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LINEAR OPERATORS AND MATRICES IN H-LOCALLY CONVEX SPACES WITH AN ORTHOGONAL BASIS

BY

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1. Introduction Let X be a complex locally convex space whose topology is generated by a family of sufficient Hilbertian seminorms $P=\{p_\alpha, \alpha\in\Delta\}$ where Δ is some directed index set. Such a space which is also complete we shall call an H-locally convex space [2], [4] (H.-l.c). We denote by $(\ ,\ )$ the corresponding semiscalar products, $\alpha\in\Delta$. We shall suppose that in such a space X, there exists a countable orthogonal basis [5], that is a system of mutually orthogonal vectors $\{e_i,\ i\in N\}$ with the property that for each $x\in X$ there is a unique number sequence $\{x_j\}$, such that $x=\sum\limits_{j=1}^{\infty} x_j e_j$, where the convergence is considered with respect to the seminorms P. For such an expansion, holds for example [5]: $x_j=(x, e_j)_\alpha/p_\alpha^2(e_j)$, $j\in N$, for each $\alpha\in\Delta$, for which $p_\alpha(e_j)\neq 0$.

We shall give the connection between continuity respectively quotient boundedness of some linear operators and their representation by a matrix in such a space. In the second part, we treat the existence of the adjoint operator.

2. On a representation of a linear operator. Let A be a linear operator on an H-l.c. space with a countable orthogonal basis $\{e_i\}$. We shall say that A has a *matrix representation* if there is a set of numbers (a_{ij}); $i, j\in N$, such that for each $x=\sum\limits_{j=1}^{\infty}x_je_j$ the series $\sum\limits_{j=1}^{\infty}a_{ij}x_j$ converges for any $i\in N$ and the following relation holds:

$$(Ax)_i=\sum_{j=1}^{\infty}a_{ij}x_j,\ i\in N. \tag{2.1}$$

Theorem 2.1. *Each continuous linear operator on an* H-l.c. *space with a countable orthogonal basis has a matrix representation.*

Proof. Let $A\in L(X)$, then there is a unique matrix (a_{ij}) such that $Ae_j=\sum\limits_{j=1}^{\infty}a_{ij}e_i$, $j\in N$. Since A is continuous and linear for any $x=\sum\limits_{j=1}^{\infty}x_je_j\in X$ we have $Ax=\sum\limits_{j=1}^{\infty}x_jAe_j$. Choose any $i\in N$ and $\alpha\in\Delta$ such that $p_\alpha(e_i)\neq 0$, then

$$(Ax)_i=(Ax, e_i)_\alpha/p_\alpha^2(e_i)=\lim_{n\to\infty}\sum_{j=1}^{n}x_j(Ae_j, e_i)_\alpha/p_\alpha^2(e_i)=\sum_{j=1}^{\infty}a_{ij}x_j.$$

This completes the proof. With some restriction we shall prove the converse assertion. First we prove:

Lemma 2.2. *Let X be an H-l.c. space with a countable basis $\{e_i\}$. If for any $x=\sum_{j=1}^{\infty}x_je_j$ the number series $\sum_{j=1}^{\infty}a_jx_j$ converges, then $f(x)=\sum_{j=1}^{\infty}a_jx_j$ is a continuous linear functional on X.*

Proof. For a given $x=\sum_{j=1}^{\infty}x_je_j\in X$, define the sequence $y=(y_k)$ of numbers $y_k=\sum_{j=1}^{k}a_jx_j$, $k=1, 2,\dots$. By assumption, it is convergent and hence bounded. Denote $y=Tx$, where T is a linear mapping from X to the space l_∞ (with the sup. norm). Let us prove that T is closed. Choose a net $\{x^\delta, \delta\in\Gamma\}$ such that $\lim_{\delta\in\Gamma}x^\delta=x$ and $\lim_{\delta\in\Gamma}Tx^\delta=z$, where Γ is some directed index set. We have $x^\delta=\sum_{j=1}^{\infty}x_j^\delta e_j$ for each $\delta\in\Gamma$. From the convergence $\lim_{\delta\in\Gamma}x^\delta=x$ and the orthogonality of the basic vectors, it follows $\lim_{\delta\in\Gamma}x_j^\delta=x_j$ for each $j\in N$. Thus, it is obvious that $\lim_{\delta\in\Gamma}\sum_{j=1}^{k}a_jx_j^\delta=\sum_{j=1}^{k}a_jx_j$, for each $k\in N$. On the other hand from the convergence $\lim_{\delta\in\Gamma}Tx^\delta=z$ in l_∞ for each $k\in N$, we have $\lim_{\delta\in\Gamma}\left|\sum_{j=1}^{k}a_jx_j-z_k\right|=0$ and consequently $\sum_{j=1}^{k}a_jx_j=z_k$, for each $k\in N$, hence $Tx=z$, thus, T is closed. Since X is reflexive it is barreled [5], [6] and from the closed graph theorem, T is continuous ([7]). There exist $p_\gamma\in P$ and $c\geqslant 0$ such that $\left|\sum_{j=1}^{k}a_jx_j\right|\leqslant\|Tx\|_\infty\leqslant c.p_\gamma(x)$, $k\in N$, $x\in X$ and hence $\left|\sum_{j=1}^{\infty}a_jx_j\right|\leqslant c.p_\gamma(x)$, $x\in X$.

Let, now, X be an H-l.c. space with countable seminorms.

Theorem 2.3. *In a Fréchet H-l.c. space with a countable orthogonal basis each linear operator which can be represented by a matrix, is continuous.*

Proof. Let (a_{ij}) ; $i, j\in N$ be a matrix which by (2.1) defines a linear operator A on X. Let us prove that A is closed. Choose a sequence $\{x^n\}\subset X$ with the property $\lim_{n\to\infty}x^n=x$ and $\lim_{n\to\infty}Ax^n=z$ in X. Since $y_i=(Ax)_i=\sum_{j=1}^{\infty}a_{ij}x_j$, $i\in N$, $x\in X$, are, by the previous lemma, continuous linear functionals on X, we have for each $i\in N$

$$\lim_{n\to\infty}(Ax^n)_i=\lim_{n\to\infty}\sum_{j=1}^{\infty}a_{ij}x_j^n=\sum_{j=1}^{\infty}a_{ij}x_j=(Ax)_i\,;$$

on the other hand for each coordinate, we have $\lim_{n\to\infty}(Ax^n)_i=z_i$, hence $Ax=z$, A is closed and from the closed graph theorem it is continuous.

We shall consider the conditions on an infinite matrix under which it defines a quotient bounded operator. An operator T is quotient bounded ($T\in Q_P(X)$) iff $p_\alpha(Tx)\leqslant c_\alpha p_\alpha(x)$, for each $\alpha\in\Delta$ and $x\in X$ ([1]), [3]).

Theorem 2.4. *Let X be an H-l.c. space with an orthogonal basis $\{e_i, i\in N\}$. Then a matrix (a_{ij}), $i, j\in N$ defines an operator $A\in Q_P(X)$ if and only if the following condition is satisfied:*

For each $p_\alpha\in P$, there is some $M_\alpha>0$, such that for any $n\in N$ and arbitrary $x_1, x_2, \dots, x_n\in C$, the following inequality holds

$$\left|\sum_{i=1}^{n}\sum_{j=1}^{n}\bar{x}_ip_\alpha^2(e_i)a_{ij}x_j\right|\leqslant M_\alpha\sum_{i=1}^{n}|x_i|^2p_\alpha^2(e_i). \tag{2.2}$$

Proof. Assume that (a_{ij}) defines an operator $A\in Q_P(X)$. Then for any $x\in X$ and $\alpha\in\Delta$: $|(Ax, x)_\alpha|\leqslant M_\alpha p_\alpha^2(x)$. In particular for $x=\sum_{j=1}^{n}x_je_j$ we obtain (2.2). We shall prove the sufficiency of this condition in some steps. First we shall state that for an arbitrary n-tuple $x_1, x_2, \dots, x_n\in C$ and for any $\alpha\in\Delta$, the following inequality holds

$$\sum_{i=1}^{n}p_\alpha^2(e_i)\left|\sum_{j=1}^{n}a_{ij}x_j\right|^2\leqslant 4M_\alpha^2\sum_{j=1}^{n}|x_j|^2p_\alpha^2(e_j). \tag{2.3}$$

It is easy to check that $L_n(x, y)=\sum_{i=1}^{n}\sum_{j=1}^{n}\bar{y}_ia_{ij}x_jp_\alpha^2(e_i)$, where $x=\sum_{j=1}^{n}x_je_j$, $y=\sum_{j=1}^{n}y_je_j$, is a sesquilinear form. By the above assumption, for any $z=\sum_{j=1}^{n}z_je_j$, the following estimation holds: $|L_n(z, z)|\leqslant M_\alpha\sum_{j=1}^{n}|z_j|^2p_\alpha^2(e_j)=M_\alpha p_\alpha^2(z)$ and consequently

$$|L_n(x, y)|\leqslant M_\alpha(p_\alpha^2(x)+p_\alpha^2(y)), \tag{2.4}$$

where we took into account that the seminorms are Hilbertian. If $p_\alpha(x)=p_\alpha(y)=1$, then $|L_n(x, y)|\leqslant 2M_\alpha$ and hence $|L_n(x, y)|\leqslant 2M_\alpha p_\alpha(x)\,p_\alpha(y)$, for any $p_\alpha\in P$ such that $p_\alpha(x)\neq 0$ and $p_\alpha(y)\neq 0$. This inequality trivially holds if $p_\alpha(x)=0$ or $p_\alpha(y)=0$. Namely, if $p_\alpha(x)=0$ then by (2.4) $|L_n(x, y)|\leqslant M_\alpha p_\alpha(y)$ for each x with the above property, hence $L_n(x, y)=0$ and similarly if $p_\alpha(y)=0$. Thus, we have

$$\left|\sum_{i=1}^{n}p_\alpha^2(e_i)\bar{y}_i\sum_{j=1}^{n}a_{ij}x_j\right|^2\leqslant 4M_\alpha^2\sum_{j=1}^{n}|x_j|^2p_\alpha^2(e_j)\sum_{j=1}^{n}|y_j|^2p_\alpha^2(e_j) \tag{2.5}$$

If we take $y_i=\sum_{j=1}^{n}a_{ij}x_j$, $i=1, 2,\dots$, where $x_1, x_2,\dots, x_n\in C$ are arbitrary, then by a simple consideration, we obtain (2.3). Let us, now, prove that for any $x=\sum_{j=1}^{\infty}x_je_j\in X$ the number series $\sum_{j=1}^{\infty}a_{ij}x_j$ are convergent for each $i\in N$. Suppose conversely, that there are some $z=\sum_{j=1}^{\infty}z_je_j\in X$ and $m\in N$ such that the series $\sum_{j=1}^{\infty}a_{mj}z_j$ is divergent, then $\sum_{j=1}^{\infty}|a_{mj}z_j|=\infty$. Let $p_{\alpha_m}\in P$ be such that $p_{\alpha_m}(e_m)\neq 0$. For each $j\in N$, let $t_j\in R$ be such that $a_{mj}z_j=|a_{mj}z_j|e^{it_j}$ where $i=\sqrt{-1}$. Using (2.3) for $n>m$ and $x=\sum_{j=1}^{n}z_je^{-it_j}e_j$ one can estimate $\left(p_{\alpha_m}(e_m)\,.\sum_{j=1}^{n}|a_{mj}z_j|\right)^2\leqslant 4M_\alpha^2\,.\sum_{j=1}^{n}|z_j|^2p_{\alpha_m}^2(e_j)$ and if $n\to\infty$ it is a contradiction with $p_{\alpha_m}(z)<\infty$. In the next step we shall prove the following inequality

$$\sum_{i=1}^{n}p_\alpha^2(e_i)\left|\sum_{j=1}^{\infty}a_{ij}x_j\right|^2\leqslant 4M_\alpha^2p_\alpha^2(x)\,;\quad x\in X,\ \alpha\in\Delta. \tag{2.6}$$

Suppose that for some $x=\sum_{j=1}^{\infty}x_j e_j \in X$, $n\in N$ and $p_\alpha \in P$ (2.6) is not valid. Then there is some integer m, we may take $m>n$, such that $\sum_{i=1}^{n} p_\alpha^2(e_i)\cdot|\sum_{j=1}^{m} a_{ij}x_j|^2 > 4M_\alpha^2 p_\alpha^2(x)$. But, then such an inequality holds also for $n=m$, and this in contradiction with (2.3). Let in (2.6) $n\to\infty$ and $y_i=\sum_{i=1}^{\infty}a_{ij}x_j$, $i=1,2,3,\dots$, then $p_\alpha^2(y)=p_\alpha^2(Ax)\leqslant 4M_\alpha^2 p_\alpha^2(x)$, $x\in X$, $\alpha\in\Delta$, hence $y\in X$ and $A\in Q_P(X)$.

With somewhat stronger conditions on the matrix (a_{ij}), which are easier to check, we can prove the following.

Theorem 2.5. *Let X be an H-l.c. space with a countable orthogonal basis $\{e_i\}$ and let for (a_{ij}) the following two conditions hold*

(i) $\sum_{j=1}^{\infty}|a_{ij}|\leqslant C$, *for all* $i\in N$,

(ii) $\sum_{i=1}^{\infty}|a_{ij}|\, p_\alpha^2(e_i)\leqslant D_\alpha p_\alpha^2(e_j)$, *for all* $j\in N$ *and* $\alpha\in\Delta$.

Then the matrix (a_{ij}) generates a quotient bounded operator A on X.

Proof. Choose arbitrary $n\in N$, $p_\alpha\in P$ and $x=\sum_{j=1}^{n}x_j e_j$, $y=\sum_{j=1}^{n}y_j e_j$ such that $p_\alpha(x)=p_\alpha(y)=1$, then $2|\sum_{i=1}^{n}\sum_{j=1}^{n}\bar{y}_i a_{ij}x_j p_\alpha^2(e_i)|\leqslant|\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}p_\alpha^2(e_i)\cdot(|y_i|^2+|x_j|^2)\leqslant \leqslant C+D_\alpha=2M_\alpha$ and for $p_\alpha(x)\neq 0$, $p_\alpha(y)\neq 0$ it follows

$$|\sum_{i=1}^{n}p_\alpha^2(e_i)\bar{y}_i\sum_{j=1}^{n}a_{ij}x_j|^2\leqslant M_\alpha^2\sum_{i=1}^{n}|x_i|^2 p_\alpha^2(e_i)\sum_{i=1}^{n}|y_i|^2 p_\alpha^2(e_i).$$

This inequality is valid also when $p_\alpha(x)=0$ or $p_\alpha(y)=0$, what can be verified in the same manner as for (2.5). Thus also (2.3) holds and from this point, the proof is same as that one of Theorem 2.4.

Let us denote with $B_P(X)$ the family of uniformly bounded linear operators on X, $A\in B_P(X)$ iff $p_\alpha(Ax)\leqslant Cp_\alpha(x)$, $x\in X$, $\alpha\in\Delta$, where C is independant of $\alpha\in\Delta$. By the preceding two theorems there follows immediately.

Corollary 2.6. *Let X be an H-l.c. space with a countable orthogonal basis $\{e_i,\ i\in N\}$, then:*

(a) *a matrix (a_{ij}) determines an operator $A\in B_P(X)$ iff the condition (2.2) is satisfied, where $M=M_\alpha$ does not depend on $\alpha\in\Delta$,*

(b) *if a matrix (a_{ij}) satisfies the conditions of Theorem 2.5, where $D=D_\alpha$ does not depend on $\alpha\in\Delta$, then it determines an operator $A\in B_P(X)$.*

3. On the existence of the adjoint operator. In [3] we defined for a linear operator A on an H-l.c. space X the adjoint operator A° by the relations

$$(Ax,y)_\alpha=(x,A^\circ y)_\alpha,\quad x,y\in X,\ \alpha\in\Delta.$$

We noted that the adjoint do not exist for all continuous operators. Our interest will be now on the question under what conditions the adjoint operator exists in an H-l.c. space with a countable orthogonal basis.

Suppose that for $A\in L(X)$ there exists an adjoint operator $A^\circ\in L(X)$. With respect to the orthogonal basis $\{e_i\}$, there exists by Theorem 2.1 two matrices (a_{ij}) and (b_{ij}) which we can estimate from the formulas

$$a_{ij}=(Ae_j,e_i)_\alpha/p_\alpha^2(e_i),\quad b_{ij}=(A^\circ e_j,e_i)_\alpha/p_\alpha^2(e_i)\ ;\quad i,j\in N. \tag{3.1}$$

where for each $i\in N$, $\alpha\in\Delta$ is such that $p_\alpha(e_i)\neq 0$. We shall give a characterisation of the existence of the adjoint operator in terms of the following generalised sequences

$$s_i=(p_\alpha(e_i),\ \alpha\in\Delta)\ ;\quad i=1,2,\dots$$

depending of the orthogonal basis and the system of seminorms on X.

Theorem 3.1. *Let X be an H-l.c. space with an orthogonal basis $\{e_i,\ i\in N\}$. If for $A\in L(X)$ there exists the adjoint operator $A^\circ\in L(X)$, then for the corresponding matrices (a_{ij}) and (b_{ij}) it follows: if $s_i\neq\lambda s_j$, $\lambda\neq 0$, then $a_{ij}=a_{ji}=b_{ij}=b_{ji}=0$.*

Proof. The following equations hold

$$a_{ij}p_\alpha^2(e_i)=\bar{b}_{ji}p_\alpha^2(e_j)\ ;\quad i,j\in N,\ \alpha\in\Delta. \tag{3.2}$$

If $p_\alpha(e_i)\neq 0$, $p_\alpha(e_j)\neq 0$ they are a consequence of (3.1) and the relations $(Ae_j,e_i)_\alpha=(e_j,A^\circ e_i)_\alpha$, in all other cases (3.2) is trivially true. Suppose, now, that $a_{ij}\neq 0$ for some $i,j\in N$, then by (3.2) $p_\alpha^2(e_i)=p_\alpha^2(e_j)\bar{b}_{ji}/a_{ij}$ for each $\alpha\in\Delta$, thus $s_i=\lambda s_j$. The same conclusion holds if a_{ji}, b_{ij} or b_{ji} is equal to 0.

For an operator $A\in L(X)$ of diagonal form with matrix (a_{ij}), where $a_{ij}=0$ if $i\neq j$, clearly the adjoint operator exists. But, there is a question if another operator has an adjoint. First we have:

Corollary 3.2. *Let X be an H-l.c. space with the orthogonal basis $\{e_i,\ i\in N\}$. If there exists no pair of indices $i,j\in N$, $i\neq j$, such that $s_i=\lambda s_j$, then only for diagonal type operators, the adjoint exists.*

In a non normable H-l.c. space with a countable orthogonal basis the set of operators which has an adjoint is not large, namely we have.

Corollary 3.3. *If in a H-l.c. space with a countable orthogonal basis each finite range operator has an adjoint, then the space is equivalent to a Hilbert space.*

Proof. Let $\{e_i,\ i\in N)$ be an orthogonal basis in X. Choose arbitrary $r,k\in N$ and $\alpha_r\in\Delta$ such that $p_{\alpha_r}(e_r)\neq 0$. Define $Ax=(x,e_r)_{\alpha_r}e_k$. Clearly $A\in L(X)$ and is finite, their matrix (a_{ij}) satisfies the relations: $a_{kr}=p_{\alpha_r}^2(e_r)\neq 0$ and $a_{ij}=0$ for $i\neq k$ and $j\neq r$. Since A has an adjoint by Theorem 3.1 $s_k=\lambda s_r$. As $r,k\in N$ are arbitrary, all sequences s_i, $i\in N$, are proportional, hence $p_\alpha(e_j)=\lambda_j p_\alpha(e_1)$, $j\in N$, $\alpha\in\Delta$ and consequently $p_\alpha^2(x)=p_\alpha^2(e_1)\sum_{j=1}^{\infty}|x_j|^2\lambda_j^2$, for each $x\in X$ and $\alpha\in\Delta$. If $\alpha_1\in\Delta$ is such that $p_{\alpha_1}(e_1)\neq 0$ then $p_\alpha(x)=K_\alpha p_{\alpha_1}(x)$, $x\in X$, $\alpha\in\Delta$, where $K_\alpha=p_\alpha(e_1)/p_{\alpha_1}(e_1)$. This completes the proof.

For the quotient bounded operator we shall prove also the converse direction of the assertion in Theorem 3.1.

Theorem 3.4. *Let X be an H-l.c. space with an orthogonal basis $\{e_i,\ i\in N\}$ and (a_{ij}) the matrix of an operator $A\in Q_P(X)$. If for each pair $i,j\in N$ such that $s_i\neq\lambda s_j$, $\lambda\neq 0$: $a_{ij}=0$, then A has the adjoint operator $A^\circ\in Q_P(X)$.*

Proof. Clearly, the condition $s_i\neq\lambda s_j$ implies also $a_{ji}=0$. We shall construct the matrix (b_{ij}) in the following way: if $s_i\neq\lambda s_j$, $\lambda\neq 0$, let $b_{ij}=b_{ji}=0$, in the other case let $b_{ij}=\bar{a}_{ji}p_\beta^2(e_j)/p_\beta^2(e_i)$, here $\beta\in\Delta$ is any of those indices for which $p_\beta(e_i)\neq 0$. The matrix (a_{ij}) satisfies the condition of Theorem 2.4

and it is easy to check that the same is true for (b_{ij}). By this theorem with this matrix is defined an operator $B \in Q_P(X)$. We shall prove the relations

$$(Ae_i, e_j)_\alpha = (e_i, Be_j)_\alpha \; ; \; \alpha \in \Delta, \; i, j \in N. \tag{3.3}$$

If $p_\alpha(e_i)=0$ or $p_\alpha(e_j)=0$ they are, clearly, fulfilled, since $A, B \in Q_P(X)$. Let $p_\alpha(e_i) \neq 0$ and $p_\alpha(e_j) \neq 0$, then, from (3.1) we obtain $(Ae_i, e_j)_\alpha = a_{ji} p_\alpha^2(e_j)$ and $(e_i, Be_j)_\alpha = \bar{b}_{ij} p_\alpha^2(e_i)$. If, now, $s_i \neq \lambda s_j$, $\lambda \neq 0$, then $a_{ji}=0$ and by the construction also $b_{ij}=0$, therefore (3.3) is true. If $s_i = \lambda s_j$, then the formula for b_{ij} implies (3.3).

Choose, now, arbitrary $x, y, \in X'$; they have the expansion $x = \sum_{j=1}^{\infty} x_j e_j$, $y = \sum_{j=1}^{\infty} y_j e_j$. Since A and B are continuous, we have $Ax = \sum_{j=1}^{\infty} x_j Ae_j$ and $By = \sum_{j=1}^{\infty} y_j Be_j$. Finally, from the continuity of semiscalar products and from (3.3) we obtain $(Ax, y)_\alpha = (x, By)_\alpha$; $x, y \in X$, $\alpha \in \Delta$. Therefore, for A there exists the adjoint operator $A^\circ = B$.

Example. Let $X = s$, be the space of all complex sequences with the topology induced by the family of seminorms $P = \{p_m, m \in N\}$, where $p_m^2(x) = \sum_{j=1}^{M} |x_j|^2$, $M = 2^m - 1$, $m \in N$. Evidently, these seminorms are Hilbertian and is an H-l.c. space with the orthogonal basis $\{e_i, i \in N\}$, where $e_i = (0,0,\ldots,1,0,\ldots)$, $i \in N$. It is easy to check that the sequences s_i have the property: for each $k \in N$ all s_i-whenever $2^{k-1} \leqq i \leqq 2^k - 1$ — are proportional. For each $i \in N$ let $k_i \in N$ be such that $2^{k_i - 1} \leqq i \leqq 2^{k_i} - 1$. Denote by (a_{ij}) the matrix of an operator $A \in Q_P(X)$. Now, we have from the above theorem : if $a_{ij} = 0$ for $i, j, \in N$ such that $2^{k_i - 1} \leqq i \leqq 2^{k_i} - 1$ and $j < 2^{k_i - 1}$ or $j \geqq 2^{k_i}$ then A has an adjoint operator $A^\circ \in Q_P(X)$. In general A is not diagonal.

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METRIC SEGMENTS AND MEAN VALUE THEOREMS

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0. Introduction. Concerning the relationships between the local and global properties of vector valued functions acting on a real interval, the basic way of expressing them is offered by the so-called *mean value theorems*. From a technical viewpoint, it must be distinguished between mean value theorems under an *inequality* form (Dieudonné [5, ch. VIII, §5], Turinici [14]) and mean value theorems under an *equality* form (Sova [13], McLeod [11]) both of them having a number of important applications to analysis (see in this direction the first one of the above references). Under this perspective, it is our main aim, in the following to extend some of the results belonging to the second category of mean value theorems to *left-continuous* vector valued functions acting on *metric segments* of a convex metric space, the basic instrument of our investigations being a *maximality principle* on ordered metric structures [16] which may be deemed as an „intrinsic" version of the variational Ekeland's one [6] or (under its quasi-metric form [15]) as a common extension of both Brézis - Browder ordering principle [3] and Brondsted's maximality principle [4]. Some further extensions of this result as well as an application to monotone semigroups theory will be given elsewhere.

1. Preliminaries. Let X be a nonempty set, „d" a *metric* on X, $\leqslant$ an *ordering* (that is, a reflexive, antisymmetric and transitive relation) on X and $\geqslant$ its *dual*. For each $x \in X$, let $X(x, \leqslant)$ $(X(x, \geqslant))$ denote the subset of all $y \in X$ with $x \leqslant y$ $(x \geqslant y)$ and for each couple $x, y \in X$ with $x \leqslant y$, let $X(x, y)$ indicate the *order-interval* between x and y (the intersection between $X(x, \leqslant)$ and $X(y, \geqslant)$). A sequence $(x_n ; n \in N)$ in X will be said to be *increasing* (*decreasing*) provided $x_i \leqslant x_j (x_i \geqslant x_j)$ whenever $i \leqslant j$, and *monotone* when it is either increasing or decreasing : in this context, the ambient ordering $\leqslant$ will be termed *interval-asymptotic* in case any monotone sequence lying in an order-interval is a Cauchy one.

Lemma 1. *Let $\leqslant$ be interval-asymptotic. Then any chain C for which there is an order-interval containing it is both $\leqslant$-Cauchy and $\geqslant$-Cauchy.*

Proof. The fact that C is $\leqslant$-Cauchy may be expressed as : to any $\varepsilon > 0$ there corresponds a $x = x(\varepsilon)$ in C with $d(x, y) < \varepsilon$ for all $y \in C(x, \leqslant)$. Suppose this does not happen : then, a number $\varepsilon > 0$ may be chosen with the property : for each $x \in C$ there exists $y \in C(x, \leqslant)$ with $d(x, y) \geqslant \varepsilon$. It immediately follows an increasing sequence $(x_n ; n \in N)$ in C (hence in the ambient order-interval) may be found with $d(x_n, x_{n+1}) \geqslant \varepsilon$, $n \in N$, contra-