These two systems are isomorphic in the following sense: defining

$$\varphi : X \to X$$

by

$$\varphi(x, y) = \left(\frac{x+y}{2}, \frac{x-y}{2} \right)$$

one obtains a bijection, which satisfies the relation:

$$\varphi(\Phi_1(t, (x, y))) = \Phi_2(t, \varphi(x, y))$$

However, we cannot obtain one system from another by a time change. Also, Φ_1 is a product system, while Φ_2 is not a product system.

The natural, resp. fine topologies for Φ_1 are strictly coarser than the product of the corresponding topologies on factors.

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THE ALGEBRA OF THE LOCALLY BOUNDED ELEMENTS IN A STANDARD H-CONE

BY

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Let S be a standard H-cone of functions on a semi-saturated set X [1]. In [6] we have proved that the set of the locally bounded elements of S is an algebra for the usual product of functions.

The purpose of this paper is to investigate some new properties of this algebra. We prove that if f has the property that there exists a covering of X with naturally open sets U_i, $i \in I$, such that $f|_{U_i}$ is a difference of bounded and continuous elements of the localisation $S(U)$ and $\frac{1}{f}$ is locally bounded, then $\frac{1}{f}$ has the same property.

It is then possible to construct a partition of the unity with elements of the previous algebra, which is subordinated to each local finite covering of X. So, a theory of the cohomology can be developped for a standard H-cone, extending the cohomology of the harmonic spaces constructed by B.Walsh [7]. Dropping the hypothesis of local compacity, the results of this paper become more general.

Let S be a standard H-cone of functions on a set X. We recall that every element $s \in S$ can be uniquely written

$$s = h + p$$

with $h \in S_H$, $p \in S_P$, where S_H represents the set of all the *subtractible elements* of S, that means

$$S_H = \{h \in S \mid \forall s \in S, h \leq s \Rightarrow s - h \in S\}$$

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and S_P is the set of all *pure potentials* of S.

$$S_P = \{p \in S \mid p \curlywedge h = 0, \ \forall h \in S_H\}$$

We suppose that $1 \in S_H$.

If $A \subset X$, we define the balayage of s on A, $s \in S$, by

$$B^A s = \bigwedge \{t \in S \mid t \geq s \text{ on } A\}$$

where

$$\bigwedge_{i \in I} s_i(x) = \widehat{\inf}_{i \in I} s_i(x).$$

Here $\hat{f}$ is the lower semi-continuous regularisation of f.

If $G \subset X$ is a finely open set, we denote by $S(G)$ the set of all positive functions f on G which are finite on a fine dense subset of G and for which there exists a sequence (s_n) in S, $s_n < \infty$ such that

$$s_n - B^{X \setminus G} s_n \uparrow f$$

where $B^{X \setminus G} s_n$ is the balayage of s_n on $X \setminus G$. From [2] we know that $S(G)$ is a standard H-cone of functions on G, for which the fine topology on G given by $S(G)$ coincides with the induced fine topology of X on G.

If G is naturally open, than a similar assertion takes place for the natural topology.

We suppose that the H-cone satisfies *the natural sheaf property*. Hence the map

$$G \to S(G), G \subseteq X, \text{ naturally open}$$

is a sheaf, that means that the two following axioms are fulfilled:

a) For any two naturally open sets $G_1, G_2 \subset X$, with $G_1 \subset G_2$, we have

$$f|_{G_1} \in S(G_1), \ \forall f \in S(G_2)$$

b) If (G_i), $i \in I$ is a family of naturally open sets and $G = \cup_{i \in I} G_i$, then $s : G \to \overline{\mathbf{R}}_+$ belongs to $S(G)$ if $s|_{G_i}$, $\forall i \in I$.

We suppose that X is semi-saturated, that means that any H-integral on S dominated by an H-measure on X (with respect to S), is also an H-measure on X. From [2] we know that if X is semi-saturated any finely open subset G of X is semi-saturated with respect to the standard H-cone $S(G)$ of functions on G.

Because X is semi-saturated for any $A \subset X$ and $x \in X$ there exists a positive measure on X, denoted $(B^A)^* \varepsilon_x$ such that

$$\forall s \in S \quad B^A s(x) = \int s d(B^A)^* \varepsilon_x$$

We denote by

$$S_M = \{s \in S \mid \exists \alpha \in \mathbf{R}, \ s \leq \alpha\}$$

and let $[S_M] = S_M - S_M$. We introduce the next set of functions:

$$R = \{f : X \to \mathbf{R} \mid \text{there exists a covering of } X, U_i, i \in I, \text{ naturally open}$$

$$\text{such that } f|_{U_i} = s - t, s, t \in S_M(U_i), \text{ naturally continuous}\}$$

We have proved in [6] that R is an algebra with the usual product of functions, which is also a lattice for the pointwise order relation. We recall a sufficient condition of apartenence of a general function to an H-cone, which was proved in [6].

Lemma 1. *If $f : X \to \mathbf{R}_+$ is naturally continuous, dominated by an element of S, and for any $x \in X$, $A \subset X$, naturally open the next inequality is valid*

$$\int_X f d(B^A)^* \varepsilon_x \leq f(x)$$

then f belongs to S.

From the proof, it is sufficient that the previous inequality is valid on the sets

$$\{x \in X | f(x) > \theta R f\}, \theta \in [0, 1).$$

where $Rf = \inf\{s \in S | \ s \geq f\}$. We recall also that a result of A.Cornea [4] states that if f is naturally continuous, Rf has the same property. We shall denote such sets by $[f > \theta R f]$.

Lemma 2. *Let $u \in S_M$ naturally continuous and $a \in \mathbf{R}$ such that there exists $m, M > 0$ for which:*

$$0 < m \leq u - a \leq M.$$

Then for any $n \in \mathbf{N}, n \geq 1$, we have:

$$\frac{1}{(u-a)^n} \in [S_M].$$

Proof. For any natural open set $A \subset X$, we can apply Hölder inequality, because $(B^A)^* \varepsilon_x \le \varepsilon_x$ is an H-measure. We get

$$B^A 1(x) = \left(\int_X d\,(B^A)^* \varepsilon_x\right)^n = \left(\int_X \frac{1}{(u-a)^{\frac{n-1}{n}}}(u-a)^{\frac{n-1}{n}}\, d\,(B^A)^* \varepsilon_x\right)^n \le$$

$$\le \left(\int_X \frac{1}{(u-a)^{n-1}}\, d\,(B^A)^* \varepsilon_x\right)\left(\int_X (u-a)\, d\,(B^A)^* \varepsilon_x\right)^{n-1} \le$$

$$\le (u-a)^{n-1}(x) \int_X \frac{1}{(u-a)^{n-1}}\, d\,(B^A)^* \varepsilon_x .$$

If we chose $A = [\alpha > p]$, with $p \in S_P$, naturally continuous then from [1], we know that

$$B^{[\alpha > p]}\alpha = \alpha$$

hence

$$B^{[\alpha > p]}1 = 1$$

and from the above inequality we have:

$$(1) \quad \frac{1}{(u-a)^{n-1}(x)} \le \int_X \frac{1}{(u-a)^{n-1}}\, d\,(B^A)^* \varepsilon_x$$

and if $\alpha = (\frac{1}{m})^n$, then

$$\int_X \left(\alpha - \frac{1}{(u-a)^{n-1}}\right) d\,(B^A)^* \varepsilon_x \le \alpha - \frac{1}{u^{n-1}}$$

Let $f = \alpha - \frac{1}{(u-a)^{n-1}}$. We show next that the set $[f > \theta R f], \theta \in [0,1)$ is included in a set of the form $[\alpha > p]$.

Indeed, let $Rf = p' + h'$, with $p' \in S_P, h' \in S_H$; because f is naturally continuous, so is Rf and h' results continuous , being majorated. Hence p' is naturally continuous. Then for any $x \in [f > \theta R f]$

$$\alpha \ge f(x) > \theta(p'(x) + h'(x)) \ge \theta p'(x).$$

From $[f > \theta R f] \subset [\alpha > \theta p']$, we deduce:

$$\int \left(\alpha - \frac{1}{(u-a)^{n-1}}\right) d\left(B^{[f > \theta R f]}\right)^* \varepsilon_x \le$$

$$\le \int \left(\alpha - \frac{1}{(u-a)^{n-1}}\right) d\left(B^{[\alpha > \theta p']}\right)^* \varepsilon_x \le \alpha - \frac{1}{(u-a)^{n-1}(x)}$$

Hence

$$\frac{1}{(u-a)^n} \in [S_M]$$

The next result was proved in [7] and has as consequence, that R is an algebra.

Lemma 3. *If $t \in S_M$ is naturally continuous and majorated by α, then $\alpha^n - (\alpha - t)^n \in S_M$.*

Hence for any $p \in S_P$, naturally continuous, we have

$$B^{[\alpha > p]}(\alpha^n - (\alpha - t)^n) \le \alpha^n - (\alpha - t)^n$$

and

$$(2) \quad (\alpha - t)^n(x) \le \int (\alpha - t)^n d\left(B^{[\alpha > p]}\right)^* \varepsilon_x$$

Theorem 1. *Let $f \in R, f \neq 0$, such that $\frac{1}{f}$ is locally bounded, then $\frac{1}{f} \in R$.*

Proof. Let $U_i, i \in I$, U_i naturally open such that $f|_{U_i} \in [S_M(U_i)]$ and it is continuous. Let $f|_{U_i} = s - t$. For any $x_0 \in U_i$, there exist $\alpha, \lambda \in \mathbb{R}_+$, $s', t' \in S$ and a naturally open neighbourhood U_0, such that

$$\lambda(s - t) = s' - \alpha - (t' - \alpha)$$

and $s' - \alpha > 2, -\frac{1}{2} < t' - \alpha < 0$.

From the natural sheaf property $f|_{U_0} \in [S_M(U_0)]$. We can write

$$\frac{1}{s-t} = \frac{\lambda}{\lambda(s-t)} = \frac{\lambda}{s' - \alpha - (t' - \alpha)} = \frac{\lambda}{s' - \alpha}\, \frac{1}{1 - \frac{t' - \alpha}{s' - \alpha}}.$$

Because $\frac{1}{s' - \alpha} < \frac{1}{2}$ and $| t' - \alpha | < \frac{1}{2}$, the next convergence is assured:

$$\frac{1}{1 - \frac{t'-\alpha}{s'-\alpha}} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{s' - \alpha}\right)^n (\alpha - t')^n =$$

$$= \frac{1}{2}\{\sum_{n=0}^{\infty} (-1)^n \left\{\left[\left(\frac{1}{s'-\alpha}\right)^n + (\alpha - t')^n\right]^2 - \left(\frac{1}{s'-\alpha}\right)^{2n} - (\alpha - t')^{2n}\right\} =$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left\{\left[\left(\frac{1}{s'-a}\right) + (a-t')^{2n}\right]^2 + \left(\frac{1}{s'-a}\right)^{4n+2} + (a-t')^{4n+2}\right\} -$$

$$-\frac{1}{2}\sum_{n=0}^{\infty}\left\{\left[\left(\frac{1}{s-a}\right)^{2n+1} + (a-t)^{2n+1}\right]^2 + \left(\frac{1}{s-a}\right)^{4n} + (a-t)^{4n}\right\}.$$

Because $\frac{1}{s'-a} < \frac{1}{2}$ and $\mid t'-a \mid < \frac{1}{2}$ the partial sums of the previous series, are uniformly majorated by a real constant, denoted by β. Then we can use Hölder inequality and the relations (1) (2) for any $n \in \mathbf{N}$ and we get:

$$\left[\left(\frac{1}{s'-a}\right)^n + (a-t')^n\right]^2 (x) \leq$$

$$\leq \left[\int \left(\left(\frac{1}{s'-a}\right)^n + (a-t')^n\right) d\left(B^{[\beta > p]}\right)^* \varepsilon_x\right]^2 \leq$$

$$\leq \int \left[\left(\frac{1}{s'-a}\right)^n + (a-t')^n\right]^2 d\left(B^{[\beta > p]}\right)^* \varepsilon_x \int 1 d\left(B^{[\beta > p]}\right)^* \varepsilon_x.$$

If we denote by s_k, t_k the partial sums then the previous inequality we get

$$s_k(x) \leq \int s_k d\left(B^{[\beta > p]}\right)^* \varepsilon_x$$

$$t_k(x) \leq \int t_k d\left(B^{[\beta > p]}\right)^* \varepsilon_x$$

Because s_k, t_k are increasing, the H-measure $(B^{[\beta > p]})^* \varepsilon_x$ commutes with the limit. Let

$$u(x) = \lim_{k\to\infty} s_k(x),\ v(x) = \lim_{k\to\infty} t_k(x)$$

then we have :

$$u(x) \leq \lim_{k\to\infty} \int s_k d\left(B^{[\beta > p]}\right)^* \varepsilon_x = \int u(x) d\left(B^{[\beta > p]}\right)^* \varepsilon_x$$

and

$$v(x) \leq \lim_{k\to\infty} \int t_k d\left(B^{[\beta > p]}\right)^* \varepsilon_x = \int v(x) d\left(B^{[\beta > p]}\right)^* \varepsilon_x$$

Hence

$$\int (\beta - u) d\left(B^{[\beta > p]}\right)^* \varepsilon_x \leq \beta - u(x)$$

$$\int (\beta - v) d\left(B^{[\beta > p]}\right)^* \varepsilon_x \leq \beta - v(x)$$

and

$$\frac{1}{1 - \frac{t'-a}{s'-a}} = u - v$$

belongs to $[S_M(U_0)]$.

From Lemma 2, it results that $\frac{1}{s'-a} \in [S_M(U_0)]$, hence $\frac{1}{f} \in [S_M(U_0)]$. Covering U, with such U_0 and using the natural sheaf property, it results that $\frac{1}{s-t} \in [S_M(U_i)]$, hence $\frac{1}{f}$ is from R.

Lemma 4. *For any $x \in X$ and any naturally neighbourhood U of x, there exists $f \in [S]$, such that*

1. $0 \leq f \leq 1$
2. $f = 0$ *on* $X \setminus U$
3. $f = 1$ *on a neighbourhood of* x.

Proof. We adapt at this case a proof of [1]. Let F_1 the cone of functions of the form

$$\inf(s, \inf_{i\in I}(s_i + \alpha_i))$$

where I is finite, and $s, s_i \in S_0, \alpha_i \in \mathbf{R}_+$.

From [1] for any x and any natural neighbourhood U of x, there exists $s, t \in F_1$, with

$$0 \leq s - t \leq 1,\quad s(x) = t(x) + 1,\quad s = t \text{ on } X \setminus U$$

Because $s - t$ is naturally continuous, there exists a natural neighbourhood V, such that $s - t \geq \frac{1}{2}$ on V. Then the function $f = \inf(2s, 1 + 2t) - 2t$, satisfies the requirements of the Lemma.

Definition. $(U_i), i \in I, U_i$ *naturally open is a locally finite, open covering of* X, *if*

1. $X = \cup_{i\in I} U_i$
2. *for any x there exists a natural neighbourhood of x, denoted by V, such that the set $\{i | V \cap U_i \neq \emptyset\}$ is finite.*

Let $(U_i), i \in I, U_i$ naturally open a locally finite covering of X.

Definition. *The family of functions $h_i : X \to \mathbf{R}$ $i \in I$, naturally continuous is called a partition of the unity subordinated to the locally finite covering if it has the properties:*

1. $h_i \geq 0,\ \forall i \in I$
2. $h_i = 0$ on $X \setminus U_i,\ \forall i \in I$

3. $\forall x \in X, \sum_{i\in I} h_i(x) = 1$.

Theorem 2. *For any locally finite, open covering of X there exists a subordinated partition of the unity (h_i), $i \in I$, with $h_i \in R$.*

Proof. Let (U_i), $i \in I$ the locally finite covering of X, with naturally open sets. For any $x \in U_i$, there exists from Lemma 4 $f_{i,x} \in R$ with the properties:

1. $0 \le f_{i,x} \le 1$
2. $f_{i,x} = 0$ on $X \setminus U_i$
3. $f_{i,x}(x) = 1$

We denote by $V_{i,x} = [f_{i,x} > 0]$. The sets $V_{i,x}$ realize an open covering of X. Because X with the natural topology is paracompact there exists a finer locally finite covering W_α: that means that for any α there exists $i(\alpha), x(\alpha)$, such that

$$W_\alpha \subset V_{i(\alpha),x(\alpha)} \subset U_{i,\alpha}$$

Let $f_\alpha = f_{i(\alpha),x(\alpha)}$; we define

$$g_i = \sum_{i(\alpha)=i} f_\alpha.$$

We observe that $g_i > 0$ on U_i.

Let us show that $g_i = 0$ on $X \setminus U_i$. If x is not from U_i, then $f_\alpha = 0$ on $X \setminus U_i$ if α is such that $i = i(\alpha)$, hence $g_i = 0$. We have also on X

$$\sum_{i\in I} g_i(x) = \sum_{i\in I} \sum_{i(\alpha)=i} f_\alpha(x) = \sum_{\alpha} f_\alpha(x) > 0$$

It results that the family

$$h_i = \frac{g_i}{\sum_{i\in I} g_i}$$

is a partition of the unity subordinated to the locally finite covering of X.

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