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ON COMPLEMENTARY 2–FORMS OF POISSON–LIE GROUPS

BY

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1. Introduction. In this Note, we assume that the reader is familiar with Poisson manifolds and Poisson-Lie groups, at the level of monograph [6], for instance, and our framework is the C^∞ category. Thus, a *Poisson structure* of a manifold M is a bivector field P such that the Schouten-Nijenhuis bracket $[P,P]=0$, and $C^\infty(M)$ becomes a Lie algebra with the bracket $\{f,g\} = P(df,dg)(f,g \in C^\infty(M))$. A *Poisson mapping* $\varphi : (M_1,P_1) \longrightarrow (M_2,P_2)$ is a mapping such that P_1 and P_2 are φ–related. A Poisson structure w of a Lie group G is a *Poisson-Lie structure* if the multiplication of G is a Poisson mapping or, equivalently, if w is *multiplicative* on G (e.g.,[6], Chapter 10). The *Poisson-Lie groups* are important because they can be seen as the classical limit of *quantum groups*. We also recall that the Poisson-Lie groups (G,w) have the fundamental property that their Lie algebra $\mathcal{G}$ is a *Lie bialgebra*, which means that w produces a certain canonical Lie algebra structure $[\alpha,\beta]^*$ of the dual vector space $\mathcal{G}^*$ of $\mathcal{G}$ $(\alpha,\beta \in \mathcal{G}^*)$, to be called the *w–dual structure* of $\mathcal{G}$. The connected and simply connected Lie group $\mathcal{G}^*$ which integrates $(\mathcal{G},[\cdot,\cdot]^*)$ is called the *w–dual group* of G. (For instance, see [6] again.)

Following [8], if (M,P) is a Poisson manifold, a 2–form of M is complementary to P if it satisfies the condition

$$\{\omega,\omega\} = 0, \tag{1.1}$$

where the bracket is the natural Schouten–Nijenhuis extension of the bracket $\{df,dg\} = d\{f,g\}$ ([3];[6], Section 4.6). The main property of ω is that it leads to a *new Lie bracket* of vector fields on M, given by [8]

$$[X,Y]' = [X,BY] + [BX,Y] - B[X,Y] + \#_P i(X)i(Y)d\omega, \tag{1.2}$$

where X, Y are vector fields of M, $\#_P : T^*M \longrightarrow TM$ is defined by $< \#_P\alpha, \beta >= P(\alpha, \beta)$, and $B : \#_P \circ \flat_\omega$ ($\flat_\omega X := i(X)\omega$). Furthermore, if one also has $d\omega = 0$, (P, B) is a *Poisson-Nijenhuis structure of M* [5], [7], and this fact relates the notion of a complementary 2-form to the theory of integrable Hamiltonian dynamical systems [5].

In particular, it is natural to study complementary 2-forms of Poisson-Lie groups (G, w), and we shall give a few results about this subject here. Namely, we will discuss left (right) invariant complementary 2-forms, and show that the bracket (1.2) associated with them yields a new Lie algebra structure on the Lie algebra $\mathcal{G}$ of G, which is dual to $(\mathcal{G}, [\cdot,\cdot]^*)$ with respect to a certain Poisson-Lie structure of the w-dual group G^* of G.

2. Invariant complementary forms. First we give a new expression of the general racket(1.2).

Proposition 2.1. *For any Poisson manifold (M, P)., and any complementary 2-form ω of P, the Lie bracket (1.2) of two vector fields X, Y of M is equal to*

$$[X, Y]' = \#_{L_X P} i(Y)\omega - \#_{L_Y P} i(X)\omega - \#_P(d(\omega(X, Y))). \tag{2.1}$$

Proof. In this formula, and in the whole paper, $\#$ and $\flat$ are always defined as for P and ω in Section 1.

Since $B = \#_P \circ \flat\omega$, we have $BX = \#_P i(X)\omega$, whence

$$\begin{aligned}[X, BY] = L_X(BY) = \#_{L_X P} i(Y)\omega + \#_P i([X, Y])\omega + \#_P(Y) L_X\omega = \\ = \#_{L_X P} i(Y)\omega + \#_P L_X i(Y)\omega.\end{aligned}$$

Using these, (1.2) becomes

$$\begin{aligned}[X, Y]' = \#_{L_X P} i(Y)w - \#_{L_Y P} i(X)w + \\ + \#_P(L_X i(Y)w - L_Y i(X)w - i([X, Y])w + i(X)i(Y)dw),\end{aligned}$$

and by applying the classical formulas of $i([X, Y])$ and L_X, we obtain (2.1) Q.e.d.

Now, we are ready for the main result.

Theorem 2.2. *Let (G, w) be a connected Poisson-Lie group, with the Lie algebra $\mathcal{G}$. Then, a left (right) invariant 2-form is a complementary 2-form of w iff $\theta := \Theta_e$ (e = the unit of G) is a solution of the classical Yang-Baxter equation of the w-dual Lie algebra $\mathcal{G}^*$. If this happens, the Lie bracket (1.2) of left (right) invariant vector fields is left (right) invariant, and it defines a Lie algebra structure of $\mathcal{G}$, which is equal to the Θ^*-dual structure of $\mathcal{G}^*$, where $\Theta^* := \theta^* - \tilde{\theta}^*$, and θ^* ($\tilde{\theta}^*$) is the left (right) invariant bivector field of the dual group G^* of G, generated by $\theta \in \wedge^2\mathcal{G}^*$.*

Proof. In this paper, we always denote by a bar left invariant objects, and by a tilde right invariant objects.

On a connected Poisson-Lie group, the bracket of two left (right) invariant 1-forms $\bar{\alpha}, \bar{\beta}$ ($\tilde{\alpha}, \tilde{\beta}$), defined by $\alpha, \beta \in \mathcal{G}$, is the left (right) invariant form $\overline{[\alpha, \beta]^*}$ ($\widetilde{[\alpha, \beta]^*}$) ([1]; [6], Proposition 10.7 and Formula (10.25$'$)). Hence, by using left (right) invariant bases, it follows that the same result is true for the general Schouten-Nijenhuis bracket of forms, and, in the case of the theorem, we have $\{\Theta, \Theta\} = \overline{[\theta, \theta]^*}$ $(= \widetilde{[\theta, \theta]^*})$. Therefore, $\{\Theta, \Theta\} = 0$ iff

$$[\theta, \theta]^* = 0, \tag{2.2}$$

which is exactly the *classical Yang-Baxter equation* of $\mathcal{G}^*$ (e.g., [6], Section 10.4).

Furthemore, in this case (2.1) becomes

$$[X, Y]' = \#_{L_X w} i(Y)\bar{\theta} - \#_{L_Y w} i(X)\bar{\theta}, \quad (X, Y \in \mathcal{G}), \tag{2.3}$$

since $\theta(X, Y) = const$. And, since the multiplicativity of w implies that $L_X w$, $L_Y w$ are left invariant fields ([4]; [6], Proposition 10.5), the result of (2.3) is a left invariant vector field. Clearly, a similar result holds in the right invariant case.

At the unit element $e \in G$, (2.3) yields a Lie algebra bracket $[X, Y]'$ on $\mathcal{G}$ such that, $\forall \alpha \in \mathcal{G}^*$, one has

$$\begin{aligned}< [X, Y]', \alpha >= (L_X w)_e(i(Y)\theta, \alpha) - (L_Y w)_e(i(X)\theta, \alpha) = \\ =< X, [i(Y)\theta, \alpha]^* > - < Y, [i(X)\theta, \alpha]^* >,\end{aligned} \tag{2.4}$$

where the last equality follows from the usual definition of the star bracket in the theory of the Poisson-Lie groups (e.g., [6], Section 10.2).

Formula (2.4) is equivalent to

$$< [X, Y]', \alpha >= -\theta(coad^*_\alpha X, Y) - \theta(X, coad^*_\alpha Y), \tag{2.5}$$

where we use the coadjoint representation of the w-dual algebra $\mathcal{G}^*$, and, in the theory of the Yang-Baxter equation, it is known that formula (2.5) provides the Θ^*-dual of$[\cdot,\cdot]^*$ (e.g., [6], Formula (10.31)). Q.e.d.

For the configuration of the Theorem 2.2, we shall denote by $\mathcal{G}_\theta$ the Lie algebra defined by (2.5), and by G_θ the corresponding connected and simply connected Lie group. These structures will be called the θ-biduals of $\mathcal{G}$ and G, respectively.

Remarks 2.3. 1) Formula (2.4) can be red also as

$$[X,Y]' = coad^*_{\sharp(X)\theta}(Y) - coad^*_{\sharp(Y)\theta}(X). \tag{2.6}$$

2) The Lie algebra $(\mathcal{G}, [\cdot,\cdot]')$ is abelian iff $\theta \in \wedge^2\mathcal{G}^*$ is a $coad^*$ invariant form.
3) The group G_θ has a natural Poisson action on G. Indeed, it is known from [8] that $\overline{[X,Y]'} = [X,Y]' = [X,Y]$. Hence, $X \mapsto X$ $(X \in \mathcal{G})$ is an infinitesimal action of G_θ on G. Furthermore, since $\forall X \in G$, $X = \#_w \flat_\theta X$, where $\flat_\theta X$ is a left invariant 1-form, it follows that X is a dressing vector field. Hence, we have a Poisson action, because of a known theorem on dressing transformations ([4]; [6], Theorem 10.31).

Some concrete examples are following.

a) take $G = Sl(2, C)$, and $\mathcal{G} = sl(2, C)$ with the basis

$$e_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ e_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and the brackets

$$[e_1, e_2] = e_3,\ [e_1, e_3] = -2e_1,\ [e_2, e_3] = 2e_2. \tag{2.7}$$

Then, $r = e_1 \wedge e_3$ is a solution of the Yang-Baxter equation of $\mathcal{G}$ (i.e., $[r,r] = 0$), and $w = \vec{r} - \overleftarrow{r}$ is a Poisson-Lie group structure of G, where the Lie algebra $\mathcal{G}^*$ of the w-dual G^* has the brackets

$$[\varepsilon^1, \varepsilon^2]^* = 2\varepsilon^2,\ \ [\varepsilon^1, \varepsilon^3]^* = 2\varepsilon^3,\ \ [\varepsilon^2, \varepsilon^3]^* = 0, \tag{2.8}$$

$(\varepsilon^1, \varepsilon^2, \varepsilon^3)$ being the dual cobasis of $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ ([2]; [6], pp. 177,171).

Now, for instance, we can take $\theta = (1/4)\varepsilon^2 \wedge \varepsilon^3$. From (2.8), we get $[\theta, \theta]^* = 0$, hence θ is a left invariant complementary 2-form of w. Using (2.6), we obtain its associated θ-bidual Lie algebra bracket:

$$[e_1, e_2]' = 0,\ [e_1, e_3]' = 0,\ [e_2, e_3]' = e_1. \tag{2.9}$$

It is well known that the equations (2.9) define the 3-dimensional Lie algebra which has a 1-dimensional central derived algebra. It can be realized by the following infinitesimal transformations of $I\!R^3$: $e_1 = (\partial/\partial x)$,

$e_2 = (\partial/\partial y)$, $e_3 = (\partial/\partial z) + y(\partial/\partial x)$, and it follows that the θ-bidual group of $Sl(2, C)$ is the group of the matrices

$$\begin{pmatrix} 1 & a & 0 & b \\ 0 & 1 & 0 & c \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix},\quad (a, b, c \in C).$$

b) The special unitar group $SU(n)$ has a well defined Poisson–Lie structure w_0 which is characterized by the fact that the w_0–dual Lie algebra $(su(n))^*$ is isomorphic to the Lie algebra $s\Delta(n, C)$ of the traceless upper triangular complex matrices with a real main diagonal ([4]; [6], p.181). This isomorphism is such that the duality pairing is expressed by

$$< \chi, \xi > = Im(trace\, \chi\xi),\quad \chi \in su(n),\ \xi \in s\Delta(n, C).$$

Now, take $\theta = \alpha \wedge \beta$, where

$$\alpha = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 & 1 \\ 0 & \lambda_2 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda_{n-1} & 0 \\ 0 & 0 & \dots & 0 & \lambda_n \end{pmatrix},\quad \beta = \begin{pmatrix} \mu_1 & 0 & \dots & 0 & 1 \\ 0 & \mu_2 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & \mu_{n-1} & 0 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

(all the missing entries are 0), and λ_i, μ_j are real numers with $\Sigma_1^n \lambda_i = 0$, $\Sigma_1^{n-1} \mu_j = 0$. Since the matrices α and β commute, we have $[\theta, \theta]^* = 0$. Hence, θ is a left invariant complementary 2–form of w_0 on $SU(n)$.

3. Closed invariant complementary forms.

Now, we shall consider the situation where Θ of Theorem 2.2 is a closed form. This case is important because, then (w, B) is a Poisson–Nijenhuis structure of G. We have

Proposition 3.1. *Let (G, w) be a connected Poisson–Lie group with the Lie algebra $\mathcal{G}$, and Θ a closed left (right) invariant complementary 2-form of w. Suppose that the derived algebra $\mathcal{G}'$ of $\mathcal{G}$ is included in the kernel of $\flat_\theta$ $(\theta = \Theta(e) \in \wedge^2\mathcal{G}^*)$. Then $= \#_w \circ \flat_\Theta$ is a multiplicative tensor field on G*

Proof. The definition of multiplicativity (e.g. [6], Section 10.1) extends to mixed tensor fields like B by the condition

$$B_{g_1 g_2} = L_{g_1^*} B_{g_2} + R_{g_2^*} B_{g_1},$$

where L_{g^*}, R_{g^*} are to be understood as L_g, R_g relatedness of tensors. Then, the usual multiplicativity conditions ([4]; [6], Proposition 10.5) continue to hold, and, in particular, since G is a connected Lie group, the multiplicativity of B is equivalent to $B(e) = 0$ and $L_{\tilde{Y}} L_{\tilde{X}} = 0$, $\forall X, Y \in \mathcal{G}$.

Since the same properties hold for w, we get $B(e) = 0$ indeed, and then

$$(3.1) \qquad L_{\tilde{Y}} L_{\tilde{X}} = \#_{L_{\tilde{Y}} w} \circ \flat_{L_{\tilde{X}} \theta}.$$

Furthermore, from (3.1) and since w is skew-symmetric we get, $\forall Z \in \mathcal{G}$ and $\forall \alpha \in \mathcal{G}^*$,

$$(3.2) \qquad < L_{\tilde{Y}} L_{\tilde{X}} B(\tilde{Z}), \tilde{\alpha} >= -(L_{\tilde{X}} \theta)(\tilde{Z}, \#_{L_{\tilde{Y}} w} \tilde{\alpha}).$$

From (3.2), it follows that $L_{\tilde{X}} \theta = 0$ implies $L_{\tilde{Y}} L_{\tilde{X}} B = 0$. But $d\theta = 0$, $L_{\tilde{X}} \bar{\theta} = 0$ hold together iff, $\forall X, Y, Z \in \mathcal{G}$, $\theta(X, [Y, Z]) = 0$.

The right invariant case is similar. Q.e.d.

Remarks 3.2. If we add the condition $\cup_{Y \in \mathcal{G}} im \#_{\varepsilon(Y)} = \mathcal{G}(\varepsilon(Y) := L_{\tilde{Y}} w(e))$ to the hypotheses of Proposition 3.1, the condition $\mathcal{G}' \subseteq \ker \flat_\theta$ becomes also a necessary condition since then, by (3.2), $L_{\tilde{Y}} L_{\tilde{X}} B = 0$ implies $L_{\tilde{X}} \bar{\theta} = 0$.

The interest of Proposition 3.1 comes from the fact that, if B is multiplicative, all the Poisson structures which belong to the Poisson hierarchy of (w, B) (e.g., [7]) are multiplicative, and G is equipped with many Poisson Lie structures.

But, of course, Proposition 3.1 is useful only if $\mathcal{G}$ has a small derived algebra $\mathcal{G}'$. For instance, if G is the additive group $\mathbb{R}^n$, the only possible Poisson–Lie structures w are the linear structures, obtained as Lie–Poisson structures of dual spaces $\mathcal{H}^*$ of n–dimensional Lie algebras $\mathcal{H}$ ($\mathbb{R}^n$ is identified with $\mathcal{H}^*$), and the w–dual algebra is exactly $\mathcal{H}$ (e.g.,[6], p.171). Then, in order to obtain invariant complementary 2–forms, we have to start with a solution $\theta \in \wedge^2 \mathcal{H}$ of the classical Yang–Baxter equation of $\mathcal{H}$. The corresponding 2-form Θ on $\mathbb{R}^n$ will have constant natural components, and it will be closed and biinvariant. The tensor B has linear natural components, and it is a multiplicative tensor field.

A less trivial example is that of a Heisenberg group H_{2n+1} i.e., a connected Lie group which has the Lie algebra $\mathcal{H}_{2n+1} := span\{X_i, Y_i, T\}$ $(i = 1, ..., n)$ with the brackets

$$(3.3) \qquad [X_i, Y_j] = \delta_{ij} Y, \; [X_i, X_j] = [Y_i, Y_j] = [X_i, T] = [Y_i, T] = 0.$$

(Notice that the Lie algebra (2.9) was of this kind.)

Then, $r = \Sigma_{i=1}^n X_i \wedge Y_i$ satisfies the generalized Yang–Baxter equation $add[r, r] = 0$ (but $[r, r] \neq 0$), and $w = r - \bar{r}$ defines a Poisson–Lie group structure on H_{2n+1} (again, see [6], Section 10.4, for instance).

It is easy to compute the w–dual $\mathcal{H}^*$ of $\mathcal{H}$ by using formula (10.13) of [6], which is equivalent to

$$(3.4) \qquad [\alpha, \beta]^* = coad_{r(\alpha)}\beta - coad_{r(\beta)}\alpha \; (\alpha, \beta \in \mathcal{H}^*).$$

Namely, if (ξ^i, η^i, τ) is the dual cobasis of (X_i, Y_i, T), we get

$$(3.5) \qquad [\xi^i, \tau]^* = \xi^i, \; [\eta^i, \tau]^* = \tau^i, \; [\xi^i, \xi^j]^* = [\xi^i, \eta^j]^* = [\eta^i, \eta^j]^* = 0.$$

Now, we see that $\theta = \Sigma_{i=1}^n \xi^i \wedge \eta^i$ is a solution of the Yang–Baxter equation $[\theta, \theta]^* = 0$ such that the corresponding left invariant form $\bar{\theta}$ of H is closed. In view of (3.3), the derived algebra $\mathcal{H}' = span\{T\} \subseteq \ker \theta$, and we are in a situation where Proposition 3.1 holds.

4. More general complementary forms.

We finish by a few remarks concerning other types of complementary 2–forms on Lie groups.

a) it is known that, if G is a Lie group and P is a right (left) invariant Poisson structure of G, any left (right) invariant 2–form ω of G is a complementary form of P [5], [8]. Formula (2.1) shows that the new bracket of left (right) invariant vector fields in zero in this case.

b) Let (G, w) be a Poisson–Lie group. In principle, it is interesting to study complementary 2–forms Θ of w on G such that the associated bracket $[\cdot, \cdot]'$ of left invariant vector fields is left invariant, while the form Θ itself is not necessarily invariant. If we assume that G is connected, this condition means $L_{\tilde{Z}}[\tilde{X}, \tilde{Y}]' = 0$, $\forall X, Y, Z \in \mathcal{G}$. Using (2.1), we see that Θ must be such that

$$(4.1) \qquad \begin{gathered} \#_{L_{\tilde{X}} w} i(\tilde{Y}) L_{\tilde{Z}} \Theta - \#_{L_{\tilde{Y}} w} i(\tilde{X}) L_{\tilde{Z}} \Theta - \\ - \#_{L_{\tilde{Z}} w}(d(\Theta(\tilde{X}, \tilde{Y}))) - \#_w d((L_{\tilde{Z}} \Theta)(\tilde{X}, \tilde{Y})) = 0. \end{gathered}$$

Unfortunately, condition (4.1) is not very illuminating.

c) Another natural question is that of finding multiplicative complementary 2–forms of a Poisson–Lie group (G, w). The simplest idea is that of trying to find a corresponding type of a generalized Yang–Baxter equation.

Namely, if $\theta \in \wedge^2 \mathcal{G}^*$, and $\Theta = \bar{\theta} - \theta$, then $\Theta(e) = 0$ and $L_{\tilde{Y}} L_{\tilde{X}} \Theta = 0$, $\forall X, Y \in \mathcal{G}$, hence, if G is connected, Θ is a multiplicative 2–form, and we should express the condition $\{\Theta,\Theta\}_w = 0$, i.e., in view of the explanation at the beginning of the proof of Theorem 2.2.

$$\overline{[\theta\theta]^*} + \widetilde{[\theta,\theta]^*} - 2\{\tilde\theta,\tilde\theta\}_w = 0. \tag{4.2}$$

In order to express the last term of (4.2), we need to express the w-bracket (e.g., [6], formula (4.6))

$$\{\tilde\alpha,\tilde\beta\} = d(w(\alpha,\tilde\beta)) + i(\#_w\alpha)d\tilde\beta - i(\#_w\tilde\beta)d\tilde\alpha, \tag{4.3}$$

for $\alpha,\beta \in \mathcal{G}^*$. Using that $\forall X \in \mathcal{G}$, $L_x w$ is left invariant, and that L_X anihilates right invariants objects, we get first

$$\begin{aligned} i(\bar X)\{\tilde\alpha,\tilde\beta\} &= (L_X w)(\tilde\alpha,\tilde\beta) - w(\tilde\alpha, i(\bar X)d\tilde\beta) = \\ &= (L_X w)(\tilde\alpha,\tilde\beta) + (\#_w\tilde\alpha)(\tilde\beta(\bar X)). \end{aligned} \tag{4.4}$$

Then, the evaluation of (4.4) at $g \in \mathcal{G}$ using the corresponding values at the unit point e of G gives

$$\begin{aligned} \{\tilde\alpha,\tilde\beta\}_g(\bar X_g) &=< L_g^*\{\tilde\alpha,\tilde\beta\}_e, X >= \\ &= [\alpha, Coad_{g^{-1}}\beta]^*(X) + (\$_{w_l(g)}\alpha)((\tilde\beta(\bar X))\circ L_g), \end{aligned} \tag{4.5}$$

where $w_l(g) := L_{g^{-1}}(w(g))$. In (4.5), the last term is the directional derivative of the function

$$\tilde\beta(\bar X)\circ L_g(\gamma) =< Coad_{\gamma^{-1}} Coad_{g^{-1}}\beta, X >,$$

where g is fixed, and γ is the variable in G. Hence, this derivative is $coad_{\#_{w_l(g)}\alpha}$, and (4.5) yields

$$\{\tilde\alpha,\tilde\beta\}(g) = L^*_{g^{-1}}(ad^*_\alpha\, Coad_{g^{-1}}\beta - coad_{\#_{w_l(g)}\alpha} Coad_{g^{-1}}\beta). \tag{4.6}$$

Now, the operators involved in (4.6) have natural extensions to forms of a higher degree. Hence, a similar formula will hold for $[\tilde\theta,\tilde\theta]$ of (4.2), and (4.2) will be equivalent to

$$\begin{aligned} &[\theta,\theta]^* + Coad_{g^{-1}}[\theta,\theta]^* - 2[\theta, Coad_{g^{-1}}\theta]^* - \\ &- coad_{\#_{w_l(g)}\alpha} Coad_{g^{-1}}\theta = 0, \ \forall g \in \mathcal{G}. \end{aligned} \tag{4.7}$$

Unfortunately again, condition (4.7) is not very illuminating either. But, the result (4.6) obtained on the way is an interesting formula for Poisson–Lie groups.

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