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GRADUAL METRICS AND OUTER MEASURES

BY

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1. Introduction. The notion of probabilistic metric (PM) spaces was introduced by K. Menger in 1942 (see [6]) for modelling those situations in which we have only probabilistic informations about the distances between points of the space ([10] is a survey in this subject; see [7] also). The E-spaces of Sherwood (see [11]) are typical examples of PM spaces. So, if $(T, \mathcal{A}, \mu)$ is a space with a probability measure μ, then to every two measurable mappings $x, y : T \longrightarrow \mathbb{R}$ we assign a distribution function $F_{xy} : \mathbb{R} \longrightarrow \mathbb{R}$, letting $F_{xy}(a) = \mu(\{t \in T : |x(t) - y(t)| < a\})$. Let $\doteq$ be the equality μ-almost everywhere and let M be the quotient space $\mathcal{M}/\doteq$, when $\mathcal{M}$ is the family of all measurable mappings of T in $\mathbb{R}$; then (M, F, T_m) is a Menger space relating to the t-norm T_m $(T_m(a,b) = \max(a+b-1, 0))$.

B. Schweizer and A. Sklar proved in [9] that the probabilistic metric on an E-space M of measurable mappings on a probability space $(T, \mathcal{A}, \mu)$ yields as much information about M as all Fréchet–Minkowski metrics on $L^p(T)$, $p > 0$; so all such Fréchet–Minkowski metrics are obtainable from a single probabilistic metric.

In section 2 of this paper we generalize E-spaces of Sherwood. So, if μ^* is an outer measure on T, then we introduce a gradual metric on the space X of all real mappings on T. In the particular case of a probability space, we obtain a probabilistic pseudometric on $M \subseteq X$ which is the equivalent form of probabilistic metric on E-space M of all measurable mappings (see [2]). Conversely, we give the conditions as a gradual metric be induced by an outer measure (Theorem 2.15). So, an outer measure on T yields the same informations as a gradual metric on X.

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In 1971, M.J. F r a n k introduced the probabilistic topological spaces for modelling those situations in which we have only probabilistic informations concerning to the statement: "the point x belongs to the closure of the set A" (see [5]). Like in general topology, every probabilistic metric induces a probabilistic topology. We completed the system of probabilistic structures by introduction in [1] of probabilistic uniformities and proximities; in [3] we initiated a study of convergence in probabilistic topological spaces. So, we introduced a probabilistic convergence function which yields "the probability that a net α converges to a point x." G.D. R i c h a r d s o n and D.C. K e n t give in [8] a generalization of this theory of convergence in the filters language. We must remark that "a probabilistic convergence function" are independent of any probabilistic determination, hence the above interpretation is not essential. So, we may introduce in section 3 an index of convergence of nets in every space with an outer measure and we obtain the generalization of some well-known results in measure theory (3.11, 3.15, 3.17, 3.18 and 3.19). The results in this section generalize some results of [3], also.

2. Gradual metrics.

In this section we define gradual metrics. Every outer measure induces a gradual metric and this mapping is an injection. We give the conditions as a gradual metric be induced by an outer measure.

Let T be a set and $\mu^* : \mathcal{P}(T) \longrightarrow \overline{I\!R}_+$ be an outer measure on T. Let $X = I\!R^T$ denote the set of all real functions on T. For every $x \in X$ and $a \in I\!R$ the sets $\{t \in T : |x(t)| > a\}$, $\{t \in T : x(t) = 0\}$, etc. will be denoted by $(|x| > a)$, $(x = \underline{o})$, respectively. The mapping $d : X \times X \longrightarrow I\!R_+$ defined by $d(x,y) = \min\{1, \inf\{a + \mu^*(|x-y| > a) : a > 0\}\}$ is a pseudo-metric on X which induces the topology of convergence in measure on X. So, a sequence $(x_n) \subseteq X$ converges in measure to $x \in X$ if and only if $d(x_n, x) \longrightarrow 0$ (equivalently, for every $\varepsilon > 0$, $\lim\limits_n \mu^*(|x_n - x| > \varepsilon) = 0$).

For every $x, y \in X$, we define a mapping $H_{xy} : I\!R \longrightarrow \overline{I\!R}_+$ letting $H_{xy}(a) = \inf\{\mu^*(|x-y| > b) : b < a\}$.

2.1. Remarks. (i) For every $a \in I\!R$ and $b, c \in I\!R$ with $b < c < a$, we have $(|x-y| > c) \subseteq (|x-y| > b)$, so that $\mu^*(|x-y| > c) \le \mu^*(|x-y| > b)$ and therefore $H_{xy}(a) = \lim\limits_{b \uparrow a} \mu^*(|x-y| > b)$.

(ii) Let $\mathcal{M}_0$ be the σ-algebra of all μ^*-measurable subsets of T, and $\mu = \mu^*|_{\mathcal{M}_0}$ be the measure induced by μ^*. If $\mu(T) = 1$ (μ is a probability on T), then, from the continuity of μ, we deduce that $H_{xy}(a) = \mu(|x-y| \ge a)$, for every measurable functions x, y on T. So, $H_{xy}(a) = 1 - \mu(|x-y| < a) = 1 - F_{xy}(a)$, where F is the probabilistic metric on the E-space (M, F, T_m) (see [11]). Therefore, in this particular case, H is the tail of F.

(iii) According to the previous remark, we may interpret $H_{xy}(a)$ as the degree that the distance between x and y be not smaller than a.

In the following theorem we give some properties of mapping H.

2.2. Theorem. *For every $x, y, z \in X$ we have:*

(H_1) H_{xy} *is a decreasing, left continuous function.*

(H_2) $H_{xy}(0) = \mu^*(T)$.

(H_3) $H_{xy}(0+) = \sup\limits_{a>0} H_{xy}(a) = 0$ *if and only if* $x = y$ *almost everywhere* ($\mu^*(x \neq y) = 0$).

(H_4) $H_{xy}(a) = H_{yx}(a)$, *for every* $a \in I\!R$.

(H_5) $H_{xy}(a+b) \le H_{xz}(a) + H_{zy}(b)$, *for every* $a, b \in I\!R$.

Proof. (H_1) It is obvious that H_{xy} is a noncreasing function; if $a_n \uparrow a$ and $H_{xy}(a) < +\infty$ then, for every $\varepsilon > 0$, there exists $b < a$ such that $H_{xy}(a) + \varepsilon > \mu^*(|x-y| > b)$. Let $n_0 \in I\!N$ such that, for every $n \ge n_0$, $b < a_n < a$; then $H_{xy}(a) + \varepsilon > \mu^*(|x-y| > b) \ge H_{xy}(a_n) \ge H_{xy}(a)$, hence $\lim\limits_{b \uparrow a} H_{xy}(b) = H_{xy}(a)$. If $H_{xy}(a) = +\infty$, then $H_{xy}(a_n) = +\infty$, for every $n \in I\!N$. The conditions (H_2) and (H_4) follow from definition.
(H_3) If $\sup\limits_{a>0} H_{xy}(a) = 0$, then, for every $a > 0$, $\mu^*(|x-y| > a) = 0$. But

$$\mu^*(x \neq y) = \mu^*(|x-y| > 0) = \mu^*\Big(\bigcup_{n=1}^{\infty}\big(|x-y| > \frac{1}{n}\big)\Big) \le \sum_{n=1}^{n} \mu^*\big(|x-y| > \frac{1}{n}\big) = 0.$$

Hence $x = y$ a.e.
(H_5) We may suppose that $H_{xz}(a) + H_{zy}(b) < +\infty$. Then, for every $\varepsilon > 0$, there exist $a_1 < a$ and $b_1 < b$ such that $\mu^*(|x-z| > a_1) < H_{xz}(a) + \varepsilon$ and $\mu^*(|z-y| > b_1) < H_{zy}(b) + \varepsilon$. Hence $\mu^*(|x-y| > a_1 + b_1) \le \mu^*(|x-z| > a_1) + \mu^*(|z-y| > b_1) < H_{xz}(a) + H_{zy}(b) + 2\varepsilon$. Because $a_1 + b_1 < a + b$ and ε is an arbitrary positive number, we have that $H_{xy}(a+b) \le H_{xz}(a) + H_{zy}(b)$.

In [2] we proved that every probabilistic metric can be characterized by a family of semi metrics – so called probabilistic pseudo-metric (see [2, Prop. 1 and Th. 1]).

In the following we define *the gradual metrics*. As we shall see, every outer measure on T induces a gradual metric on X.

2.3. Definition. *A gradual metric on X is a family $\gamma = \{d_s : s \in (0, +\infty]\}$ of semi-metrics on X (a semi-metric on X is a symmetric* *function defined on $X \times X$ assuming non-negative real values, and zero on the diagonal of $X \times X$) satisfying:*

(γ_1) $d_\infty(x,y)=0$, *for every* $x,y\in X$.

(γ_2) $d_{s+t}(x,y) \le d_s(x,z)+d_t(z,y)$, *for every* $x,y,z\in X$, $s,t>0$.

(γ_3) $\lim\limits_{\substack{t\to s\\ t>s}} d_t(x,y) = d_s(x,y)$, *for every* $x,y\in X$, $s\in(0,+\infty)$.

2.4. Remarks. (i) Generally, a semi-metric $d_s\in\gamma$ is not a pseudometric; the triangle inequality acts on the whole family γ but not on every $d_s\in\gamma$, individually.

(ii) $d_t(x,y)\le d_s(x,y)$, for every $x,y\in X$ and $0<s<t$. Indeed, from (γ_2), $d_t(x,y) = d_{s+(t-s)}(x,y) \le d_s(x,y)+d_{t-s}(y,y) = d_s(x,y)$.

(iii) From the previous remark and (γ_3) it follows that $d_s(x,y) = \sup\limits_{t>s} d_t(x,y)$, for every $x,y\in X$, $s>0$.

2.5. Theorem. *For every* $s\in(0,+\infty]$ *we define* $d_s: X\times X\to \mathbb{R}_+$ *letting* $d_s(x,y)=\min\{1,\inf\{a\ge0: H_{xy}(a)\le s\}\}$. *Then* $\gamma=\{d_s: s\in(0,+\infty]\}$ *is a gradual metric on* X.

Proof. We denote by $A_s(x,y)=\{a\ge0: H_{xy}(a)\le s\}$. If $A_s(x,y)=\emptyset$ then $\inf A_s(x,y) = +\infty$, hence $d_s(x,y)=1$. From (H_1) it follows that $d_s(x,y)=d_s(y,x)$, for every $x,y\in X$, $s>0$.

We remark that $A_s(x,x)\supseteq(0,+\infty)$, hence $d_s(x,x)=0$, for every $x\in X$, $s>0$. Therefore $\gamma=\{d_s: s>0\}$ is a family of semi-metrics.

(γ_1) is a consequence of definition of d_∞. Let $x,y,z\in X$ and $s,t>0$. Because $d_{s+t}(x,y)\le1$, we may suppose that $d_s(x,z)+d_t(z,y)<1$. For every $\varepsilon>0$, there exist $a<d_s(x,z)+\varepsilon$ and $b<d_t(z,y)+\varepsilon$ such that $H_{xz}(a)\le s$ and $H_{zy}(b)\le t$. From (H_5), it follows that $H_{xy}(a+b)\le s+t$, hence $d_{s+t}(x,y)\le a+b<d_s(x,z)+d_t(z,y)+2\varepsilon$ and we have (γ_2). From Remark 2.4 (ii), it follows that $\ell = \lim\limits_{\substack{t\to s\\ t>s}} d_t(x,y) = \sup\limits_{t>s} d_t(x,y)\le d_s(x,y)$.

Because $d_s(x,y)\le1$, we may suppose that $\ell<1$. If $\ell<d_s(x,y)$, then there exists $\varepsilon>0$ such that $\ell+\varepsilon<d_s(x,y)\le\inf A_s(x,y)$. Hence $H_{xy}(\ell+\varepsilon)>s$. Let t such that $H_{xy}(\ell+\varepsilon)>t>s$. It follows that $\ell\ge d_t(x,y)\ge\ell+\varepsilon$ and this is a contradiction.

2.6. Remarks. (i) For every $s>0, x,y\in X$, $d_s(x,y)=\min\{1,\inf\{a\ge0:\mu^*(|x-y|>a)\le s\}\}$. Indeed, if $A_s(x,y)=\{a\ge0:H_{xy}(a)\le s\}$ and $B_s(x,y)=\{a\ge0:\mu^*(|x-y|>a)\le s\}$, then $A_s(x,y)\subseteq B_s(x,y)$, so that $\inf B_s(x,y)\le\inf A_s(x,y)$. For every $a\in B_s(x,y)$ with $a>\inf B_s(x,y)$ there exists $b\in B_s(x,y)$ with $b<a$, so that $H_{xy}(a)\le\mu^*(|x-y|>b)\le s$.

Therefore $b\in A_s(x,y)$, hence $\inf A_s(x,y)\le b<a$; it follows that $\inf A_s(x,y)\le\inf B_s(x,y)$.

(ii) $d_s(x,y)=0$ if and only if $\sup\limits_{a>0}\mu^*(|x-y|>a)\le s$.

(iii) Let $d_0(x,y) = \sup\limits_{s>0} d_s(x,y) = \lim\limits_{s\to0} d_s(x,y)\le1$. Then d_0 is a pseudo metric on X. Indeed, letting $s,t\to0$ in (γ_2), we obtain the triangle inequality for d_0.

(iv) In particular case of Remark 2.1 (ii), $d_s(x,y)=\inf\{a\ge0: F_{xy}(a)\ge1-s\}$, so that $\delta=\{d_{1-s}: s\in(0,1]\}$ is the probabilistic pseudo-metric induced by probabilistic metric F on T (see [2]).

2.7. Definition. *We say that gradual metric γ defined in 2.5 is induced by outer measure μ^*. In the following proposition we compare the pseudo metric of convergence in measure with semi-metrics d_s of γ.*

2.8. Proposition. *For every* $x,y\in X$ *and* $s\in(0,\infty]$,

$$\min\{s,d_s(x,y)\}\le d(x,y)\le d_s(x,y)+s.$$

Proof. From Remark 2.6 (i), $d_s(x,y)=\min\{1,\inf B_s(x,y)\}$ where $B_s(x,y)=\{a\ge0:\mu^*(|x-y|>a)\le s\}$.

If $B_s(x,y)=\emptyset$ then $\inf B_s(x,y)=+\infty$ so that $d_s(x,y)=1$. For every $a\ge0$, $\mu^*(|x-y|>a)>s$, hence $a+\mu^*(|x-y|>a)>s$, so that $\min\{s,d_s(x,y)\}=\min\{s,1\}\le d(x,y)\le1+s=d_s(x,y)+s$.

If $B_s(x,y)\ne\emptyset$ let $b_s=\inf B_s(x,y)\in\mathbb{R}_+$. For every $a\in(0,b_s)$, $\mu^*(|x-y|>a)>s$, so that $a+\mu^*(|x-y|>a)>a+s$ and $\inf\{a+\mu^*(|x-y|>a): a\in(0,b_s)\}\ge s\ge\min\{s,d_s(x,y)\}$. Obviously, $\inf\{a+\mu^*(|x-y|>a): a\ge b_s\}\ge b_s\ge d_s(x,y)\ge\min\{s,d_s(x,y)\}$. Hence $\min\{s,d_s(x,y)\}\le d_s(x,y)$. Now, for every $a\in B_s(x,y)$, $\mu^*(|x-y|>a)\le s$, hence $a+\mu^*(|x-y|>a)\le a+s$, for every $a\ge b_s$. Therefore we have $d(x,y)\le b_s+s$ and $d(x,y)\le1+s$, so that $d(x,y)\le d_s(x,y)+s$.

2.9. Remarks. (i) For every $s>d(x,y)$, we obtain from 2.8, $d_s(x,y)\le d(x,y)$.

(ii) $d\le d_0=\sup\limits_{s>0} d_s$.

As we shall see in the following example, we can't improve this last inequality.

2.10. Example. For every $A,B\subseteq T$, $d_s(\chi_A,\chi_B)=1$ if $0<s<\mu^*(A\Delta B)$ and $d_s(\chi_A,\chi_B)=0$, otherwise (χ_A is the characteristic function of A, and $A\Delta B=(A-B)\cup(B-A)$). $d(\chi_A,\chi_B)=\min\{1,\mu^*(A\Delta B)\}$. If $\mu^*(A\Delta B)\in(0,1)$ then $d(\chi_A,\chi_B)=\mu^*(A\Delta B)<1=d_0(\chi_A,\chi_B)$.

2.11. Remark. $d_s(x,y)=0$ for every $s>0$ iff $x=y$, a.e. Indeed, if $d_s(x,y)=0$, for every $s>0$, then, from 2.9 (ii), $d(x,y)=0$, hence $x=y$ almost everywhere. Conversely, $x=y$ a.e. implies, from 2.2 (H_3), that $H_{xy}(a)=0$, for every $a>0$, hence $A_s(x,y)\supseteq(0,+\infty)$, for every $s>0$. Therefore $d_s(x,y)=0$, for every $s>0$.

In the following theorem we add to (γ_1)-(γ_3) a few other properties of a gradual metric induced by an outer measure.

2.12. Theorem. *Let μ^* be an outer measure on T and $\gamma=\{d_s:s\in(0,+\infty]\}$ be the gradual metric induced by μ^*. Then we have:*

(γ_4) $d_s(x,y)=d_s(x-y,\underline{0})$, *for every* $s>0$, $x,y\in X$.

(γ_5) $d_s(x,\underline{0})\le d_s(y,\underline{0})$, *for every* $s>0$ *and* $x,y\in X$, *with* $|x|\le|y|$.

(γ_6) $d_s(x,\underline{0})\le\sum_{n=1}^{\infty}d_{s_n}(x_n,\underline{0})$, *for every* $(s_n)\subseteq(0,+\infty]$, $(x_n)\subseteq X$,

with $s=\sum_{n=1}^{\infty}s_n$ *and* $x=\sum_{n=1}^{\infty}x_n\in X$ $\quad(\sum_{n=1}^{\infty}x_n(t)=x(t)$, *for every* $t\in T$).

(γ_7) $d_s(x,\underline{0})=\min\{1,\inf\{a\ge 0:d_s(x^a,\underline{0})=0\}\}$, *for every* $s>0$ *and* $x\in X$, *where* $x^a=\chi_{(|x|>a)}$, $a>0$.

Proof. For every $s>0$, $d_s(x,y)=\min\{1,\inf B_s(x,y)\}$, where $B_s(x,y)=\{a\ge 0:\mu^*(|x-y|>a)\le s\}$. So (γ_4) follows directly from definition. If $x,y\in X$ with $|x|\le|y|$ then $(|x|>a)\subseteq(|y|>a)$ so that $B_s(y,\underline{0})\subseteq B_s(x,\underline{0})$, for every $s>0$ and we obtain (γ_5). Let $s=\sum_{n=1}^{\infty}s_n>0$ and $x=\sum_{n=1}^{\infty}x_n\in X$. Because $d_s(x,\underline{0})\le 1$, we may suppose that $\sum_{n=1}^{\infty}d_{s_n}(x_n,\underline{0})<1$.

For every $a>\sum_{n=1}^{\infty}d_{s_n}(x_n,\underline{0})$ there exists a sequence $(a_n)\subseteq(0,+\infty)$ such that $a=\sum_{n=1}^{\infty}a_n$ and $a_n>d_{s_n}(x_n,\underline{0})$, for every $n\in I\!N$.

Therefore, there exists $(b_n)\subseteq I\!R$ such that $b_n<a_n$ and $\mu^*(|x|>b_n)\le s_n$, for every $n\in I\!N$.

Let $b=\sum_{n=1}^{\infty}b_n<a$. From $\bigcap_{n=1}^{\infty}(|x_n|\le b_n)\subseteq(\sum_{n=1}^{\infty}|x_n|\le\sum_{n=1}^{\infty}b_n)$ it follows that $\mu^*(|x|>b)=\mu^*(|\sum_{n=1}^{\infty}x_n|>b)\le\mu^*(\bigcup_{n=1}^{\infty}(|x_n|>b_n))\le$ $\le\sum_{n=1}^{\infty}\mu^*(|x_n|>b_n)\le\sum_{n=1}^{\infty}s_n=s$, so that $d_s(x,\underline{0})\le b<a$. Then we obtain $d_s(x,\underline{0})\le\sum_{n=1}^{\infty}d_{s_n}(x_n,\underline{0})$. From 2.6 (ii), $d_s(x^a,\underline{0})=0$ iff $\sup_{\alpha>0}\mu^*(|x^a|>\alpha)\le s$, but this is equivalent with $\mu^*(|x|>a)\le s$. Therefore $\{a\ge0:d_s(x^a,\underline{0})=0\}=B_s(x,\underline{0})$ and, from 2.6 (i), we obtain (γ_7).

2.13. Remark. The properties (γ_1)-(γ_7) are not independent; for example $(\gamma_3)+(\gamma_4)+(\gamma_6)\Rightarrow(\gamma_2)$. Indeed, for every $s,t>0$, $x,y,z\in X$ and $\varepsilon>0$, let us denote $s_1=s$, $s_2=t$, $x_1=x-z$, $x_2=z-y$ and $s_n=\varepsilon/2^{n-2}$, $x_n=\underline{0}$, for every $n\ge 3$; then $\sum_{n=1}^{\infty}s_n=s+t+\varepsilon$ and $\sum_{n=1}^{\infty}x_n=x-y$. From (γ_6) we obtain $d_{s+t+\varepsilon}(x-y,\underline{0})\le d_s(x-z,\underline{0})+d_t(z-y,\underline{0})$. Letting $\varepsilon\longrightarrow 0$ and using (γ_3), $d_{s+t}(x-y,\underline{0})\le d_s(x-z,\underline{0})+d_t(z-y,\underline{0})$ and from (γ_4) we obtain (γ_2).

2.14. Definition. *We say that a gradual metric $\gamma=\{d_s:s>0\}$ which satisfies the properties (γ_4)-(γ_7) from Theorem 2.12 is a gradual metric in measure.*

Therefore a *gradual metric in measure* is a family of semi-metrics $\gamma=\{d_s:s>0\}$ which satisfies:

(γ_1) $d_\infty(x,y)=0$, for every $x,y\in X$.

(γ_2) $d_{s+t}(x,y)\le d_s(x,z)+d_t(z,y)$, for every $x,y,z\in X$, $s,t>0$.

(γ_3) $\lim_{t\searrow s}d_t(x,y)=d_s(x,y)$, for every $x,y\in X$, $s\in(0,+\infty)$.

(γ_4) $d_s(x,y)=d_s(x-y,\underline{0})$, for every $x,y\in X$, $s>0$.

(γ_5) $d_s(x,\underline{0})\le d_s(y,\underline{0})$, for every $s>0$ and $x,y\in X$ with $|x|\le|y|$.

(γ_6) $d_s(x,\underline{0})\le\sum_{n=1}^{\infty}d_{s_n}(x_n,\underline{0})$, for every $s=\sum_{n=1}^{\infty}s_n>0$, $x=\sum_{n=1}^{\infty}x_n\in X$.

(γ_7) $d_s(x,\underline{0})=\min\{1,\inf\{a\ge 0:d_s(x^a,\underline{0})=0\}\}$, for every $s>0$ and $x\in X$, where $x^a=\chi_{(|x|>a)}$, $a>0$.

In the following theorem we prove that the properties (γ_1)–(γ_7) characterize gradual metrics induced by outer measures.

2.15. Theorem. *The mapping $\Psi : \mu^* \longrightarrow \gamma$, where $\Psi(\mu^*) = \gamma$ is the gradual metric induced by the outer measure μ^* is a bijection between the set of all outer measures on T and the set of all gradual metrics in measure.*

Proof. We recall that $\Psi(\mu^*) = \gamma = \{d_s : s > 0\}$, where $d_s(x, y) = \min\{1, \inf\{a \geq 0 : \mu^*(|x - y| > a) \leq s\}\}$, for every $x, y \in X$ and $s \in (0, \infty]$. From Theorems 2.5 and 2.12 it follows that $\Psi(\mu^*) = \gamma$ is a gradual metric in measure. We must prove that for every gradual metric in measure γ there exists exactly one outer measure μ^* such that $\Psi(\mu^*) = \gamma$.

Let $\gamma = \{d_s : s > 0\}$ be a gradual metric in measure; so, the family $\{d_s : s > 0\}$ satisfies the properties (γ_1)–(γ_7) from 2.14. We define $\mu^* : \mathcal{P}(T) \longrightarrow \overline{I\!R}_+$ letting $\mu^*(A) = \inf\{s > 0 : d_s(\chi_A, \underline{0}) = 0\}$.

From (γ_1), $d_\infty(\chi_A, \underline{0}) = 0$, for every $A \subseteq T$, so that μ^* is well-defined.

$\mu^*(\emptyset) = \inf\{s > 0 : d_s(\underline{0}, \underline{0}) = 0\} = 0$. For every $A, B \subseteq T$ with $A \subseteq B$, $\chi_A \leq \chi_B$; from (γ_5) we have $d_s(\chi_A, \underline{0}) \leq d_s(\chi_B, \underline{0})$, for every $s > 0$, so that $\mu^*(A) \leq \mu^*(B)$.

Let $(A_n)_{n \in I\!N}$ be a sequence of subsets of T, with $A_n \cap A_m = \emptyset$, for every $m \neq n$ and let $A = \bigcup_{n=1}^{\infty} A_n$; then $\chi_A = \sum_{n=1}^{\infty} \chi_{A_n}$. If $\sum_{n=1}^{\infty} \mu^*(A_n) = +\infty$ then we have $\mu^*(A) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$. We may suppose that $\sum_{n=1}^{\infty} \mu^*(A_n) < +\infty$; for every $\varepsilon > 0$ and $n \in I\!N$ there exists $s_n > 0$ such that $s_n < \mu^*(A_n) + \frac{\varepsilon}{2^n}$ and $d_{s_n}(\chi_{A_n}, \underline{0}) = 0$. Let $s = \sum_{n=1}^{\infty} s_n < \sum_{n=1}^{\infty} \mu^*(A_n) + \varepsilon < +\infty$. From (γ_6), $d_s(\chi_A, \underline{0}) \leq \sum_{n=1}^{\infty} d_{s_n}(\chi_{A_n}, \underline{0}) = 0$, so that $\mu^*(A) \leq s = \sum_{n=1}^{\infty} s_n < \sum_{n=1}^{\infty} \mu^*(A_n) + \varepsilon$, for every $\varepsilon > 0$. Now, if (A_n) is an arbitrary sequence of subsets of T we denote by $B_1 = A_1$ and $B_n = A_n - \bigcup_{k=1}^{n-1} A_k$. Then $B_n \cap B_m = \emptyset$, for every $n \neq m$ and $A = \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$. Hence $\mu^*(A) \leq \sum_{n=1}^{\infty} \mu^*(B_n) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$, because $B_n \subseteq A_n$, for every $n \in I\!N$. Therefore μ^* is an outer measure on T; let $\gamma' = \{d'_s : s > 0\} = \Psi(\mu^*)$ be gradual metric induced by μ^*. From Remark 2.6 (i), $d'_s(x, \underline{0}) = \min\{1, \inf\{a \geq 0 : \mu^*(|x| > a) \leq s\}\}$.

$$\mu^*(|x| > a) = \inf\{s > 0 : d_s(\chi_{(|x| > a)}, \underline{0}) = 0\} = \inf\{s > 0 : d_s(x^a, \underline{0}) = 0\}.$$

Now, from (γ_3), there exists $s_a > 0$ such that $s_a = \mu^*(|x| > a)$ and $d_{s_a}(x^a, \underline{0}) = 0$. Therefore

$$d'_s(x, \underline{0}) = \min\{1, \inf\{a \geq 0 : s_a \leq s\}\} = \min\{1, \inf\{a \geq 0 : d_s(x^a, \underline{0}) = 0\}\}.$$

So that, from (γ_7), $d'_s(x, \underline{0}) = ds(x, \underline{0})$.

Now, using (γ_4), we obtain $d_s(x, y) = d'_s(x, y)$, for every $x, y \in X$ and $s > 0$, so that $\Psi(\mu^*) = \gamma$.

If μ_1^* is another outer measure on T with $\Psi(\mu_1^*) = \gamma$, then, for every $A \subseteq T$, we have $d_s(\chi_A, \underline{0}) = \min\{1, \inf\{a \geq 0 : \mu_1^*(|\chi_A| > a) \leq s\}\}$ hence $d_s(\chi_A, \underline{0}) = 0$ iff $\mu_1^*(\chi_A > a) \leq s$, for every $a > 0$, or equivalently iff $\mu_1^*(A) \leq s$. Therefore $\mu^*(A) = \inf\{s > 0 : d_s(\chi_A, \underline{0}) = 0\} = \inf\{s > 0 : \mu_1^*(A) \leq s\}$, so that $\mu^* = \mu_1^*$.

3. The index of convergence in measure. Let γ be the gradual metric induced by the outer measure μ^* on T. For every $s > 0$ and $r > 0$ let us denote $U_s(r) = \{(x, y) : d_s(x, y) < r\}$ and for every $s \geq 0$ let $\mathcal{U}_s = \{U \subseteq X \times X : \exists r > 0, \exists t > s \text{ such that } U_t(r) \subseteq U\}$.

3.1. Theorem. *The family $\mathcal{U} = \{\mathcal{U}_s : s \geq 0\}$ has the following properties:*

(U_1) *For every $s \geq 0$, $\mathcal{U}_s$ is a semi-uniformity on X ($\mathcal{U}_s$ is a filter on $X \times X$, $\Delta \subseteq U$ and $U^{-1} \in \mathcal{U}$ for every $U \in \mathcal{U}_s$).*

(U_2) $\mathcal{U}_s = \bigcup_{t > s} \mathcal{U}_t$, *for every $s \geq 0$.*

(U_3) *For every $s \geq 0$ and $U \in \mathcal{U}_{2s}$ there exists $V \in \mathcal{U}_s$ such that $V \circ V \subseteq U$.*

Proof. For every $s \geq 0$ and $U_1, U_2 \in \mathcal{U}_s$ there exist $t_2 > s$, $t_2 > s$, $r_1 > 0$, $r_2 > 0$ such that $U_{t_1}(r_1) \subseteq U_1$ and $U_{t_2}(r_2) \subseteq U_2$. Let $t = \min\{t_1, t_2\} > s$ and $r = \min\{r_1, r_2\} > 0$. Then $d_t \geq d_{t_1}$, $d_t \geq d_{t_2}$ (see 2.4 (ii)) so that $U_t(r) \subseteq U_{t_1}(r) \cap U_{t_2}(r) \subseteq U_1 \cap U_2$. It follows that $\mathcal{U}_s$ is a filter on $X \times X$. Obviously, $\Delta \subseteq U$, for every $U \in \mathcal{U}_s$ and, because $U_t^{-1}(r) = U_t(r)$, it follows that $U^{-1} \in \mathcal{U}_s$, for every $U \in \mathcal{U}_s$. Hence $\mathcal{U}_s$ is a semi-uniformity, for every $s \geq 0$. From definition, it follows that $\mathcal{U}_t \subseteq \mathcal{U}_s$ if $t > s$, so that $\mathcal{U}_s \supseteq \bigcup_{t > s} \mathcal{U}_t$. If $U \in \mathcal{U}_s$ then $U_t(r) \subseteq U$ for some $t > s$ and $r > 0$. Let u such that

$t > u > s$; then $U \in \mathcal{U}_u \subseteq \bigcup_{t>s} \mathcal{U}_t$. Now, for every $s \geq 0$ and $U \in \mathcal{U}_{2s}$, there exist $t > s$ and $r > 0$ such that $U_{2t}(2r) \subseteq U$. For every $(x, y) \in U_t(r) \circ U_t(r)$, there exists $z \in X$ such that $d_t(x, z) < r$ and $d_t(z, y) < r$. From (γ_2) we have $d_{2t}(x, y) \leq d_t(x, z) + d_t(z, y) < 2r$, hence $(x, y) \in U_{2t}(2r)$; therefore $U_t(r) \circ U_t(r) \subseteq U$ and we remark that $U_t(r) \in \mathcal{U}_s$.

3.2. Remarks. (i) $\mathcal{U}_0$ is an uniformity on $X \times X$ and $\mathcal{U}_s \subseteq \mathcal{U}_0$, for every $s \geq 0$.

(ii) $\mathcal{U}$ is a pseudo-uniform structure on X (see [4]).

Using 2.8 and 2.9 we may prove:

3.3. Proposition. *Let $\mathcal{U}_d$ be the uniformity induced by pseudo metric of convergence in measure d; then $\mathcal{U}_0 = \mathcal{U}_d$.*

Proof. For every $U \in \mathcal{U}_0$ there exist $s > 0$ and $r > 0$ such that $U_s(r) \subseteq U$; obviously we may suppose that $r \leq s$. For every $(x, y) \in U_d(r)$, $d(x, y) < r$ and from 2.9 (i) we obtain $d_r(x, y) \leq d(x, y) < r$. Because $r \leq s$ we have $d_s(x, y) \leq d_r(x, y) < r$, hence $(x, y) \in U_s(r) \subseteq U$. Therefore $U_d(r) \subseteq U$, so that $U \in \mathcal{U}_d$. It follows that $\mathcal{U}_0 \subseteq \mathcal{U}_d$. For every $U \in \mathcal{U}_d$ there exists $r > 0$ such that $U_d(2r) \subseteq U$. If $(x, y) \in U_r(r)$ then $d_r(x, y) < r$ and from 2.8 we obtain $d(x, y) \leq d_r(x, y) + r < 2r$, hence $(x, y) \in U_d(2r) \subseteq U$. Therefore $U_r(r) \subseteq U$, hence $U \in \mathcal{U}_r \subseteq \mathcal{U}_0$.

3.4. Definition. *We say that $\mathcal{U}$ is the gradual uniformity of convergence in measure on X.*

Now, we introduce an index of convergence of sequences in X.

Let $\mathcal{X}$ be the set of all sequences on X; for every $\alpha \in \mathcal{X}$ and $x \in X$ let $\alpha \xrightarrow{s} x$ denote the situation in which the sequence α converges to x in semi-uniform space $(X, \mathcal{U}_s)$.

3.5. Theorem. *Let $\alpha \in \mathcal{X}$, $x \in X$ and $s \geq 0$; the following are equivalent:*

(i) $\alpha \xrightarrow{s} x$.

(ii) $d_t(\alpha, x) \longrightarrow 0$, *for every* $t > s$.

(iii) $\sup\limits_{a>0} \overline{\lim\limits_{n}}\, \mu^*(|\alpha(n) - x| > a) \leq s$.

Proof. (i) $\Rightarrow$ (ii) : If $\alpha \xrightarrow{s} x$ then, for every $t > s$ and $r > 0$, there exists $n_0 \in \mathbb{N}$ such that $d_t(\alpha(n), x) < r$, for any $n \geq n_0$, hence $\lim\limits_{n} d_t(\alpha(n), x) = 0$, for every $t > s$.

(ii) $\Rightarrow$ (iii) : If $s < \sup\limits_{a>0} \overline{\lim\limits_{n}}\, \mu^*(|\alpha(n) - x| > a)$ then there exist $t > s$, $a > 0$ and $(k_n) \subseteq \mathbb{N}$, $k_n \uparrow +\infty$ such that $\mu^*(|\alpha(k_n) - x| > a) > t$, for every $n \in \mathbb{N}$. We choose a number b such that $0 < b < \min\{1, a\}$; then $\mu^*(|\alpha(k_n) - x| > b) > t$. From 2.6 (i) it follows that $d_t(\alpha(k_n), x) > b$, for every $n \in \mathbb{N}$, so that $d_t(\alpha, x) \not\to 0$.

(iii) $\Rightarrow$ (i) : For every $t > s$ and $r > 0$, $\overline{\lim\limits_{n}}\, \mu^*(|\alpha(n) - x| > \frac{r}{2}) < t$, hence there exists $n_0 \in \mathbb{N}$ such that $\mu^*(|\alpha(n) - x| > \frac{r}{2}) < t$, for every $n \geq n_0$; so that $d_t(\alpha(n), x) \leq \frac{r}{2} < r$, for every $n \geq n_0$.

We obtain $(\alpha(n), x) \in U_t(r)$ for every $n \geq n_0$, hence $\alpha \xrightarrow{s} x$.

3.6. Remarks. (i) If $\alpha \xrightarrow{s} x$ then $\overline{\lim\limits_{n}}\, d(\alpha(n), x) \leq s$. Indeed, from 3.5, it follows that $d_t(\alpha, x) \longrightarrow 0$, for every $t > s$. But from 2.8, $d(\alpha, x) \leq \leq d_t(\alpha, x) + t$ hence, $\overline{\lim\limits_{n}}\, d(\alpha(n), x) \leq t$, for every $t > s$.

(ii) α converges in measure to x if and only if $d_s(\alpha, x) \to 0$, for every $s > 0$. Indeed, from 3.5, $d_s(\alpha, x) \longrightarrow 0$, for every $s > 0$ if and only if $\alpha \xrightarrow{0} x$; but $\mathcal{U}_0 = \mathcal{U}_d$ (see 3.3).

We say that $\alpha \in \mathcal{X}$ is s-*Cauchy* if α is a Cauchy sequence in the semi uniform space $(X, \mathcal{U}_s)$.

The proof of the following theorem is analogous to the proof of Theorem 3.5 and we omit it.

3.7. Theorem. *Let $\alpha \in \mathcal{X}$ and $s \geq 0$; the following conditions are equivalent:*

(i) *α is s Cauchy.*

(ii) *α is Cauchy sequence in the semi-metric space (X, d_t), for every $t > s$.*

(iii) $\sup\limits_{a>0} \overline{\lim\limits_{m,n}}\, \mu^*(|\alpha(m) - \alpha(n)| > a) \leq s$.

3.8. Definition. *For every $\alpha \in \mathcal{X}$ and $x \in X$ we define*

$$c(\alpha, x) = \sup_{a>0} \overline{\lim_{n\in\mathbb{N}}}\, \mu^*(|\alpha(n) - x| > a),$$
$$c(\alpha) = \sup_{a>0} \overline{\lim_{m,n}}\, \mu^*(|\alpha(m) - \alpha(n)| > a)$$

$(\overline{\lim\limits_{m,n}}\, a_{m,n} = \inf\limits_{p\in\mathbb{N}} \sup\limits_{m>n\geq p} a_{m,n}$, *for every* $(a_{m,n}) \subseteq \overline{\mathbb{R}}$).

We say that $c(\alpha, x)$ $(c(\alpha))$ is *the index of convergence in measure* of sequence α to x (*the index of Cauchyness in measure* of α).

3.9. Remarks. (i) If we denote $\mathcal{U}_\infty = \{X \times X\}$ then, for every $\alpha \in \mathcal{X}$ and $x \in X$, there exists $s \in [0, \infty]$ such that $\alpha \xrightarrow{s} x$ (or α is s–Cauchy). From 3.5 (respectively, 3.7) it follows that $c(\alpha, x) \leq s$ ($c(\alpha) \leq s$), hence $c(\alpha, x) = \inf\{s \geq 0 : \alpha \xrightarrow{s} x\}$ and $c(\alpha) = \inf\{s \geq 0 : \alpha \text{ is } s\text{–Cauchy}\}$.

(ii) $c(\alpha, x) = 0$ ($c(\alpha) = 0$) if and only if α converges in measure to x (α is Cauchy in measure).

(iii) If $\mathcal{M}_0$ is the σ–algebra of all μ^*–measurable subsets of T and $\mu = \mu^*|_{\mathcal{M}_0}$ is a probability on T, then $p(\alpha, x) = 1 - c(\alpha, x)$ is the sequential probabilistic convergence function on the space of all measurable functions on T relating to the t–norm $T_m : I \times I \longrightarrow I$, $T_m(a, b) = \max(a + b - 1, 0)$ (see [3; sec. 5 and 6]).

Now, for every $\alpha \in \mathcal{X}$, let $\mathcal{X}_\alpha$ be the set of all subsequences of α. The following two results are extensions of similar results in E–spaces (see [3; sec. 4 and 6]) and we omit their proofs.

3.10. Theorem. *For every $\alpha \in \mathcal{X}$ and $x \in X$ we have:*

(C_1) $c(\alpha, x) = 0$, *when* $\alpha \subseteq \{x\}$.

(C_2) $c(\alpha, x) = \sup\{\inf\{c(\gamma, x) : \gamma \in \mathcal{X}_\beta\} : \beta \in \mathcal{X}_\alpha\}$.

(C_3) *For every* $(\beta^n)_{n \in \mathbb{N}} \subseteq \mathcal{X}$ *there exists a strictly increasing sequence of natural numbers* $(k_n)_{n \in \mathbb{N}}$ *such that*

$$c(\delta, x) \leq c(\alpha, x) + \sup\{c(\beta^n, \alpha(n)) : n \in \mathbb{N}\},$$

where $\delta \in \mathcal{X}$, $\delta(n) = \beta^n(k_n)$, *for every* $n \in \mathbb{N}$.

3.11. Proposition. *For every* $\alpha \in \mathcal{X}$, $\beta \in \mathcal{X}_\alpha$ *and* $x, y \in X$ *we have:*

(i) $|c(\alpha, x) - c(\alpha, y)| \leq \mu^*(x \neq y)$.

(ii) $c(\alpha) \leq 2\,c(\alpha, x)$.

(iii) $c(\alpha, x) \leq c(\beta, x) + c(\alpha)$.

3.12. Remarks. (i) If $x = y$ a.e. $c(\alpha, x) = c(\alpha, y)$, for every $\alpha \in \mathcal{X}$ (see 3.11 (i)).

(ii) If α converges in measure to x then α is Cauchy in measure (see 3.11 (ii)).

(iii) If α is Cauchy in measure and it has a subsequence β which converges in measure to x, then α converges in measure to x (see 3.11 (iii)).

3.13. Definition. *We say that a sequence $\alpha \in \mathcal{X}$ is Cauchy in measure on $A \subseteq T$ if, for every $a > 0$,* $\lim\limits_{m,n} \mu^*(A \cap (|\alpha(m) - \alpha(n)| > a) = 0$.

For every $\alpha = (x_n) \in \mathcal{X}$ let us denote $C(\alpha) = \bigcap\limits_{a>0} \bigcap\limits_{n \in \mathbb{N}} \bigcup\limits_{p>q\geq n} (|x_p - x_q| < a)$ and $A(\alpha) = \{t \in T : (x_n(t))_{n\in\mathbb{N}} \text{ converges in } \mathbb{R}\}$. We define $x : T \longrightarrow \mathbb{R}$ letting $x(t) = \lim\limits_{n\to\infty} x_n(t)$ if $t \in A(\alpha)$ and $x(t) = 0$ otherwise. Then we obtain $A(\alpha) = \bigcap\limits_{a>0} \bigcup\limits_{n\in\mathbb{N}} \bigcap\limits_{m\geq n} (|x_m - x| < a)$.

3.14. Remarks. (i) $A(\alpha) \subseteq C(\alpha)$, for every $\alpha \in \mathcal{X}$. Indeed, $A(\alpha) = \bigcap\limits_{a>0} \bigcup\limits_{n\in\mathbb{N}} \bigcap\limits_{p>q\geq n} [(|x_p - x| < a) \cap (|x_q - x| < a)] \subseteq \bigcap\limits_{a>0} \bigcup\limits_{n\in\mathbb{N}} \bigcap\limits_{p>q\geq n} (|x_p - x_q| < 2a) \subseteq C(\alpha)$.

(ii) $C(\beta) \subseteq C(\alpha)$, for every $\beta \in \mathcal{X}_\alpha$.

(iii) If α is Cauchy in measure on A, then $A \dot{\subseteq} C(\alpha)$ (we say that $A \dot{\subseteq} B$ if $\mu^*(A - B) = 0$). Indeed, there exists $\beta \in \mathcal{X}_\alpha$ such that β converges to x a.e. on A; so $A \dot{\subseteq} A(\beta) \subseteq C(\beta) \subseteq C(\alpha)$. Therefore $C(\alpha)$ contains every set $A \subseteq T$ with α Cauchy in measure on A.

The following theorem gives a sufficient condition such that α to be Cauchy in measure on $C(\alpha)$.

3.15. Theorem. *If $\alpha \in \mathcal{X}$ satisfies the condition*

$$(*) \qquad \sup_{a>0} \inf_{n\in\mathbb{N}} \sup_{p>q\geq n} \mu^* \left[\bigcup_{i>j\geq n} (|x_i - x_j| < a) - (|x_p - x_q| < a) \right] = 0$$

then α is Cauchy in measure on $C(\alpha)$.

Proof. For every $a > 0$, $n \in \mathbb{N}$ and $p, q \in \mathbb{N}$, $C(\alpha) - (|x_p - x_q| < a) \subseteq \bigcup\limits_{i>j\geq n} (|x_i - x_j| < a) - (|x_p - x_q| < a)$ so that $\mu^*(C(\alpha) - (|x_p - x_q| < a)) \leq \mu^*(\bigcup\limits_{i>j\geq n} (|x_i - x_j| < a) - (|x_p - x_q| < a))$. Therefore $\overline{\lim\limits_{p,q}}\, \mu^*(C(\alpha) - (|x_p - x_q| < a)) \leq \sup\limits_{p>q\geq n} \mu^*[\bigcup\limits_{i>j\geq n} (|x_i - x_j| < a) - (|x_p - x_q| < a)]$.

The proof follows immediately by letting $n \to \infty$ and using $(*)$.

3.16. Remark. Let $\alpha \in \mathcal{X}$ be a sequence which satisfies $(*)$ of Theorem 3.15. Then, from 3.15 and 3.14 (iii), there exists $\beta \in \mathcal{X}_\alpha$ such that $A(\beta) \doteq C(\alpha)$. So, we obtain an extension of a well-known result of Riesz, because $C(\alpha) = T$ in the case where α is Cauchy in measure on T.

In the case of measurable sequences we obtain:

3.17. Theorem. *Let $\alpha \in \mathcal{X}$ be a sequence of μ^* measurable functions. Then the following conditions are equivalent:*

(i) α it satisfies $(*)$.

(ii) $c(\alpha) = \mu^*(T - C(\alpha))$.

(iii) *α is Cauchy in measure on $C(\alpha)$.*

Proof. If $\alpha = (x_n)$ is a sequence of measurable functions, then μ^* is a measure on the sets $\{(|x_p - x_q| \geq a) : p, q \in \mathbb{N}\}$. Then $(*)$ is equivalent with $\overline{\lim\limits_{p,q}}\, \mu^*(|x_p - x_q| \geq a) = \sup\limits_{n \in \mathbb{N}} \mu^*(\bigcap\limits_{i > j \geq n} (|x_i - x_j| \geq a))$, for every $a > 0$; letting $a \longrightarrow 0$ and using the continuity of measure μ^* we obtain that (i) is equivalent with (ii).

From 3.15 we obtain (i) $\Rightarrow$ (iii). Conversely, if α is Cauchy in measure on $C(\alpha)$, then $c(\alpha) = \sup\limits_{a>0} \overline{\lim\limits_{p,q}}\, \mu^*(|x_p - x_q| \geq a) \leq \lim\limits_{a \to 0} [\overline{\lim\limits_{p,q}}\, \mu^*(C(\alpha) \cap$ $\cap\, (|x_p - x_q| \geq a)) + \mu^*(T - C(\alpha))]$. Therefore $c(\alpha) \leq \mu^*(T - C(\alpha)) =$ $= \lim\limits_{a \to 0} \mu^*(\underline{\lim\limits_{p,q}}\, |x_p - x_q| \geq a)) \leq \lim\limits_{a \to 0} \overline{\lim\limits_{p,q}}\, \mu^*(|x_p - x_q| \geq a) = c(\alpha)$.

3.18. Corollary. *If α is a sequence of μ^* measurable functions which satisfies $(*)$ then there exists $\beta \in \mathcal{X}_\alpha$ such that $\mu^*(T - A(\beta)) = c(\alpha)$.*

Proof. See Remark 3.16 and Theorem 3.17.

We remark that 3.18 generalizes Theorem 6.6 from [3].

3.19. Theorem. *Let μ^* be a finite outer measure on T and $\alpha \in \mathcal{X}$ be a sequence of μ^*-measurable functions on T which satisfies $(*)$. Then there exists $x \in X$ such that $c(\alpha, x) \leq 2c(\alpha)$.*

Proof. From Corollary 3.18 there exist $\beta \in \mathcal{X}_\alpha$ and $x \in X$ such that $c(\alpha) = \mu^*(T - A(\beta)) = \mu^*(\bigcup\limits_{a>0} \bigcap\limits_{n \in \mathbb{N}} \bigcup\limits_{m \geq n} (|\beta(m) - x| \geq a)) =$ $= \sup\limits_{a>0} \mu^*(\overline{\lim\limits_{n}}(|\beta(n) - x| \geq a)) \geq \sup\limits_{a>0} \overline{\lim\limits_{n}}\, \mu^*(|\beta(n) - x| \geq a) = c(\beta, x)$. From 3.11 (iii) it follows that $c(\alpha, x) \leq 2c(\alpha)$.

3.20. Remark. A sufficient condition for $(*)$ is that the series

$$\sum_{j=1}^{\infty} \sum_{i=j}^{\infty} \mu^*[(|x_i - x_j| < a) \cap (|x_p - x_q| \geq a)]$$

to be convergent uniformly for $p, q \in \mathbb{N}$ and $a > 0$.

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