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CONVERGENCE OF A FINITE-DIFFERENCE SCHEME FOR WEAKLY-ELLIPTICAL EQUATIONS

BY

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In this paper we study the convergence of a finite-difference scheme approximating a boundary value problem for a class of weakly elliptical differential equations.

Let be $\Omega \subset \mathbf{R}^2$ a bounded and open subset with a continuous Lipschitz boundary Γ.

We consider a boundary value problem for the third order weakly elliptical equation with variable coefficients:

$$\mathcal{L}u = f \text{ in } \Omega, \tag{1}$$

where

$$\mathcal{L}u = -\sum_{i=0}^{1}\sum_{k=0}^{2} \frac{\partial}{\partial x_1^{1-i}\,\partial x_2^{i}} \left(a_{ik}(x_1,x_2)\frac{\partial^2 u}{\partial x_1^{2-k}\,\partial x_2^{k}}\right) + a(x_1,x_2)\left(\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2}\right). \tag{2}$$

We impose the following conditions:

$$\begin{aligned} & a_{ik} \in \mathbf{W}_p^{s-2}(\Omega),\ p > 2/(s-2),\ 0 < s-2 \le 1,\ i = 0,1,\ k = 0,1,2; \\ & a = a_0 + \sum_{i=1}^{2} D_i a_i(x),\ a_0 \in \mathbf{L}^{2+\epsilon}(\Omega),\ \epsilon > 0,\ a_i \in \mathbf{W}_p^{s-2}(\Omega),\ i = 1,2; \\ & a(x) > 0,\ \forall x \in \Omega. \end{aligned} \tag{3}$$

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These conditions are sufficient that the coefficients belong to the space of multipliers:

$$a_{ik} \in M(\mathbf{W}_2^{s-2}(\Omega)),\ a \in M(\mathbf{W}_2^{s-1}(\Omega), \mathbf{W}_2^{s-3}(\Omega)).$$

If we note

$$\text{(4)}\qquad b_{ik} = \begin{cases} a_{0k} & \text{if } i = 0 \\ a_{i-1,k} & \text{if } i = 2,\ k = 0,1,2, \\ a_{i-1,k} + a_{ik} & \text{if } i = 1 \end{cases}$$

we suppose that

$$\text{(5)}\qquad \sum_{i,k=0}^{2} b_{ik}(x_1, x_2)\,\xi_i\xi_k \ \geq\ \mu^2 \,(\sum_{k=0}^{2} C_2^k \xi_k)^2,\ \forall (x_1, x_2) \in \Omega,$$

and $\forall \xi = (\xi_0, \xi_1, \xi_2) \in \mathbf{R}^3$.

We consider the boundary value problem:

$$\text{(6)}\qquad \begin{cases} \mathcal{L}u = f & \text{in } \Omega \\ u = 0,\ \dfrac{\partial u}{\partial x_i} = 0,\ i = 0, 1, & \text{on } \Gamma \end{cases}$$

We assume that the generalized solution of the problem (6) belongs to the Sobolev space $W_2^s(\Omega)$, $2 < s \leq 3$.

Let be the boundary value problem (6) for the third order weakly elliptical equation in the square $\Omega = (0,1) \times (0,1)$.

Let $\bar{\omega}$ be the uniform mesh in Ω with the step h and $\gamma = \Gamma \cap \bar{\omega}$.

Finite differences are defined as usual:

$$\text{(7)}\qquad v_{x_i} = \frac{v^{+i} - v}{h},\quad v_{\bar{x}_i} = \frac{v - v^{-i}}{h},$$

where $v^{\pm i} = v(x \pm he_i)$ and e_i is the unit vector on the x_i-axis.

On the set of the functions defined on ω we can define the inner scala

$$(v, w)_{\mathbf{L}^2(\omega)} = h^2 \sum_{x \in \omega} v(x) w(x)$$

and the norm

$$\|v\|_{\mathbf{L}^2(\omega)} = h(\sum_{x \in \omega} v^2(x))^{1/2}$$

We introduce the averaging operators with the step h

$$T_i^+ f(x) = \int_0^1 f(x + hte_i)dt = T_i^- f(x + he_i) = T_i f(x + \frac{h}{2} e_i),$$
$$i = 1, 2,\ x = (x_1, x_2).$$

These operators map the partial derivates onto differences

$$T_i^+ (\frac{\partial}{\partial x_i}) f(x) = \frac{\partial}{\partial x_i}(T_i^+ f(x)) = f_{x_i}(x)$$

and

$$T_i f(x) = \int_{-1/2}^{1/2} f(x + hte_i)dt,\ i = 1, 2.$$

We shall construct a stable difference scheme for the problem (6). We start from the identity

$$\text{(8)}\qquad a(u, v) = \int_\Omega (\sum_{i,k=0}^{2} b_{ik}(x) \frac{\partial^2 u}{\partial x_1^{2-i} \partial x_2^i} \frac{\partial^2 v}{\partial x_1^{2-k} \partial x_2^k} + a(x)\delta u\, \delta v)dx =$$
$$= \int_\Omega f \delta v\, dx,$$

where $\delta u = \dfrac{\partial u}{\partial x_1} + \dfrac{\partial u}{\partial x_2}$.

The integrals over Ω are replaced by

$$\text{(9)}\qquad h^2 \sum_{x \in \omega} (\sum_{i,k=0}^{2} \tilde{b}_{ik}(x) u_{x_1^{2-i} x_2^i} \tilde{v}_{x_1^{2-k} x_2^k} + \tilde{a}(x)(\tilde{u}_{x_1} + \tilde{u}_{x_2})(\tilde{v}_{x_1} + \tilde{v}_{x_2})) =$$
$$= h^2 \sum_{x \in \omega} \tilde{f}(x)(\tilde{v}_{x_1} + \tilde{v}_{x_2}).$$

Using the formula summation by parts we have:

$$\text{(10)}\qquad -\frac{h^2}{2} \sum_{x \in \omega} \sum_{i=0}^{1} \sum_{k=0}^{2} ((\tilde{a}_{ik}(x)\tilde{u}_{x_1^{2-k} x_2^k})_{x_1^{1-i} x_2^i} + (\tilde{a}_{ik}(x)\tilde{u}_{x_1^{2-k} x_2^k})_{x_1^{1-i} x_2^i}) +$$
$$+ h^2 \sum_{x \in \omega} \tilde{a}(x)(\tilde{u}_{x_1} + \tilde{u}_{x_2}) = h^2 \sum_{x \in \omega} \tilde{f}(x).$$

We set

$$\text{(11)}\qquad \mathcal{L}_h \tilde{u} = -\frac{1}{2} \sum_{i=0}^{1} \sum_{k=0}^{2} ((\tilde{a}_{ik}(x)\tilde{u}_{x_1^{2-k} x_2^k})_{x_1^{1-i} x_2^i} + (\tilde{a}_{ik}(x)\tilde{u}_{x_1^{2-k} x_2^k})_{x_1^{1-i} x_2^i}) +$$
$$+ \tilde{a}(x)(\tilde{u}_{x_1} + \tilde{u}_{x_2}) = \tilde{f}(x)$$

where $\tilde u$ are the grid functions defined to be zero outside ω. The coefficients $\tilde a_{ik}$, $\tilde a$, and $\tilde f$ are averages for a_{ik}, a and f.

We must couple this scheme with the boundary conditions:

(12)
$$\begin{cases} \tilde u = 0 & \text{on } \gamma \\ \tilde u_{x_i} = 0 & \text{on } \gamma,\ i = 1,2,\ \gamma = \Gamma \cap \bar\omega. \end{cases}$$

The equations (11)+(12) form a linear algebraic system containing as many unknowns as there are equations. The unique solvability of this system, as well as the stability of the scheme, follow from a priori bounds for $\tilde v$.

Theorem 1. *Let $\tilde v$ be a solution of the problem* (11)+(12). *Then the following estimate*

(13)
$$C\mu^2 \| \tilde v \|_{\mathbf{L}^2(\omega)} \le \| f \|_{\mathbf{L}^2(\Omega)}$$

holds.

Proof. We set $\tilde u = \tilde v$ in (9) and by the structure of the grid functions $\tilde b_{ik}$, $\tilde a$ at the points of ω and the conditions (5) we obtain

(14)
$$\sum_{i,k=0}^{2} \tilde b_{ik}(x)\tilde v_{x_1^{2-i}x_2^{i}}\tilde v_{x_1^{2-k}x_2^{k}} + \tilde a(x)(\tilde v_{x_1} + \tilde v_{x_2})^2 \ge \mu^2\Big(\sum_{k=0}^{2} C_2^k \tilde v_{x_1^{2-k}x_2^{k}}\Big)^2.$$

Then

(15)
$$\mu^2 h^2 \sum_{x\in\omega}\Big(\sum_{k=0}^{2} C_2^k \tilde v_{x_1^{2-k}x_2^{k}}\Big)^2 \le a(\tilde v,\tilde v) = h^2 \sum_{x\in\omega} \tilde f(\tilde v_{x_1} + \tilde v_{x_2}) \le$$
$$\le \|\tilde f\|_{\mathbf{L}^2(\omega)} \|\tilde v_x\|_{\mathbf{L}^2(\omega)}.$$

On the other hand, for any grid function $\tilde v$ prescribed on ω which satisfies the conditions (12) and equal to zero outside ω we have

$$C h^2 \sum_{x\in\omega} \tilde v^2 \le C_1 h^2 \sum_{x\in\omega}(\tilde v_{x_1} + \tilde v_{x_2})^2 \le h^2 \sum_{x\in\omega}\Big(\sum_{k=0}^{2} C_2^k \tilde v_{x_1^{2-k}x_2^{k}}\Big)^2,$$

and

$$C\mu^2 \|\tilde v\|_{\mathbf{L}^2(\omega)} \le \|\tilde f\|_{\mathbf{L}^2(\omega)} \le \|f\|_{\mathbf{L}^2(\Omega)}.$$

Let u be the solution of the boundary problem (6) and v be the solution of the finite-difference scheme (11)+(12). For $s > 1$, $u(x)$ is continuous and

the error $z = u - v$ is defined at the nodes of the mesh $\bar\omega$. So we obtain the problem

(16)
$$\mathcal{L}_h z = -\sum_{i=0}^{1}\sum_{k=0}^{2}\Big(T_{i+1}^{+}T_{2-i}^{2}\Big(a_{ik}\frac{\partial^2 u}{\partial x_1^{2-k}\partial x_2^{k}}\Big)\Big)_{x_1^{1-i}x_2^{i}} + T_1^2T_2^2\Big(a\sum_{i=1}^{2}\frac{\partial u}{\partial x_i}\Big) +$$
$$+ \sum_{i=0}^{1}\sum_{k=0}^{2}\frac{1}{2}\big(\tilde a_{ik}u_{x_1^{2-k}x_2^{k}} + \tilde a_{ik}^{+i}u^{+i}_{x_1^{2-k}x_2^{k}}\big)_{x_1^{1-i}x_2^{i}} - \tilde a\sum_{i=1}^{2}u_{x_i}$$

and $z = 0$, $z_{x_i} = 0$ on γ, $i = 1,2$.

Using the notation

(17)
$$\eta_{ik} = T_{i+1}^{+}T_{2-i}^{2}\Big(a_{ik}\frac{\partial^2 u}{\partial x_1^{2-k}\partial x_2^{k}}\Big) - \frac{1}{2}\big(\tilde a_{ik}u_{x_1^{2-k}x_2^{k}} + \tilde a_{ik}^{+i}u^{+i}_{x_1^{2-k}x_2^{k}}\big)$$

and

(18)
$$\eta = T_1^2T_2^2\Big(a\sum_{i=1}^{2}\frac{\partial u}{\partial x_i}\Big) - \tilde a\sum_{i=1}^{2}u_{x_i}$$

the right-hand-side terms of (16) can be written as:

(19)
$$\mathcal{L}_h z = -\sum_{i=0}^{1}\sum_{k=0}^{2}(\eta_{ik})_{x_1^{1-i}x_2^{i}} + \eta.$$

Lemma 1. *The following a priori estimate*

(20)
$$\|z\|_{\mathbf{L}^2(\omega)} \le C\Big(\|\eta\|_{\mathbf{L}^2(\omega)} + \sum_{i=0}^{1}\sum_{k=0}^{2}\|\eta_{ik}\|_{\mathbf{L}^2(\omega)}\Big)$$

holds.

Proof. By the energy method it follows that

$$(\mathcal{L}_h z, z_{x_1} + z_{x_2})_{\mathbf{L}^2(\omega)} = -\sum_{i=1}^{2}\sum_{k=0}^{2}\big((\eta_{ik})_{x_1^{1-i}x_2^{i}}, z_{x_1} + z_{x_2}\big)_{\mathbf{L}^2(\omega)} +$$
$$+ (\eta, z_{x_1} + z_{x_2})_{\mathbf{L}^2(\omega)}.$$

Using the formula summation by parts and Cauchy's inequality we obtain the inequality (20).

Theorem 2. *If conditions (3), (4), (5) are satisfied, then the finite-difference scheme (11) and (12) converges and the estimate*

$$\|u - v\|_{\mathbf{L}^2(\omega)} \le C\, h^{s-2} \|u\|_{\mathbf{W}_2^s(\Omega)}, \quad 2 < s \le 3 \tag{21}$$

holds.

Proof. In order to obtain the convergence rate estimate for the finite-difference scheme (11)+(12) it is sufficient to estimate the terms in the sum on the right-hand-side of the inequality (20). We represent η_{ik} in the following manner:

$$\eta_{ik} = \eta_{ik}^1 + \eta_{ik}^2 + \eta_{ik}^3 + \eta_{ik}^4, \quad i = 0, 1, \quad k = 0, 1, 2,$$

where

(22)

$$\eta_{ik}^1 = T_{i+1}^+ T_{2-i}^2 \Big(a_{ik} \frac{\partial^2 u}{\partial x_1^{2-k} \partial x_2^k}\Big) - T_{i+1}^+ T_{2-i}^2 (a_{ik}) \cdot T_{i+1}^+ T_{2-i}^2 \Big(\frac{\partial^2 u}{\partial x_1^{2-k} \partial x_2^k}\Big),$$

$$\eta_{ik}^2 = \Big(T_{i+1}^+ T_{2-i}^2 (a_{ik}) - \frac{1}{2}(\bar{a}_{ik} + \bar{a}_{ik}^{+i})\Big) T_{i+1}^+ T_{2-i}^2 \Big(\frac{\partial^2 u}{\partial x_1^{2-k} \partial x_2^k}\Big),$$

$$\eta_{ik}^3 = \frac{1}{2}(\bar{a}_{ik} + \bar{a}_{ik}^{+i}) \Big(T_{i+1}^+ T_{2-i}^2 \Big(\frac{\partial^2 u}{\partial x_1^{2-k} \partial x_2^k}\Big) - \frac{1}{2}\big(u_{x_1^{2-k} x_2^k} + u_{x_1^{2-k} x_2^k}^{+i}\big)\Big)$$

$$\eta_{ik}^4 = \frac{1}{4}(-\bar{a}_{ik} + \bar{a}_{ik}^{+i})\big(u_{x_1^{2-k} x_2^k} - u_{x_1^{2-k} x_2^k}^{+i}\big)$$

for $i = 0, 1$, $k = 0, 1, 2$.

We set

$$\eta_0 = T_1^2 T_2^2 \Big(a_0 \sum_{i=1}^{2} \frac{\partial u}{\partial x_i}\Big) - (T_1^2 T_2^2 a_0) \sum_{i=1}^{2} u_{x_i} \tag{23}$$

$$\eta_i = T_1^2 T_2^2 \Big(\sum_{j=1}^{2} \frac{\partial u}{\partial x_j} D_i a_i\Big) - (T_1^2 T_2^2 D_i a_i) \sum_{j=1}^{2} u_{x_j}, \quad i = 1, 2. \tag{24}$$

We introduce the cells

$$e_0 = \{\zeta \in \omega;\ |\zeta_i - x_i| \le h,\ i = 1, 2\}$$

$$e_i = \{\zeta \in \omega;\ x_i < \zeta_i < x_i + h, |\zeta_{3-i} - x_{3-i}| < h,\ i = 1, 2\}.$$

Using the linear transformation

$$\zeta_1 = x_1 + h x_1^*, \quad \zeta_2 = x_2 + h x_2^*,$$

we map the cells $e_0(x)$, $e_i(x)$ on standard cells $E_0(x)$ and respectively $E_i(x)$:

$$E_0(x) = \{\, x^*: |x_i^*| < 1,\ i = 1, 2 \,\}$$

$$E_i(x) = \{\, x^*: 0 < x_i^* < 1,\ |x_{3-i}^*| < 1,\ i = 1, 2 \,\}.$$

The Jacobian of this linear transformation is $J = h^2$.

We use the following regularity hypothesis: if $v \in W_2^s(e_k)$ and the diameter of the element e_k is h then it follows that there exists constant C such that

$$|v|_{\mathbf{W}_2^s(E_k)} \le C\, J^{-1/2}\, h^s \|v\|_{\mathbf{W}_2^s(e_k)} \tag{25}$$

for $i = 0, 1, 2$, (the diameter of a quadrilater cells is the length of the longer diagonal). In deriving bounds on bilinear forms we use the generalized Bramble–Hilbert Lemma, [2].

Let be

$$a_{ij}^*(x^*) = a_{ij}(x + h x^*),$$
$$a^*(x^*) = a(x + h x^*),$$
$$u^*(x^*) = u(x + h x^*).$$

We estimate η_{01}^1 at the node $x \in \omega$.

$$\begin{aligned} \eta_{01}^1(x) = {} & \frac{1}{h^2} \int_{E_1} (1 - |x_2^*|) a_{01}^*(x^*) \frac{\partial^2 u^*}{\partial x_1^* \partial x_2^*}\, dx_1^* dx_2^* - \\ & - \int_{E_1} (1 - |x_2^*|) a_{01}^*(x^*) dx_1^* dx_2^* \cdot \frac{1}{h^2} \int_{E_1} (1 - |x_2^*|) \frac{\partial^2 u^*}{\partial x_1^* \partial x_2^*}\, dx_1^* dx_2^*. \end{aligned} \tag{26}$$

$\eta_{01}^1(x)$ is the bounded bilinear functional of $(a_{01}^*, u^*) \in \mathbf{W}_p^\alpha(E_1) \times \mathbf{W}_q^\beta(E_1)$ for $\alpha > 0$, $\beta \ge 2$, $q > 2$. $\eta_{01}^1 = 0$ if u^* is a second degree-polynomial or a_{01}^* is a constant. Using generalized Bramble-Hilbert Lemma, we obtain:

$$|\eta_{01}^1| \le \frac{C}{h^2} |a_{01}^*|_{\mathbf{W}_p^\alpha(E_1)} \cdot |u^*|_{\mathbf{W}_q^\beta(E_1)} \quad \text{for } 0 < \alpha \le 1,\ 2 \le \beta \le 3.$$

By the regularity hypothesis we have:

$$|\eta_{01}^1| \le \frac{C\, h^{\alpha+\beta-1}}{h^2} \|a_{01}\|_{\mathbf{W}_p^\alpha(e_1)} \cdot \|u\|_{\mathbf{W}_q^\beta(e_1)}. \tag{27}$$

Summating over the nodes of the mesh ω one obtain

$$\|\eta_{01}^1\|_{\mathbf{L}^2(\omega)} = h\left[\sum_{x\in\omega}(\eta_{01}^1(x))^2\right]^{1/2} \leq Ch^{\alpha+\beta-2}\|a_{01}\|_{\mathbf{W}_p^{\alpha}(\Omega)}\cdot\|u\|_{\mathbf{W}_q^{\beta}(\Omega)}.$$

Setting $\alpha = s-2, \beta = 2, q = 2p/(p-2)$ from the imbedding Sobolev Theorem we have:

$$\|\eta_{01}^1\|_{\mathbf{L}^2(\omega)} \leq Ch^{s-2}\|a_{01}\|_{\mathbf{W}_p^{s-2}(\Omega)}\cdot\|u\|_{\mathbf{W}_2^s(\Omega)}. \tag{28}$$

Similar estimates hold for $\eta_{00}^1, \eta_{02}^1, \eta_{10}^1, \eta_{11}^1, \eta_{12}^1, \eta_{ik}^2, \eta_{ik}^4, \eta^1$ and η^2 for $i = 0,1$; $k = 0,1,2$.

η_{ik}^3 is a bounded bilinear form of $(a_{ik}, u) \in \mathbf{C}(e_i) \times \mathbf{W}_2^s(e_i)$ which vanishes if u is a second degree polynomial. We obtain for these bilinear functionals an estimate of the form (28).

Similarly $\eta_0(x)$ is a bounded bilinear form of $(a_0, u) \in \mathbf{L}^q(e_0) \times \mathbf{W}_{2q/(q-2)}^{s-1}(e_0)$ which vanishes if u is a first degree polynomial, where $2 < q < \min\{2+\epsilon, \frac{2}{s-2}\}$. We obtain for η_0 an estimate of the form:

$$\|\eta_0\|_{\mathbf{L}^2(\omega)} \leq Ch^{s-2}\|a_0\|_{\mathbf{L}^{2+\epsilon}(\Omega)}\|u\|_{\mathbf{W}_2^s(\Omega)}.$$

Thus we proved Theorem 2.

Remark 1. As usual, the same letter C stands for various constants independent of h and the various functions involved.

Remark 2. Starting from the same principle of approximating the identity (8) we can construct other difference schemes. They are different from (11) by the way of replacement of the derivatives $\frac{\partial^2}{\partial x_1^{2-k}\partial x_2^k}$ by the difference quotients.

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