonempty , closed and bounded so from a, b, d and Corollary 4.1.2, Theorem 4.1 and Corollary 4.1.2. respectively from [2] we get $\mathrm{Min}(B_d|Z_+) \neq \emptyset$.

Now, let $\alpha \in \mathrm{Min}(B|Z_+)$. If $\alpha = a$ then $A \subset \mathrm{Sup}_1\, A - Z_+$. If $\alpha \neq a$ let $C = (a + Z_+) \cap (\alpha - Z_+)$, which is nonempty, closed and $C \neq \{\alpha\}$.

There is no $\beta \in \mathrm{Sup}\, A \cap C$, $\beta \neq \alpha$ because $\alpha \in \mathrm{Min}(B|Z_+)$. Let $\varepsilon \in Z_+\backslash\{0\}$ and $D = \{\lambda a + (1 - \lambda)\alpha, \lambda \in [0, 1]\} \subset C$. Since $(\alpha - \varepsilon, \alpha + \varepsilon)$ is a convex neighbourhood of α, we get $\mu \in D \cap (\alpha - \varepsilon, \alpha + \varepsilon)$ and thus there is $\mu \in C$, $\mu \neq \alpha$, $\mu > \alpha - \varepsilon$ so $\alpha - \varepsilon \notin \mathrm{Sup}\, A$ and $\alpha \in \mathrm{Sup}_1\, A$. Thus we get $A \subseteq \mathrm{Sup}_1\, A - Z_+ \subseteq \bigcap_{\varepsilon \in Z_+\backslash\{0\}} (\mathrm{Sup}_1\, A - Z_+ + \varepsilon) = \overline{\mathrm{Sup}_1\, A - Z_+} \backslash\{+\infty\} =$
$= (\mathrm{Sup}_1\, A \cup \overline{\mathrm{Sup}_1\, A - Z_+\backslash\{0\}})\backslash\{+\infty\}$.

Of course $A - Z_+\backslash\{0\} = \mathrm{Sup}_1\, A - Z_+\backslash\{0\}$.

We can prove that $\mathrm{Sup}_1\, A - Z_+ = \overline{A - Z_+\backslash\{0\}}\backslash\{+\infty\}$ which gives that $A \subset \mathrm{Sup}_1\, A - Z_+$. Since $\mathrm{Sup}\, A \neq \{+\infty\}$ we have that $+\infty \notin \mathrm{Sup}_1\, A$.

Let $\alpha \in \mathrm{Sup}_1\, A - Z_+$, then $\alpha \in \mathrm{Sup}_1\, A$ or $\alpha \in \mathrm{Sup}_1\, A - Z_+\backslash\{0\}$.

If $\alpha \in \mathrm{Sup}_1\, A - Z_+\backslash\{0\}$ then $\alpha \in A - Z_+\backslash\{0\} \subset \overline{A - Z_+\backslash\{0\}}$.

If $\alpha \in \mathrm{Sup}_1\, A$ then for all $\varepsilon \in Z_+\backslash\{0\}$ there is $a \in A$ with $a > \alpha - \varepsilon$ and thus $\alpha \in \bigcup_{\varepsilon \in Z_+\backslash\{0\}} (A - Z_+\backslash\{0\} + \varepsilon) = \overline{A - Z_+\backslash\{0\}}$.

Now we have $\mathrm{Sup}_1\, A - Z_+ \subseteq \overline{A - Z_+\backslash\{0\}}$ and let $\alpha \in \overline{A - Z_+\backslash\{0\}}\backslash\{+\infty\}$. If $\alpha \in A - Z_+\backslash\{0\} = \mathrm{Sup}_1\, A - Z_+\backslash\{0\}$ then $\alpha \in \mathrm{Sup}_1\, A - Z_+$.

If $\alpha \notin A - Z_+\backslash\{0\}$ then $\alpha \in \operatorname{Sup} A$. Let $\varepsilon \in Z_+\backslash\{0\}$, so there is $a' \in A - Z_+\backslash\{0\}$, $a' > \alpha - \varepsilon$. We can found $a \in A$, $a > \alpha - \varepsilon$ and thus $\alpha \in \operatorname{Sup}_1 A \subset \operatorname{Sup}_1 A - Z_+$.

Finally we get $\operatorname{Sup}_1 A - Z_+ = \overline{A - Z_+\backslash\{0\}}\backslash\{+\infty\}$ and thus A has property (P'). Similar we get that A has property (P) in i).

Let $\mu \in \operatorname{Sup} A\backslash\{+\infty\}$, $\beta \in \operatorname{Sup}_1 A \neq \emptyset$ and $\gamma < \mu$, $\gamma < \beta$.

Since $\beta \in \operatorname{Sup}_1 A$ and $\gamma < \beta$ we get that $\gamma \notin \operatorname{Sup} A$.

The set $B = (\mu - Z_+) \cap \operatorname{Sup} A \cap (\gamma + Z_+)$ is nonempty, closed and $\gamma \leq b \leq \mu$ for all $b \in B$ so $\operatorname{Min}(B|Z_+) \neq \emptyset$.

Let $\alpha \in \operatorname{Min}(B|Z_+)$ $(\alpha \neq \gamma,\ \gamma \notin \operatorname{Sup} A)$.

The set $C = (\alpha - Z_+) \cap (\mu - Z_+) \cap (\gamma + Z_+) = (\alpha - Z_+) \cap (\gamma + Z_+)$ does not contain any $\eta \in \operatorname{Sup} A$ since $\alpha \in \operatorname{Min}(B|Z_+)$.

For $\varepsilon \in Z_+\backslash\{0\}$, $(\alpha - \varepsilon, \alpha + \varepsilon)$ is a convex neighbourhood for α and thus

$$\operatorname{Sup} A\backslash\{+\infty\} \subseteq \operatorname{Sup}_1 A + Z_+ \subseteq \bigcap_{\varepsilon \in Z_+\backslash\{0\}} \operatorname{Sup}_1 A + Z_+ - \varepsilon = \overline{\operatorname{Sup}_1 A + Z_+} \subset$$

$$\subset \bigcap_{\varepsilon \in Z_+\backslash\{0\}} (\operatorname{Sup} A - \varepsilon) = \overline{\operatorname{Sup} A + Z_+} = \operatorname{Sup} A.$$

Thus $\operatorname{Sup} A = \overline{\operatorname{Sup}_1 A + Z_+} \cup \{+\infty\}$.

But $\overline{\operatorname{Sup}_1 A + Z_+}\backslash\{+\infty\} = \operatorname{Sup}_1 A \cup \overline{\operatorname{Sup}_1 A + Z_+\backslash\{0\}}\backslash\{+\infty\}$. (i)

$\operatorname{Sup}_1 A \cup (\operatorname{Inf}_1 \operatorname{Sup}_1 A + Z_+)$.

Let $\alpha \in \operatorname{Inf}_1 \operatorname{Sup}_1 A \subset \operatorname{Sup} A$. If there is $\varepsilon \in Z_+\backslash\{0\}$ such that $\alpha - \varepsilon \in \operatorname{Sup} A \subset \operatorname{Sup}_1 A + Z_+ - \varepsilon$ we get $\alpha \in \operatorname{Sup}_1 A + Z_+$. But $\alpha \in \operatorname{Inf} \operatorname{Sup}_1 A$ and so $\alpha \in \operatorname{Sup}_1 A$, contradiction. So $\operatorname{Inf}_1 \operatorname{Sup}_1 \subset \operatorname{Sup}_1 A$ and $\overline{\operatorname{Sup}_1 A + Z_+} \subseteq (\operatorname{Sup}_1 A + Z_+) \cup \{+\infty\}$.

Finally we get $(\operatorname{Sup}_1 A + Z_+)\backslash\{+\infty\} = \operatorname{Sup}_1 A + Z_+$ and $\operatorname{Sup} A = (\operatorname{Sup}_1 A + Z_+) \cup \{+\infty\}$.

In a similar way we get $\operatorname{Inf} A = (\operatorname{Inf}_1 A - Z_+) \cup \{-\infty\}$.

4. Let X and Y be two real linear spaces:

Definition 4.1. *Let $F : X \to Z$ a multifunction such that $-\infty \notin f(x)$ for all $x \in D(f)$. Here $D(f) = \{x \in X_1, f(x) \neq \emptyset\}$. We define the ε-vectorial subdifferential of f in $x_0 \in D(f)$ the set denoted by $\partial_v^\varepsilon f(x_0)$ given by $\partial_{vf}^\varepsilon(x_0) = \{T \in L(X, Z)$ such that there exists $y_0 \in f(x_0)$ with $Tx_0 - y_0 \not< Tu - y - \varepsilon$, $\forall u \in D(f)$, $y \in f(u)\}$.*

Throughout this paragraph we shall consider that $L(X, Z)$ contains the linear and continuous operator from X to Z whenever X and Z are linear topological spaces.

For $\varepsilon = 0$ we obtain the vectorial subdifferential for a multifunction introduced in [2].

Observation 4.1. We can notice that there exists a strong connection between the approximate vectorial subdifferential of a multifunction and the clasical approximate subdifferential of a function.

Indeed, if X is a real topological vector space and $\varphi : X \to \bar{R}$ is a map, we can define the multifunction $\varphi : X \to R$ given by

$$\varphi(x) = \begin{cases} \{t \in (-\infty, +\infty],\ t \geq f(x)\} & \forall x \in D(\varphi) \\ +\infty & x \notin D(\varphi) \end{cases}$$

We can see that $D(\varphi) = X$ and $-\infty \notin \varphi(x)$, $\forall x \in X$. It can be shown that $\partial_v^\varepsilon \varphi(x_0) = \partial^\varepsilon f(x_0)$, $\forall x_0 \in D(\varphi)$, $\varepsilon > 0$. Indeed for $x_0 \in D(\varphi)$ we have $\partial_v^\varepsilon \varphi(x_0) = \{T \in L(X, R)$ such that there is $y_0 \geq \varphi(x_0)$ with $y_0 - Tx_0 \leq \leq y - Tu + \varepsilon$, $\forall u \in X$, $y \in f(u)\} = \{T \in L(x, R)$ such that there is $y_0 \geq \varphi(x_0)$ with $y_0 - Tx_0 \leq y - Tu + \varepsilon$, $\forall u \in D(\varphi)$, $y \geq \varphi(u)\}$.

Let $T \in \partial^\varepsilon \varphi(x_0)$ then $\varphi(x_0) - Tx \leq \varphi(u) - Tu + \varepsilon$, $\forall\ u \in D(\varphi)$. So $T \in \partial^\varepsilon \varphi(x_0)$ and thus $\partial^\varepsilon \varphi(x_0) \subseteq \partial^\varepsilon \varphi(x_0)$.

Now let $T \in \partial^\varepsilon \varphi(x_0)$ then $\varphi(x_0) - Tx_0 \leq f(u) - Tu + \varepsilon$, $\forall\ u \in D(\varphi)$. Thus for all $t \geq \varphi(u)$ we have $\varphi(x_0) - Tx_0 \leq t - Tu + \varepsilon$, $\forall\ u \in D(\varphi)$. We can get $y_0 = \varphi(x_0) \in \varphi(x_0)$ such that for all $u \in X$ and $t \in \varphi(u)$ we have $y_0 - Tx_0 \leq t - Tu + \varepsilon$. Finally we get $\partial^\varepsilon \varphi(x_0) \subseteq \partial^\varepsilon \varphi(x_0)$ and thus $\partial^\varepsilon \varphi(x_0) = \partial^\varepsilon \varphi(x_0)$.

Observation 4.2. The multifunction $f : X \to \overline{Z}$ with the property that $-\infty \notin f(x)$, $\forall\ x \in D(f)$ is ε-vectorial subdifferential in $x_0 \in D(f)$ with $f(x_0) \neq \{+\infty\}$ if and only if there is $y_0 \in f(x_0)$ and $T \in L(X, Z)$ such that

$$y_0 - Tx_0 \in {}^\varepsilon\operatorname{Min}\{y - Tu,\ \ u \in D(f), y \in f(u)\}$$

Observation 4.3. If $M \subset X$ is a nonempty set of a real linear space X, we consider the set

$$M_v^\varepsilon = \{T \in L(X, Z),\ Tx \not< -\varepsilon,\ \forall x \in M\}$$

Let $f : X \to Z$ be a multifunction and $G(f) = \{(x, y); y \in f(x), x \in D(f)\}$. We have $T \in \partial_v^\varepsilon f(x_0)$ if and only if there is $y_0 \in f(x_0)$ such that $Tx_0 - y_0 < < Tu - y - \varepsilon$ for all $y \in f(u)$, $u \in D(f)$. Thus $T \in \partial_v^\varepsilon f(x_0)$ if and only if $(-T, I)(u - x_0, y - y_0) \not< -\varepsilon$ and finally

$$T \in \partial_v^\varepsilon f(x_0) \ \text{ if and only if } \ (-T, I) \in (G(f) - (x_0, y_0))_v^\varepsilon$$

for some $y \in f(x_0)$ and $I : Z \to Z$, $Iz = z$, $\forall z \in Z$.

Proposition 4.1. *If (Z, τ) is a locally convex space, ordered by a convex, pointed cone Z_+ with $Z_+ = \{0\} \cup \overset{\circ}{Z}_+$ and if $f : X \to Z$ is a multifunction* *with the property that* $-\infty \notin f(x)$, $\forall x \in D(f)$ *then for* $x_0 \in D(f)$ *with* $f(x_0)$ *a compact set we have*

$$\bigcap_{\varepsilon \in Z_+ \setminus \{0\}} \partial_v^\varepsilon f(x_0) = \partial_v f(x_0)$$

Proof. Obviously, we have always $\partial_v f(x_0) \subseteq \partial_v^\varepsilon f(x_0)$, $\forall \varepsilon \in Z_+$ and thus $\partial_v f(x_0) \subseteq \bigcap_{\varepsilon \in Z_+ \setminus \{0\}} \partial_v^\varepsilon f(x_0)$. Now let $T \in \bigcap_{\varepsilon \in Z_+ \setminus \{0\}} \partial_v^\varepsilon f(x_0)$ then for all $\varepsilon \in Z_+ \setminus \{0\}$ there is $y_0^\varepsilon \in f(x)$ such that

$$Tx_0 - y_0^\varepsilon \not< Tu - y - \varepsilon, \quad \forall u \in D(f),\ y \in f(u).$$

It means that

$$Tx_0 - Tu + y \in C(-Z_+) \cup \{0\} - \varepsilon + y_0^\varepsilon.$$

For any u, y fix we get $Z_\varepsilon \in C(-Z_+) \cup \{0\}$ such that $Tx_0 - Tu + y = = z_\varepsilon - \varepsilon + y_0^\varepsilon$. For any $\varepsilon \in Z_+ \setminus \{0\}$ we have $Tx_0 - Tu + y = z_\varepsilon + y_0 - \varepsilon$. Since $f(x_0)$ is compact, then $(y_0^\varepsilon)_\varepsilon$ has a convergent subsequence and unless generality we can suppose that $(y_0^\varepsilon)_\varepsilon$ is convergent. Thus $(z_\varepsilon)_\varepsilon$ is a convergent sequence and his limit belongs to $C(-Z_+) \cup \{0\}$ which is a closed set in Z because $Z_+ = \{0\} \cup \overset{\circ}{Z}_+$. So there is $z \in C(-Z_+) \cup \{0\}$, $y_0 \in f(x_0)$ such that $Tx_0 - Tu + y = z + y_0$ and thus for all $u \in D(f)$, $y \in f(u)$, $Tu - Tx_0 \not< y - y_0$ which means that $T \in \partial f(x_0)$ and $\bigcup_{\varepsilon \in Z_+ \setminus \{0\}} \partial_v^\varepsilon f(x_0) \subseteq \partial f(x_0)$.

These two conclusions proves the equality.

Definition 4.2. i) [9] *For a multifunction* $f : D(f) \subseteq X \to Z$ *we define* $f^* : D(f^*) \subseteq L(X, Z) \to Z$ *by* $f^*(T) = \mathrm{Sup}_1 \{Tu - y,\ u \in D(f),\ y \in f(u)\}$ *called the vectorial conjugate of f, where* $D(f^*) = \{T \in L(X, Z)$ *with* $\mathrm{Sup}_1 \{Tu - y,\ u \in D(f),\ y \in f(u)\} \neq \emptyset\}$.

ii) *For a multifunction* $f : X \to Z$ *we define* $\tilde{f}^* : L(X, Z) \to Z$ *by* $\tilde{f}^*(T) = \mathrm{Sup}\{Tu - y, u \in D(f), y \in f(u)\}$ *called the extension vectorial conjugate of f.*

Proposition 4.2. *A multifunction* $f : X \to Z$ *with* $-\infty \notin f(x)$, $\forall x \in D(f)$ *is* ε*-vectorial subdifferentiable at* $x_0 \in D(f)$ *with* $f(x_0) \neq \{+\infty\}$ *if and only if there is* $y_0 \in f(x_0)$ *and* $T \in L(X, Z)$ *such that* $Tx_0 - y_0 \in \in \tilde{f}^*(T) - \varepsilon$.

Proof. First we suppose that f is ε-vectorial subdifferential in $x_0 \in D(f)$, $f(x_0) \neq \{+\infty\}$, so there is $y_0 \in f(x_0)$ and $T \in L(X, Z)$ such that

$$Tx_0 - y_0 \not< Tu - y - \varepsilon, \quad \forall u \in D(f),\ y \in f(u).$$

Thus, $Tx_0 - y_0 + \varepsilon \in \mathrm{Sup}\{Tu - y, u \in D(f), y \in f(u)\} = \tilde{f}^*(T)$. Converselly let $y_0 \in f(x_0)$ and $T \in L(X, Z)$ such that $Tx_0 - y_0 \in \tilde{f}^*(T) - \varepsilon$.

This means that $Tx_0 - y_0 + \varepsilon \not< Tu - y$, $\forall u \in D(f)$, $y \in f(u)$ and implies that $T \in \partial_v^\varepsilon f(x_0)$.

Observation 4.4. If $\varepsilon = 0$ we can replace $\tilde{f}^*$ by f^* and we obtain the Theorem 2.1 by [2].

Proposition 4.3. *If* $f : X \to Z$ *is a multifunction with* $-\infty \notin f(x)$, $\forall x \in D(f)$ *we have:*

i) *for all* $x_0 \in D(f)$ *with* $f(x_0) \neq \{+\infty\}$ *and every* $T \in L(X, Z)$ *with the property that there exist* $y_0 \in f(x_0)$ *and* $y^* \in \tilde{f}^*(T)$ *such that* $y_0 + y^* = Tx_0 + \varepsilon$ *we have* $T \in \partial_v^\varepsilon f(x_0)$.

ii) *for all* $x_0 \in D(f)$ *with* $\partial_v^\varepsilon f(x_0)$ *and every* $T \in \partial^\varepsilon f(x_0) \cap D(f^*)$ *there is* $y_0 \in f(x_0)$ *such that*

$$Tx_0 \not< y_0 + y^* - \varepsilon', \quad \forall y^* \in f^*(T), \quad \forall \varepsilon' > \varepsilon$$

Proof. i) Follows directly from proposition 4.2.

ii) Since $T \in \partial_v^\varepsilon f(x_0)$ we get $y_0 \in f(x_0)$ such that $Tx_0 - y_0 \not< Tu - y - \varepsilon$, $\forall u \in D(f)$, $\forall y \in f(u)$. Suppose by the contrary that there is $y^* \in f^*(T)$ such that $Tx_0 - y_0 < y^* - \varepsilon$ with $\varepsilon' > \varepsilon$.

Since $y^* \in f^*(T)$ if follows that for all $\varepsilon' > \varepsilon$, there is $u \in D(f)$, $y \in f(u)$ such that

$$Tx_0 - y_0 + \varepsilon < y^* - \varepsilon' + \varepsilon < Tu - y.$$

This contradiction proves the proposition.

Proposition 4.5. *If f is multifunction* $f : Y \to Z$ *with* $f(y) \neq \emptyset$, $f(y) \neq \{+\infty\}$, $\forall y \in Y$ *and* $-\infty \not< f(y)$, $\forall y \in Y$, *then* $0 \in \partial_v^\varepsilon f(x_0)$ *if and only if* $f(y_0) \cap {}^\varepsilon\mathrm{Min}\, f(y) \neq \emptyset$.

Proof. $0 \in \partial_v^\varepsilon f(y_0)$ if and only if there is $z_0 \in f(y_0)$ such that

$$-z_0 \not< -z - \varepsilon \quad \forall z \in f(y),\ y \in Y$$

and this is exactly the conclusion.

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