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AN EXAMPLE IN THE AXIOMATIC POTENTIAL THEORY: THE SEMI–DYNAMICAL SYSTEMS

BY

EUGEN POPA and LILIANA POPA*

1. Introduction. The semi–dynamical systems were introduced in the framework of the axiomatic potential theory by H m i s s i [5] and, in the more general form used in this note, by B u c u r–B e z z a r g a [2], [3].

The purpose of this note is to continue the study of semi–dynamical systems from the viewpoint of axiomatic potential theory. Mainly, two new notions for the theory of semi–dynamical systems are defined and studied. The first one is that of a time change. It describes essentially the situation when the excessive functions for one system are also excessives for another one (see th. 3.3 and 3.4.)

The second one is that of boundaries. Namely, the entrance and exit boundaries for semi–dynamical systems are defined. The relation with minimal functions is considered (see th. 4.11.). The harmonic functions and the potentials appear now in a natural way.

The last two paragraphs present a method to obtain new semi–dynamical systems (namely the tensor product) and a list of simple examples of semi–dynamical systems.

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2. Preliminaries. We recall the following notions and results from [2]. Let $(X, \mathcal{X})$ be a separated and countably generated measurable space

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[4]. $\omega \in X$ is a fixed element; we denote by $X_0 = X \setminus \{\omega\}$.

Definition 2.1. *A semi-dynamical system is a measurable map*

$$\Phi : [0, +\infty) \times X \to X$$

which has the following properties:

1. $\Phi(0, x) = x,\ \forall x \in X$.
2. $\Phi(s + t, x) = \Phi(s, \Phi(t, x)),\ \forall s, t \in [0, +\infty),\ \forall x \in X$.
3. $\Phi(t, x) = \omega$ *and* $t' \geq t \Rightarrow \Phi(t', x) = \omega$.
4. $\Phi(t, x) = \Phi(t, y),\ \forall t > 0 \Rightarrow x = y$.

Remark. If $\Phi(t_0, x) = \Phi(t_0, y)$, then:

$$\Phi(s + t_0, x) = \Phi(s, \Phi(t_0, x)) = \Phi(s, \Phi(t_0, y)) = \Phi(s + t_0, y)$$

hence $\Phi(t, x) = \Phi(t, y),\ \forall t \geq t_0$.

Definition 2.2. *We call the life time (of the system Φ) the map* [4] $\zeta : X \to [0, +\infty]$, *defined by*

$$\zeta(x) = \inf\{t \geq 0 | \Phi(t, x) = \omega\}$$

Eliminating those points $x \in X_0$ for which $\zeta(x) = 0$, we may and do suppose that $\zeta(x) > 0,\ \forall x \in X_0$.

Remark. If $\zeta(x) < +\infty$, then $\Phi(\zeta(x), x) = \omega$. Indeed, if

$$x_0 = \Phi(\zeta(x), x) \in X_0$$

then

$$\forall t > 0 : \quad \Phi(t, x_0) = \Phi(t, \Phi(\zeta(x), x)) = \Phi(t + \zeta(x), x) = \omega$$

hence $\zeta(x_0) = 0$, contradiction. So, for each $x \in X_0$ we have either

$$\zeta(x) = +\infty \text{ or } \Phi(\zeta(x), x) = \omega$$

Definition 2.3. *For each $x \in X_0$, we denote by*

$$\Gamma_x = \{\Phi(t, x) | t \in [0, \zeta(x))\}$$

(and call the trajectory starting from the point x).

Definition 2.4. *The semi-dynamical system Φ is called transient if there exists $A_n \in \mathcal{X}$ such that $X_0 = \bigcup_{n \in \mathbb{N}} A_n$ and:*

$$\forall x \in X_0 : \ m(\{t \in [0, +\infty) | \Phi(t, x) \in A_n\}) < +\infty$$

(m denoting the Lebesgue measure).

Remark. There exist semi-dynamical systems, which are not transient, cf. ex. 6(e).

Definition 2.5. *We denote $x \leq_\Phi y$ if there exists $t \in [0, +\infty)$ such that $y = \Phi(t, x)$.*

If the sistem under consideration is transient, then $x \leq_\Phi y$ is an order relation.

Definition 2.6. *The fine topology, denoted by τ_Φ consists of the sets $D \subseteq X_0$ for which:*

$$\forall x \in D\ \exists \tau \in (0, \zeta(x)) \ \textit{ such that } \ \Phi(t, x) \in D,\ \forall t \in (0, \tau)$$

Remark. With every semi-dynamical system Φ one associates the semi-group $\mathcal{P} = (P_t)_{t>0}$ (markovian, of kernels on $(X, \mathcal{X})$), defined by $P_t f(x) = f(\Phi(t, x))$. The same definition gives a *sub*-markovian semi-group on X_0: if f is defined, numerical, positive on X_0, then we extend it to X by $f(\omega) = 0$.

The resolvent $\mathcal{V} = (V_\alpha)_{\alpha>0}$ considered in [2] is that associated with this semi-group:

$$V_\alpha f(x) = \int_0^\infty e^{-\alpha t} P_t f(x) dt$$

Proposition 2.7. (a) *Φ is transient if and only if V_0 is a proper kernel.*

(b) *The excessive functions associated to the semi-group $\mathcal{P}$ (or with the resolvent $\mathcal{V}$) are exactly those functions $f : X_0 \to [0, +\infty]$ which are: measurable, decreasing (with respect to the associated order relation $\leq_\Phi$) and right continuous (i.e. $\lim_{t \searrow 0} f(\Phi(t, x)) = f(x), \forall x \in X_0$ or equivalently, continuous in the fine topology).*

We denote by $\mathcal{E}_\Phi$ the set of excessive functions with respect to the semi-group $\mathcal{P}$ on X_0.

Remark. H m i s s i [5] considers a locally compact space X, with countable base. Φ is supposed to be continuous and with properties 1) and 2). In this case, the associated semi-group is Feller, but not strongly Feller.

Proposition 2.8. *The fine topology τ_Φ is separated.*

If the system is transient, then every trajectory Γ_x is both open and closed in the fine topology.

Definition 2.9. *The natural topology, denoted by τ_Φ^0 is formed of the sets $D \subseteq X_0$ such that:*

$$\forall x \in X_0 \quad and \quad t_0 \in [0, \zeta(x)) \quad for\ which \quad \Phi(t_0, x) \in D, \ \exists \varepsilon > 0$$

$$such\ that: \quad \forall t \in (t_0 - \varepsilon, t_0 + \varepsilon) \cap [0, \zeta(x)) \Rightarrow \Phi(t, x) \in D$$

Remark. For each $x \in X_0$, the map $t \mapsto \Phi(t, x)$ is continuous on $[0, \zeta(x))$, in the natural topology.

3. Time change in the semi-dynamical systems.

Definition 3.1. *Let Φ be a semi-dynamical system on $(X, \mathcal{X})$. We define a time change (for Φ) as any measurable map*

$$\tau : [0, +\infty) \times X \to [0, +\infty)$$

such that:

1. $\forall x \in X_0 : \ \tau(t, x) = 0 \iff t = 0$
2. $\tau(s, \Phi(\tau(t, x), x)) + \tau(t, x) = \tau(s + t, x)$
3. $\forall x \in X_0 : \ \lim\limits_{t \searrow 0} \tau(t, x) = 0$

Remark. (i) If the other two properties hold, then the first one is equivalent to $\tau \not\equiv 0$. Indeed, let us consider:

$$t_0 = \sup\{t \geq 0 \mid \tau(t, x) = 0\}$$

If $t_0 \in (0, +\infty)$, then taking $t = s = \frac{2t_0}{3}$, we get:

$$\tau(\frac{4t_0}{3}, x) = \tau(\frac{2t_0}{3}, \Phi(\tau(\frac{2t_0}{3}, x), x)) + \tau(\frac{2t_0}{3}, x) = 0$$

which is a contradiction. Hence either $\tau(t, x) = 0, \ \forall t \in [0, +\infty)$, or [4] $\tau(t, x) = 0 \iff t = 0$.

(ii) It results that the map $t \mapsto \tau(t, x)$ is strictly increasing on $[0, \zeta(x))$. Indeed, if $t' > t$ then:

$$\tau(t', x) = \tau(t, x) + \tau(s, \Phi(\tau(t, x), x)) > \tau(t, x)$$

(iii) For each $t_0 \in [0, +\infty)$ we have $\lim\limits_{t \nearrow t_0} \tau(t, x) = \tau(t_0, x)$. Indeed, let us denote by $x_0 = \Phi(\tau(t_0, x), x)$ (when $\tau(t_0) < \zeta(x)$). Then we have

$$\tau(t + t_0, x) = \tau(t_0, x) + \tau(t, x_0)$$

hence the conclusion.

Example 3.2. Searching for a time change of the form $\tau(t, x) = f(x).t$, one finds out that the function f must be constant on every trajectory.

Theorem 3.3. *Let τ be a time change for the semi-dynamical system Φ. Then $\Psi(t, x) := \Phi(\tau(t, x), x)$ is a semi-dynamical system on $(X, \mathcal{X})$, for which $\mathcal{E}_\Phi \subseteq \mathcal{E}_\Psi$ and $\zeta_1(x) < +\infty \Longrightarrow \zeta(x) < +\infty$, (where ζ_1 denotes the life time of the system Ψ).*

Proof. Ψ is a measurable map and

$$\Psi(0, x) = \Phi(\tau(0, x), x) = \Phi(0, x) = x$$

Next:

$$\Psi(s + t, x) = \Phi(\tau(s + t, x), x) = \Phi(\tau(t, x) + \tau(s, \Phi(\tau(t, x), x)), x) =$$

$$= \Phi(\tau(t, x), \Phi(\tau(s, \Phi(\tau(t, x), x)))) = \Psi(t, \Psi(s, x))$$

If $\Psi(t, x) = \omega$ and $t' > t$, then $\tau(t', x) \geq \tau(t, x)$ hence $\Phi(\tau(t, x), x) = \omega \Rightarrow \Psi(t', x) = \Phi(\tau(t', x), x) = \omega$. If $\Psi(t, x) = \Psi(t, y)$, $\forall t > 0$, then $\Phi(\tau(t, x), x) = \Phi(\tau(t, y), y)$. Let us denote by $s_n = \tau(\frac{1}{n}, x)$, $t_n = \tau(\frac{1}{n}, y)$. Hence $s_n, t_n \to 0$. If we write $s_n = s_{n+1} + \alpha_n$ and $t_n = t_{n+1} + \beta_n$, then:

$$\Phi(s_n, x) = \Phi(s_{n+1} + \alpha_n, x) = \Phi(\alpha_n, \Phi(s_{n+1}, x)) = \Phi(\alpha_n, \Phi(t_{n+1}, y)) =$$

$$= \Phi(t_{n+1} + \alpha_n, y) = \Phi(t_{n+1} + \beta_n, y)$$

From the injectivity [2, prop.1] we obtain that $\alpha_n = \beta_n$. Hence, if for example $s_1 = t_1 + a$, then $s_n = t_n + a, \forall n$. It results that $a = 0$, hence $s_n = t_n, \forall n$ and $\Phi(s_n, x) = \Phi(s_n, y)$. Using the remark 2.1., it results $\Phi(s, x) = \Phi(s, y), \forall s > 0$, and finally $x = y$.

If $\zeta_1(x) < +\infty$, then:

$$\omega = \Psi(\zeta_1(x), x) = \Phi(\tau(\zeta_1(x), x), x)$$

which proves that $\tau(\zeta_1(x), x) = \zeta(x)$. Particularly $\forall x \in X_0 : \zeta_1(x) > 0$.

Let be $f \in \mathcal{E}_\Phi$. If $x \leq_\Psi y$, then there exists $t \geq 0$ such that $y = \Psi(t, x) = \Phi(\tau(t, x), x)$. Hence $x \leq_\Phi y$ which implies $f(x) \geq f(y)$. Finally since $\lim\limits_{t \searrow 0} \tau(t, x) = 0$, it follows that:

$$\lim_{t \searrow 0} f(\Psi(t, x)) = \lim_{t \searrow 0} f(\Phi(\tau(t, x), x)) = f(x)$$

In fact, $\zeta_1(x)$ is uniquely determined by:

$$\lim_{t \nearrow \zeta_1(x)} \tau(t, x) \leq \zeta(x) \leq \tau(\zeta_1(x), x)$$

Indeed, if not, there would exists a $t_0 < \zeta_1(x)$ for which $\zeta(x) \leq \tau(t_0, x)$. Hence $\omega = \Phi(\tau(t_0, x), x) = \Psi(t_0, x)$ and $t_0 \geq \zeta_1(x)$, which is a contradiction.

Theorem 3.4. *Let $(X, \mathcal{X})$ be a lusin space; and Φ, Ψ be two semi-dynamical systems, such that: $\mathcal{E}_\Phi \subseteq \mathcal{E}_\Psi$ and $\zeta_1(x) < +\infty \Longrightarrow \zeta(x) < +\infty$. Then, there exists a time change τ for Φ, such that $\Psi(t, x) = \Phi(\tau(t, x), x)$.*

Proof. For $x \in X_0$ and $t < \zeta_1(x)$, let us denote $y = \Psi(t, x)$. Hence $y \in X_0$ and $x \leq_\Psi y$. The characteristic function $\chi_{X_0 \setminus \Gamma_x}$ (the trajectories being considered with respect to the system Φ), is measurable, decreasing and right continuous: it follows that $\chi_{X_0 \setminus \Gamma_x} \in \mathcal{E}_\Phi$. Hence $\chi_{X_0 \setminus \Gamma_x} \in \mathcal{E}_\Psi$, which implies that $y \in \Gamma_x$. There exists then a unique $\tau(t, x) \in [0, +\infty)$ such that $y = \Phi(\tau(t, x), x)$. From the unicity, we get $\tau(0, x) = 0$. For $t + s < \zeta_1(x)$ we have:

$$\Phi(\tau(t + s, x), x) = \Psi(t + s, x) = \Psi(s, \Psi(t, x)) =$$

$$= \Phi(\tau(s, \Psi(t, x)), \Phi(\tau(t, x), x)) = \Phi(\tau(t, x) + \tau(s, \Phi(\tau(t, x), x), x)$$

and again from unicity we get the formula

$$\tau(t + s, x) = \tau(t, x) + \tau(s, \Phi(\tau(t, x), x))$$

In order to prove the measurability of τ, let us denote by

$$D_\Phi = \{(t, x) | \, t < \zeta(x)\} \; ; \; \mathrm{Im}_\Phi = \{(y, x) | y \in \Gamma_x\}$$

The map $\Phi_1 : [0, +\infty) \times X \to X \times X$ defined by $\Phi_1(t, x) = (\Phi(t, x), x)$ is injective on the set D_Φ:

$$(\Phi(t, x), x) = (\Phi(t', y), y) \Longleftrightarrow \Phi(t, x) = \Phi(t', y) \text{ and } x = y \Longleftrightarrow$$

$$\Longleftrightarrow x = y \text{ and } t = t'$$

Using L u s i n's theorem [4], τ appears as the composition of measurable functions:

$$D_\Psi \xrightarrow{\Psi_1} Im_\Psi \longrightarrow Im_\Phi \xrightarrow{\Phi_1^{-1}} [0, +\infty) \times X \xrightarrow{pr} [0, +\infty)$$

The function τ is extended now from the set D_Ψ to the whole $[0, +\infty) \times X$ by:

$$\tau(t, x) = \zeta(x) \text{ if } \zeta_1(x) < +\infty \text{ and } t \geq \zeta_1(x)$$

$$\tau(t, \omega) = 0, \forall t \in [0, +\infty)$$

The definition is correct, and the measurability is clear. The property 2 from Def. 3.1 is also valid: it holds for $\zeta_1(x) < +\infty$ if $t < \zeta_1(x)$ and $s < \zeta_1(y)$, where $y = \Phi(\tau(t, x), x)$; now, for $\zeta_1(x) = +\infty$, we have also $\zeta_1(y) = +\infty, \forall s \in [0, +\infty)$. Finally, for $\zeta_1(x) < +\infty$ and $t \geq \zeta_1(x)$, then $\tau(t, x) = \zeta(x)$ hence $\Phi(\tau(t, x), x) = \omega$ and the relation 2 becomes $\tau(t + s, x) = \tau(t, x) (= \zeta(x))$, $\forall s \in [0, +\infty)$. The case $x = \omega$ is trivial.

4. Boundaries for the semi–dynamical systems.

Proposition 4.1. *By: $x \sim_f y$ if $\Gamma_x \cap \Gamma_y \neq \emptyset$ one defines an equivalence relation on X_0.*

Proof. The relation defined above means that, from a point on, the trajectories coincide:

$$a \in \Gamma_x \cap \Gamma_y \Longleftrightarrow a = \Phi(t, x) = \Phi(s, y)$$

hence $\Phi(\tau, a) = \Phi(\tau + t, x) = \Phi(\tau + s, y)$. The reflexivity and the symmetry being obvious, let us prove the transitivity. Let us suppose that $\Gamma_x \cap \Gamma_y \neq \emptyset$ and $\Gamma_y \cap \Gamma_z \neq \emptyset$. There exist then $t, s, s', u \in [0, +\infty)$ such that $\Phi(t, x) = \Phi(s, y)$ and $\Phi(s', y) = \Phi(u, z)$. For $v = \max(s, s')$ we have $\Phi(v, y) \in \Gamma_x \cap \Gamma_z$.

Definition 4.2. *The equivalence classes for the relation $\sim_f$ will be called final points (of the semi-dynamical system Φ). We shall denote by $\hat{x}$ the equivalence class of the point $x \in X_0$ with respect to the relation $\sim_f$.*

The set of all final points is called the exit boundary of the system Φ and will be denoted by $\mathcal{O}$.

We have: $x \leq_\Phi y \Longrightarrow x \sim_f y$, but the converse is not true (cf.ex. 6(b)).

Definition 4.3. *Let $\mathcal{T}$ be the set of all parts of X, which are totally ordered with respect to the relation $\leq_\Phi$. Clearly, each Γ_x belongs to $\mathcal{T}$. Using Zorn's lemma, we conclude the existence of maximal elements in $\mathcal{T}$.*

These elements will be called *initial points* for the system Φ. The set of the initial points will be called the *entrance boundary* of the system Φ and will be denoted by $\mathcal{I}$.

Proposition 4.4. *There exists a canonical, surjective, map.*

$$\mathcal{I} \to \mathcal{O}.$$

Proof. Let be $\alpha \in \mathcal{I}$. We define $\varphi(\alpha) = \hat{x}$, where $x \in \alpha$. The definition is correct, since $x, y \in \alpha$ means that x and y are comparable for the order relation $\leq_\Phi$, for example $x \leq_\Phi y$. In this case, we have $\Gamma_y \subseteq \Gamma_x$, hence $\hat{x} = \hat{y}$.

We prove now that φ defined above is a surjective map: if $\beta \in \mathcal{O}$, and $x \in \beta$, then, Γ_x being a totally ordered part, it follows that there exists $\alpha \in \mathcal{I}$ such that $\Gamma_x \subseteq \alpha$. It results that $\varphi(\alpha) = \beta$.

Definition 4.5. *Any measurable map $h : X_0 \to [0, +\infty)$, such that:*

$$h(\Phi(t, x)) = h(x), \ \forall t \geq 0, x \in X_0$$

(that is, h is constant on every trajectory) is called harmonic function.

The harmonic functions belong to $\mathcal{E}_\Phi$ and coincide with the *invariant* functions with respect to the semi-group $\mathcal{P}$.

Within the classical (semi-)dynamical systems, the harmonic functions are exactly the prime integrals of the vector field.

Definition 4.6. *We extend each function $f \in \mathcal{E}_\Phi$ to $\mathcal{I}$ and to $\mathcal{O}$ through:*

$$\alpha \in \mathcal{I} \Rightarrow \check{f}(\alpha) = \sup_{x \in \alpha} f(x)$$

and:

$$\hat{x} \in \mathcal{O} \Rightarrow \hat{f}(\hat{x}) = \inf_{y \in \hat{x}} f(y)$$

We call a potential every function $p \in \mathcal{E}_\Phi$ for which:

$$\forall x \in X_0 \Rightarrow p(\hat{x}) = 0$$

Proposition 4.7. *For each function $f : \mathcal{O} \to [0, +\infty)$ we denote by $H_f : X_0 \to [0, +\infty)$ the function defined as: $H_f(x) = f(\hat{x})$. Then:*

a) *H_f is a harmonic function.*

b) *Let h be a harmonic function. Let us denote $f = \hat{h}|_{\mathcal{O}}$. Then $h = H_f$.*

c) *For each $s \in \mathcal{E}_\Phi$ let us denote: $g = \hat{s}|_{\mathcal{O}}$ and $h = H_g$. Then $p = s - h$ is a potential.*

Proof. a) If $x \in X_0$, then $\Gamma_x \subseteq \hat{x}$, which proves that the function H_f is constant on every trajectory.

b) Let be $x \in X_0$, hence

$$H_f(x) = f(\hat{x}) = \hat{h}(\hat{x}) = \inf_{y \sim_f x} h(y)$$

Since for any $y \sim_f x$ there exist $s, t \in [0, +\infty)$ with $\Phi(t, x) = \Phi(s, y)$ it results that we have

$$h(y) = h(\Phi(s, y)) = h(\Phi(t, x)) = h(x)$$

c) For each $x \in X_0$ we have $p(\hat{x}) = s(\hat{x}) - h(\hat{x}) = 0$.

Definition 4.8. a) *$s \in \mathcal{E}_\Phi$ is called minimal when:*

$$s_1, s_2 \in \mathcal{E}_\Phi \ \text{ and } \ s = s_1 + s_2 \Longrightarrow \exists \alpha \in [0, 1] \ \text{ such that } \ s_1 = \alpha . s$$

b) *For each $\beta \in \mathcal{O}$ we denote by $h_\beta : X_0 \to [0, +\infty)$ the function defined as:*

$$h_\beta(x) = \begin{cases} 1, & x \in \beta \\ 0, & x \notin \beta \end{cases}$$

c) *For each $x_0 \in X_0$ we denote by $p_{x_0} : X_0 \to [0, +\infty)$ the function defined as:*

$$p_{x_0}(x) = \begin{cases} 1, & x \leq_\Phi x_0, x \neq x_0 \\ 0, & \text{elsewhere} \end{cases}$$

Definition 4.9. *$x_0 \in X_0$ is called a ramification point if there exist $x \neq y$ and $t, s > 0$ such that $\Phi(t, x) = x_0 = \Phi(s, y)$ while $\Phi(t', x) \neq \Phi(s', y)$, $\forall t' \in [0, t), s' \in [0, s)$.*

Proposition 4.10. a) *h_β is a harmonic, minimal function.*

b) *If x_0 is not a ramification point, then p_{x_0} is a minimal potential.*

Proof. a) Let be $x \in X_0$ and $t \geq 0$. Since $x \sim_f \Phi(t,x)$, we obtain that: for $x \in \beta$, we have $\Phi(t,x) \in \beta$, hence $h_\beta(\Phi(t,x)) = 1 = h_\beta(x)$; while for $x \notin \beta$, then $\Phi(t,x) \notin \beta$, which ensures that $h_\beta(\Phi(t,x)) = 0 = h_\beta(x)$. Hence h_β is a harmonic function.

Let be $s_1, s_2 \in \mathcal{E}_\Phi$ such that $h_\beta = s_1 + s_2$. For $x \notin \beta$, we have $s_1(x) = s_2(x) = 0$. If $x \in \beta$, then for any $t \geq 0$ we obtain:

$$1 = h_\beta(\Phi(t,x)) = s_1(\Phi(t,x)) + s_2(\Phi(t,x)) = s_1(x) + s_2(x) = h_\beta(x)$$

Since $s_1(\Phi(t,x)) \leq s_1(x)$ and $s_2(\Phi(t,x)) \leq s_2(x)$, it follows the existence of $\alpha_x \in [0,1]$ (depending on x), such that $s_1(x) = \alpha_x$, $s_2(x) = 1 - \alpha_x$. Let us consider then $y \in \beta$. Since $x \sim_f y$, it follows the existence of $s, t \geq 0$, such that $\Phi(t,x) = \Phi(s,y)$. Then:

$$\alpha_y = s_1(y) = s_1(\Phi(s,y)) = s_1(\Phi(t,x)) = s_1(x) = \alpha_x$$

hence $s_1 = \alpha h_\beta$.

b) p_{x_0} is decreasing: indeed, if $p_{x_0}(x) = 0$ and $x \leq_\Phi y$, then $y \leq_\Phi x_0$ cannot hold, since the transitivity would imply that $x \leq_\Phi x_0$ and $p_{x_0}(x) = 1$. Hence $p_{x_0}(y) = 0$. p_{x_0} is clearly right continuous in every point. Finally, p_{x_0} is a potential, since clearly $\tilde{p}_{x_0}(\beta) = 0$ for $\beta \neq \hat{x}_0$, while $\tilde{p}_{x_0}(\hat{x}_0) = 0$, since for any $t \geq 0$ we have $p_{x_0}(\Phi(t,x_0)) = 0$.

Let be $s_1, s_2 \in \mathcal{E}_\Phi$, such that $p_{x_0} = s_1 + s_2$. For $x \not\leq_\Phi x_0$, we have $s_1(x) = s_2(x) = 0$. Let be $x, y \leq_\Phi x_0$. Since x_0 is not a ramification point, there exist t, s such that $\Phi(t,x) = \Phi(s,y) = a \leq_\Phi x_{x_0}$, $a \neq x_0$. Hence:

$$1 = p_{x_0}(x) = s_1(x) + s_2(x) \geq s_1(a) + s_2(a) = p_{x_0}(a) = 1$$

$$1 = p_{x_0}(y) = s_1(y) + s_2(y) \geq s_1(a) + s_2(a) = p_{x_0}(a) = 1$$

and it follows that $s_1(x) = s_1(a)$, hence the existence of an $\alpha \in [0,1]$ such that $s_1(x) = \alpha, \forall x \leq_\Phi x_0, x \neq x_0$.

Theorem 4.11. a) *Let $s \in \mathcal{E}_\Phi$ be a minimal function. Then s is either harmonic, or a potential.*

b) *Let h be a harmonic, minimal function. There exists then $\beta \in \mathcal{O}$ and $\lambda \geq 0$ such that $h = \lambda \cdot h_\beta$.*

c) *Let p be a minimal potential. There exists then $x_0 \in X_0$, which is not a ramification point and $\lambda \geq 0$ such that $p = \lambda \cdot p_{x_0}$.*

Proof. a) Let $s = h + p$, with h a harmonic function and p a potential. The minimality of s implies the existence of an $\alpha \in [0,1]$ such that: $h = \alpha s$;

$p = (1-\alpha)s$. If $\alpha = 1$, then $s = h$ and s is a harmonic function. If $\alpha \in [0,1)$, then $h = \dfrac{\alpha}{1-\alpha} p$, hence $\tilde{h}(\hat{x}) = 0, \forall x$. We obtain $h \equiv 0$ and $s = p$ results a potential.

b) Let h be a minimal, harmonic function. If there exists $\beta_1 \neq \beta_2 \in \mathcal{O}$, such that $h(\beta_1) \neq 0$ and $h(\beta_2) \neq 0$, then we can define f_1 and f_2 by:

$$f_1(\beta) = \begin{cases} h(\beta_1), & \text{if } \beta = \beta_1 \\ 0, & \text{if not} \end{cases}$$

and

$$f_2(\beta) = \begin{cases} 0, & \text{if } \beta = \beta_1 \\ h(\beta), & \text{if not} \end{cases}$$

Since $h = H_{f_1} + H_{f_2}$, while H_{f_1} and h are not proportional, one contradicts the minimality. If $h(\beta) = 0, \forall \beta \in \mathcal{O}$, we would obtain $h \equiv 0$. Hence, there exists a unique $\beta \in \mathcal{O}$, for which $h(\beta) \neq 0$. So, we obtain $h = \lambda h_\beta$, with $\lambda = h(\beta)$.

c) Let p be a minimal potential. Supposing $p \not\equiv 0$, let x_0 be such that $p(x_0) \neq 0$. If $x_0 \leq_\Phi x$, then $p(x) = 0$. Indeed, if not, let us denote $\lambda = p(x) \neq 0$; we can write $p = \lambda p_x + (p - \lambda p_x)$, and this would contradict the minimality of p. Let then $\mu = p(x_0)$. Again $p = \mu p_{x_0} + (p - \mu p_{x_0})$, toghether with the minimality of p implies that $p = \mu p_{x_0}$.

If x_0 would be a ramification point, then the set of all points from $\mathcal{I}$ which contain x_0 could be divided into two non–void, disjoint parts, denoted by $\mathcal{I}_x$ resp. $\mathcal{I}_y$, one being formed of the points which contain x (the notation from def.9). Defining f, resp. g as the characteristic functions of the sets $[z \in \mathcal{I}_x, z <_\Phi x_0]$, resp. $[z \in \mathcal{I}_y, z <_\Phi x_0]$, we would obtain $p = f + g$, contradicting once again the minimality.

Remark. $\mathcal{O}$ may thus be considered as the Martin boundary of the system, while the set of points which are not fine accumulation points of ramification points, may be regarded as the canonical space of representation [1], for the system.

5. Tensor product of semi-dynamical systems.

Definition 5.1. *Let Φ', Φ'' be two semi-dynamical systems on $(X', \mathcal{X}')$ resp. $(X'', \mathcal{X}'')$. We denote by: $X = (X_0' \times X_0'') \cup \{\omega\}$ and $\omega = (\omega', \omega'')$ (hence $X_0 = X_0' \times X_0''$). We define the tensor product of Φ' and Φ'' by $\Phi : [0, +\infty) \times X \to X$ where $\Phi(t, (x', x'')) = (\Phi'(t, x'), \Phi''(t, x''))$.*

Theorem 5.2. (a) *Φ is a semi dynamical system on $(X, \mathcal{X})$.*

(b) *We have:* $\zeta(x', x'') = \min(\zeta'(x'), \zeta''(x''))$.

(c) *If at least one of the factors is transient, then the tensor product is also transient.*

(d) *The product topologies* $\tau'_{\Phi'} \times \tau''_{\Phi''}$, *resp.* $\tau'^{0}_{\Phi'} \times \tau''^{0}_{\Phi''}$, *are coarser than the topology* τ_Φ, *resp.* τ^0_Φ, *on* X_0.

Proof. (a) The verification is immediate.

(b) Indeed: $\Phi(t,(x,y)) = \omega \iff \Phi'(t,x) = \omega$ and $\Phi''(t,y) = \omega$ hence the conclusion.

(c) It is enough to consider the following sequence of measurable sets: $(A_n \times X'')_n$ in order to get the conclusion:

$$\forall (x,y): \; m(\{t \in [0,+\infty) | \Phi(t,(x,y)) \in A_n \times X''\}) < +\infty$$

(d) If D' and D'' are open sets, then, for each $(x,y) \in D' \times D''$, $\exists \varepsilon \in (0, \zeta(x,y))$ such that $\Phi'(t,x) \in D'$ and $\Phi''(t,y) \in D''$, $\forall t \in [0,\varepsilon)$.

As for the natural topologies, let D' and D'' be naturally open sets.

It follows that, for each $x = (x', x'')$ and $t_0 \in [0, \zeta(x', x''))$, with $\Phi(t_0, x) \in D' \times D''$ we have $t_0 < \min(\zeta'(x'), \zeta''(x''))$ and $\Phi'(t_0, x') \in D'$, $\Phi''(t_0, x'') \in D''$. Hence, there exists $\varepsilon > 0$ such that: $\forall t \in (t_0 - \varepsilon, t_0 + \varepsilon) \cap \cap [0, \zeta(x))$ we have $\Phi'(t, x') \in D'$ and $\Phi''(t, x'') \in D''$. Hence $\Phi(t,x) \in D' \times D''$, so that $D' \times D''$ is naturally open with respect to Φ.

Remark. (i) The order relation associated with the tensor product **is not** the product of the order relations on the factors. $(x', x'') \leq_\Phi (y', y'')$ means that there exists $t \in [0,+\infty)$ such that $y' = \Phi'(t,x')$ and $y'' = \Phi''(t,x'')$. Hence $(x', x'') \leq_\Phi (y', y'') \Rightarrow (x' \leq_{\Phi'} y'$ and $x'' \leq_{\Phi''} y'')$, but the converse is not true.

We may say that one does the product of the trajectories, with the same speed.

(ii) It is possible that $\mathcal{E}_{\Phi'}$ and $\mathcal{E}_{\Phi''}$ be standard H–cones [1], while $\mathcal{E}_{\Phi' \times \Phi''}$ has no more this property (cf. ex. Σ 6(d)).

6. Examples.

We are going to illustrate the preceding notions on some simple examples.

(a) $X = \mathbf{R} \cup \{\omega\}$ and $\Phi(t,x) = t + x$ is a semi–dynamical system for which $\mathcal{E}_\Phi$ is a *standard H cone* [1]. The entrance and the exit boundary contain each one a point.

(b) Let us denote:

$$X = \{(-x,x) | x > 0\} \cup \{(-x,-x) | x > 0\} \cup \{(x,0) | x \geq 0\} \cup \{\omega\}$$

and define $\Phi : [0,+\infty) \times X \to X$ by:

$$\Phi(t,a) = \begin{cases} (-x+t, x-t), & a = (-x,x),\ t \in [0,x] \\ (t-x, 0), & a = (-x,x),\ t \in (x,+\infty) \\ (-x+t, -x+t), & a = (-x,-x),\ t \in [0,x] \\ (t-x, 0), & a = (-x,-x),\ t \in (x,+\infty) \\ (x+t, 0), & a = (x,0) \\ \omega, & a = \omega \end{cases}$$

This semi dynamical system has the entrance boundary formed of two points, and the exit boundary formed of one point.

$(0,0)$ is a ramification point; if one considers $X_1 = X \setminus \{(0.0)\}$, and $\Phi_1 : [0,+\infty) \times X_1 \to X_1$ is defined by

$$\Phi_1(t,a) = \begin{cases} (-x+t, x-t), & a = (-x,x),\ t \in [0,x] \\ \omega, & a = (-x,x),\ t \in (x,+\infty) \\ (-x+t, -x+t), & a = (-x,-x),\ t \in [0,x] \\ \omega, & a = (-x,-x),\ t \in (x,+\infty) \\ (x+t, 0), & a = (x,0) \\ \omega, & a = \omega \end{cases}$$

then one obtains a semi–dynamical system, which has the entrance and exit boundary formed each one of three points.

c) We consider now $X = \{z \in \mathbf{C} | \, |z| = 1\} \cup \{\omega\}$ and define $\Phi : [0,+\infty) \times X \to X$ by:

$$\Phi(t,z) = \begin{cases} z \cdot e^{it}, & z \in \mathbf{C} \\ \omega, & z = \omega \end{cases}$$

One obtains in this way a *periodic* semi–dynamical system, which **is not** transient.

d) Let $X = \mathbf{R}^2 \cup \{\omega\}$. We define two semi–dynamical systems by $\Phi_k : [0,+\infty) \times X \to X$, $k = 1,2$

$$\Phi_1(t,(x,y)) = (x+t, y+t), \quad \text{resp.}$$

$$\Phi_2(t,(x,y)) = (x+t, y)$$

These two systems are isomorphic in the following sense: defining

$$\varphi : X \to X$$

by

$$\varphi(x, y) = \left(\frac{x+y}{2}, \frac{x-y}{2}\right)$$

one obtains a bijection, which satisfies the relation:

$$\varphi(\Phi_1(t,(x,y))) = \Phi_2(t, \varphi(x,y))$$

However, we cannot obtain one system from another by a time change. Also, Φ_1 is a product system, while Φ_2 is not a product system.

The natural, resp. fine topologies for Φ_1 are strictly coarser than the product of the corresponding topologies on factors.

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Universitatea "Al.I.Cuza"
Seminarul Matematic
6600 - Iaşi
ROMANIA
and
Universitatea Tehnică "Gh.Asachi"
Catedra de matematică
6600 - IAŞI
ROMANIA

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THE ALGEBRA OF THE LOCALLY BOUNDED ELEMENTS IN A STANDARD H-CONE

BY

LILIANA POPA*

Let S be a standard H-cone of functions on a semi-saturated set X [1]. In [6] we have proved that the set of the locally bounded elements of S is an algebra for the usual product of functions.

The purpose of this paper is to investigate some new properties of this algebra. We prove that if f has the property that there exists a covering of X with naturally open sets U_i, $i \in I$, such that $f|_{U_i}$ is a difference of bounded and continuous elements of the localisation $S(U)$ and $\frac{1}{f}$ is locally bounded, then $\frac{1}{f}$ has the same property.

It is then possible to construct a partition of the unity with elements of the previous algebra, which is subordinated to each local finite covering of X. So, a theory of the cohomology can be developped for a standard H-cone, extending the cohomology of the harmonic spaces constructed by B.Walsh [7]. Dropping the hypothesis of local compacity, the results of this paper become more general.

Let S be a standard H-cone of functions on a set X. We recall that every element $s \in S$ can be uniquely written

$$s = h + p$$

with $h \in S_H$, $p \in S_P$, where S_H represents the set of all the ***subtractible elements*** of S, that means

$$S_H = \{h \in S \mid \forall s \in S, h \leq s \Rightarrow s - h \in S\}$$

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