$$0=d^2 s_{ac}=s_{aa}d\tilde{\eta}_{b}-s_{ab}d\tilde{\eta}_{a}+ds_{aa}\wedge\tilde{\eta}_{b}-ds_{ab}\wedge\tilde{\eta}_{a}=s_{aa}(d\tilde{\eta}_{b}+\tilde{\eta}_{c}\wedge\tilde{\eta}_{a})-$$
$$-s_{ab}(d\tilde{\eta}_{a}+\tilde{\eta}_{b}\wedge\tilde{\eta}_{c})$$

are satisfied according to (48).

Thus, for the connection $\tilde{\nabla}$ and the local canonical basis F'_a we have:

$$\tilde{\nabla}_i(F'_a)^k_j=\partial_i(F'_a)^k_j=0,\ a=1,2,3;\ i,j,k=1,\dots,4m,$$

hence the local expressions of F'_a are constant. By a linear coordinate transformation we can give to the matrices associated with F'_a the usual canonical expressions. Hence the almost quaternal structure is integrable. Conversely if the structure is integrable there exist local charts such that the local expressions of F_a in these local charts are constant. For such a chart consider the local connection having trivial connections coefficients. This connection is obviously adapted with the corresponding 1-forms $\eta_a=0$. Then $T=0$, $R=0$, thus the local expressions of S and W are zero. Due to their definition, they are globally trivial.

A manifold carrying an integrable almost quaternal structure is called a *quaternion manifold*. The typical examples of quaternion manifolds are the quaternion projective spaces.

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TWO-POINT BOUNDARY PROBLEMS IN THE LINEAR THEORY OF RODS

BY

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1. Introduction. In the present paper, by rod we mean a body B comprising a material line and two deformable vectors (called „directors"), attached to every point of line (this mathematical model is called also „generalized rod" [1]). For such a body the basic equations are presented and a strain energy functions is proposed. By using the geometrical and constitutive equations, the displacement equations of equilibrium are written. For some self-adjoint boundary value problems, existence and uniqueness theorems are proved.

2. Notations. Basic equations. Let the particle of the material line of the body B be identified with coordinate S. In a reference configuration of B, let C be the material line $\mathbf{R}(S)$ the position vector of the points of C and $\mathbf{D}_\alpha$ be the values of the directors on C. The coordinate S is the arc length parameter of C. Let c be the present material line of B, $\mathbf{r}$ the position vector of the points of c and $\mathbf{d}_\alpha$ the directors at $\mathbf{r}$. In the present paper the vectors and the tensor are refered to the following base vectors; the unit principal normal $\mathbf{A}_1$, the unit binormal $\mathbf{A}_2$ and the tangent vector $\mathbf{A}_3=d\mathbf{R}/dS$.

We consider an initial straight elastic rod with variable directors, which are normal to each other and to the straight line C. So that we have

(2.1) $$\mathbf{D}_\alpha=D_{(\alpha)}\mathbf{A}_\alpha,\ \text{do not sum on } \alpha,\ \alpha=1,2.$$

As $D_{(\alpha)}$, $\alpha=1,2$, depend on S, this mathematical model describes a rod which has nonuniform thickness in the reference configuration.

We present now the basic equations of the theory [1].

Geometrical equations

(2.2) $$\mathbf{r}=\mathbf{R}+\mathbf{u},\ \mathbf{d}_\alpha=\mathbf{D}_\alpha+\bar{\delta}_\alpha,$$

(2.3) $$\begin{cases}\gamma_{\alpha\beta}=D_{(\alpha)}\bar{\delta}_{\beta\alpha}+D_{(\beta)}\bar{\delta}_{\alpha\beta},\quad \gamma_{\alpha3}=\bar{\delta}_{\alpha3}+D_{(\alpha)}u'_\alpha,\quad \gamma_{33}=2\bar{\delta}_{33}=2u'_3,\\ k_{\alpha\beta}=D_{(\beta)}\bar{\delta}'_{\alpha\beta}+D'_{(\alpha)}\bar{\delta}_{\beta\alpha},\quad k_{\alpha3}=\bar{\delta}'_{\alpha3}+D'_{(\alpha)}u'_\alpha,\end{cases}$$

where $\mathbf{u}$, $\bar{\delta}_\alpha$ are the displacement vectors and γ_{ij}, k_{ij} are the deformation measures' denotes the derivative with respect to S.

Constitutive equations

(2.4) $$\begin{cases}=n^3=2\rho_0(\partial\psi/\partial\gamma_{33}),\quad \bar{\pi}^\alpha=2(n^\alpha-D'_{(\alpha)}p^{\alpha3})/D_{(\alpha)}=2\rho_0(\partial\psi^*/\partial\gamma_{\alpha3}),\\ \bar{\pi}^{\lambda\mu}=(D_{(\lambda)}\pi^{\lambda\mu}+D_{(\mu)}\pi^{\mu\lambda}-D'_{(\lambda)}p^{\mu\lambda}-D'_{(\lambda)}p^{\lambda\mu})/2D_{(\lambda)}D_{(\mu)}=2\rho_0(\partial\psi^*/\partial\gamma_{\lambda\mu}),\\ p^{\alpha\beta}=\rho_0D_{(\beta)}(\partial\psi^*/\partial k_{\alpha\beta}),\quad p^{\alpha3}=\rho_0(\partial\psi^*\partial k_{\alpha3}),\quad n^\alpha=D_{(\alpha)}\pi^{\alpha3}+D'_{(\alpha)}p^{\alpha3},\\ D_{(1)}\pi^{12}-D_{(2)}\pi^{21}=D'_{(2)}p^{21}-D'_{(1)}p^{12},\end{cases}$$

where ρ_0 is the mass density in the reference configuration, n^i is the contact force, $p^{\alpha i}$ are the contact directors couples, $\pi^{\alpha i}$ are the intrinsic director couples and ψ^* is the Helmholtz free energy. We assume that ψ^* is quadratic in the kinematic variables γ_{ij}, $k_{\alpha i}$. Therefore we have

$$
\begin{aligned}
\rho_0\psi = {} & A_{\alpha\beta\lambda\delta}\gamma_{\alpha\beta}\gamma_{\lambda\delta}+A_{\alpha3\beta3}\gamma_{\alpha3}\gamma_{\beta3}+A_{3333}\gamma_{33}\gamma_{33}+A_{\alpha\beta\lambda3}\gamma_{\alpha\beta}\gamma_{\lambda3}+\\
&+A_{\alpha\beta33}\gamma_{\alpha\beta}\gamma_{33}+A_{\alpha333}\gamma_{\alpha3}\gamma_{33}+B_{\alpha\beta\lambda\delta}\gamma_{\alpha\beta}k_{\lambda\delta}+B_{\alpha\beta\lambda3}\gamma_{\alpha\beta}k_{\lambda3}+\\
&+B_{\alpha3\lambda\delta}\gamma_{\alpha3}k_{\lambda\delta}+B_{\alpha3\lambda3}\gamma_{\alpha3}k_{\lambda3}+B_{33\alpha\beta}\gamma_{33}k_{\alpha\beta}+B_{33\alpha3}\gamma_{33}k_{\alpha3}+\\
&+C_{\alpha\beta\lambda\delta}k_{\alpha\beta}k_{\lambda\delta}+C_{\alpha\beta\lambda3}k_{\alpha\beta}k_{\lambda3}+C_{\alpha3\beta3}k_{\alpha3}k_{\beta3},
\end{aligned}
\tag{2.5}
$$

where the coefficients A_{ijkl}, B_{ijkl}, C_{ijlk} satisfy the conditions

$$
\begin{cases}
A_{\alpha i\beta j}=A_{\beta j\alpha i}, \ A_{\alpha\beta\gamma\delta}=A_{\beta\alpha\gamma\delta}, \ A_{\alpha\beta i3}=A_{\beta\alpha i3},\\
B_{\alpha\beta\gamma j}=B_{\beta\alpha\gamma j}, \ C_{\alpha i\beta j}=C_{\beta j\alpha i};\ \alpha,\beta=1,2;\quad i,j=1,2,3.
\end{cases}
\tag{2.6}
$$

Equilibrium equations

$$
\{(n^i)'+\rho_0 f^i=0, \quad (p^{\alpha i})'+\rho_0 l^{\alpha i}=\pi^{\alpha i}, \ \alpha=1,2;\ i=1,2,3. \tag{2.7}
$$

where f^i is the assigned body force and $l^{\alpha i}$ are the assigned director couples.

3. The displacement equations of equilibrium and the boundary value problems. By using the geometrical equations and the constitutive equations, the equilibrium equations can be written in the form

$$
[\mathbf{T}(S)\mathbf{w}'+\mathbf{Q}(S)\mathbf{w}]'-[\bar{\mathbf{Q}}(S)\mathbf{w}'+\mathbf{P}(S)\mathbf{w}]+\mathbf{f}(S)=0, \tag{3.1}
$$

where $\mathbf{w}=(u_i, \delta_{\alpha i})$, $\mathbf{f}=(\rho_0 f^i, \rho_0 l^{\alpha i})$; $i=1,2,3$; $\alpha=1,2$.
Concerning the system (3.1), we have the following result.

Lemma 1. *If the conditions (2.6) hold, then the matrices* $\mathbf{T}$ *and* $\mathbf{P}$ *are hermitian and the matrix* $\bar{\mathbf{Q}}$ *is the adjoint matrix of* $\mathbf{Q}$ $(\bar{\mathbf{Q}}=\mathbf{Q}^*)$.

Proof. If we writte completely the system (3.1), the proof results by direct verification.

We suppose now that $\mathbf{T}(S)$ is a nonsingular matrix. Let $\mathbf{p}$ be the following vector $\mathbf{p}=(n^i, p^{\alpha i})=\mathbf{T}(S)w'+\mathbf{Q}(S)\mathbf{w}$. If further we set $\mathbf{A}=\mathbf{T}^{-1}\mathbf{Q}$, $\mathbf{B}=\mathbf{T}^{-1}$, $\mathbf{C}=\mathbf{P}-\mathbf{Q}^*\mathbf{T}^{-1}\mathbf{Q}$, $\mathbf{T}=\mathbf{B}^{-1}$, $\mathbf{Q}=-\mathbf{B}^{-1}\mathbf{A}$, $\mathbf{P}=\mathbf{C}+\mathbf{A}^*\mathbf{B}^{-1}\mathbf{A}$, then the system (3.1) becomes

$$
\mathbf{w}'-\mathbf{A}(S)\mathbf{w}-\mathbf{B}(S)\mathbf{p}=0, \quad -\mathbf{p}'-\mathbf{A}^*(S)\mathbf{p}+\mathbf{C}(S)\mathbf{w}-\mathbf{f}=0. \tag{3.2}
$$

Concerning the system (3.2) we formulate the following boundary value problems

$$
(\mathcal{B})\begin{cases}
(a)\begin{cases}\mathbf{L}_1[\mathbf{w},\ \mathbf{p}]\equiv\mathbf{L}^*[\mathbf{p}](S)+\mathbf{C}(S)\mathbf{w}(S)=\mathbf{f},\\
\mathbf{L}_2[\mathbf{w},\ \mathbf{p}]\equiv\mathbf{L}[\mathbf{w}](S)-\mathbf{B}(S)\mathbf{p}(S)=0,\quad S\in[a,b],\end{cases}\\
(b)\quad \mathbf{M}_1\mathbf{w}(a)+\mathbf{M}_2\mathbf{p}(a)+\mathbf{M}_3\mathbf{w}(b)+\mathbf{M}_4\mathbf{p}(b)=0,
\end{cases}
$$

where $\mathbf{L}[\mathbf{w}](S)=\mathbf{w}'(S)-\mathbf{A}(S)\mathbf{w}(S)$, $\mathbf{L}^*[\mathbf{p}](S)=-\mathbf{p}'(S)-\mathbf{A}^*(S)\mathbf{p}(S)$, and $\mathbf{M}_\beta$, $\beta=1,2,3,4$ are 18×9 matrices such that the 18×36 matrix $\mathcal{M}=[M_1M_2M_3M_4]$ has rank 18.

Remark 1.a) The components of the matrices $\mathbf{A}(S)$, $\mathbf{B}(S)$, $\mathbf{C}(S)$ are functions of the constitutive coefficients and of the functions $D_{(\alpha)}$ and $D'_{(\alpha)}$. Hence, these matrices depend on constitutive and geometrical properties of the considered body.

b) By an appropriate choice of the matrices $\mathbf{M}_\beta$, we can formulate various boundary conditions. These conditions consist on the specification of either $\mathbf{w}$ or $\mathbf{p}$ or a suitable combinations of these quantities at the ends of the line C. For example, if

$$
\mathbf{M}_1=\begin{bmatrix}\mathbf{E}_9\\ 0_9\end{bmatrix},\quad \mathbf{M}_2=\begin{bmatrix}0_9\\ 0_9\end{bmatrix},\quad \mathbf{M}_3=\begin{bmatrix}0_9\\ \mathbf{E}_9\end{bmatrix},\quad \mathbf{M}_4=\begin{bmatrix}0_9\\ 0_9\end{bmatrix}.
$$

we obtain the condition

$$
\mathbf{w}(a)=\mathbf{w}(b)=0. \tag{3.3}
$$

If $\mathbf{M}_1=\begin{bmatrix}\mathbf{E}_9\\ 0_9\end{bmatrix}$, $\mathbf{M}_2=\begin{bmatrix}0_9\\ 0_9\end{bmatrix}$, $\mathbf{M}_3=\begin{bmatrix}0_9\\ 0_9\end{bmatrix}$, $\mathbf{M}_4=\begin{bmatrix}0_9\\ \mathbf{E}_9\end{bmatrix}$, we get the mixed condition

$$
\mathbf{w}(a)=\mathbf{p}(b)=0. \tag{3.4}
$$

In both above cases, the rank of the matrix $\mathcal{M}$ is 18.

We make now the following hypothesis. (h): We suppose that A_{ijkl}, B_{ijkl}, $C_{ijkl}\in\mathcal{L}^\infty[a,b]$ and $D_{(\alpha)}\in C^1[a,b]$, $\alpha=1,2$.

From hypothesis (h) it results that $\mathbf{A}(S)$, $\mathbf{B}(S)$, $\mathbf{C}(S)\in\mathcal{L}^\infty_{99}[a,b]$ (the spaces used here are defined in [3]). Also, from Lemma 1 it follows that $\mathbf{B}(S)$ and $\mathbf{C}(S)$ are hermitian matrices.

With the help of the vector $y=(y_\alpha)$, $\alpha=\overline{1,18}$, where $y_i=w_i$, $y_{9+i}=p_i$; $i=\overline{1,9}$, the problem $(\mathcal{B})$ becomes

$$
(\mathcal{B}')\ (a)\ \mathcal{L}[y](S)=\mathcal{J}\mathbf{y}'(S)+\mathcal{A}(S)\mathbf{y}(S)=\mathcal{F},\ S\in[a,b].\ (b)\,\mathcal{M}\hat{\mathbf{y}}=0,
$$

where the hermitian matrix function $\mathcal{A}(S)$ and the skew-hermitian matrix $\mathcal{J}$ are

$$
\mathcal{J}=\begin{bmatrix}0_9 & -\mathbf{E}_9\\ \mathbf{E}_9 & 0_9\end{bmatrix},\quad \mathcal{A}(S)=\begin{bmatrix}\mathbf{C}(S) & -\mathbf{A}(S)\\ \mathbf{A}(S) & -\mathbf{B}(S)\end{bmatrix},
$$

$\mathcal{F}$ is given by $\mathcal{F}^i=f^i$, $\mathcal{F}^{9+i}=0$, $i=\overline{1,9}$ and $\hat{\mathbf{y}}=(\hat{y}_\alpha)$ is defined by $\hat{y}_\alpha=y_\alpha(a)$, $y_{18+\alpha}=y_\alpha(b)$; $\alpha=\overline{1,18}$.

Let $(\mathcal{B}_0)$ be the boundary value problem $(\mathcal{B})$ when $\mathbf{f}=0$. By using the theorem 8.1 [3] we can prove the following lemma.

Lemma 2. *If (2.6) hold, the problems* $(\mathcal{B}_0-(3.3))$ *and* $(\mathcal{B}_0-(3.4))$ *are self-adjoint problems.*

4. Existence and uniqueness theorems. Theorem 1. (*Uniqueness theorem*). *If the strain energy function is positive definite i.e.*

$$
\int_a^b \rho_0\psi^*\,dS\geq c\int_a^b\sum_{\alpha,j=1}^{2,3}(\gamma_{\alpha i}^2+k_{\alpha i}^2)\,dS,\quad c>0,\ c=\text{const}. \tag{4.1}
$$

then if the boundary value problem formulated by $(\mathcal{B}-a)$ *together with*

(4.2) $$\mathbf{w}(a)=\mathbf{w}_a,\ \mathbf{w}(b)=\mathbf{w}_b \text{ or } \mathbf{w}(a)=\mathbf{w}_a,\ \mathbf{p}(b)=\mathbf{p}_b,$$

has a solution, it is unique.

Proof. Let $\mathbf{w}$ denotes the difference of two solutions $\mathbf{w}=\mathbf{w}_1-\mathbf{w}_2$. The vector $\mathbf{w}$ satisfies the zero boundary conditions

(4.3) $$\mathbf{w}(a)=\mathbf{w}(b)=0 \text{ or } \mathbf{w}(a)=\mathbf{p}(b)=0,$$

and the equations (2.7) for $\mathbf{f}=0$. We multiply the equation $(2.7)_1$ by u_i and $(2.7)_2$ by $\bar{\delta}_{\alpha i}$. Then we add all these equations and we integrate. We obtain

(4.4) $$2\int_a^b \rho_0\psi^* dS=(\mathbf{nu}+\mathbf{p}\bar{\delta}_\alpha)\Big|_a^b.$$

In virtue of (4.1) and (4.3) it results that $\gamma_{\alpha i}=k_{\alpha i}=0$. From (2.3) we obtain $u_1=bS+c$, $u_2=dS+e$, $u_3=a$, $\bar{\delta}_{11}=0$, $\bar{\delta}_{22}=0$, $\bar{\delta}_{12}=fD_{(1)}$, $\bar{\delta}_{21}=gD_{(2)}$, $\bar{\delta}_{13}=-D_1 b$, $\bar{\delta}_{23}=-D_{(2)}d$, where a, b, c, d, e, f, g are constants. Hence, from (4.3) it results $\mathbf{w}=0$, $S\in[a, b]$, and the theorem is proved.

Remark 2. In conformity with this theorem, the problem $(\mathcal{B}_0-a)$ and (3.3) (3.4) has only the null solution.

Theorem 2. (*Existence and uniqueness theorem*) *If* (2.6), (4.1) *hold* E *and* $\rho_0, D_{(1)}, D_{(2)}\in C^1[a, b]$, $A_{ijkl}, B_{ijkl}, C_{ijkl}\in\mathcal{L}^\infty[a, b]$ *and* $f^i, l^{\alpha i}\in\mathcal{L}^2[a, b]$ *then there exists a* 9×9 *matrix function* (G r e e n *function*) *so that the problem* $(\mathcal{B}-a)$ *and* (3.3), (3.4) *has a unique solution given by*

$$\mathbf{w}(S)=\int_a^b \mathbf{G}(S, T)\mathbf{f}(T)dT.$$

$\mathbf{G}(T, S)$ *is continuous in* (T, S) *on* $\square=[a, b]\times[a, b]$, *is absolutely continuous in each argument on* $[a, b]$ *for fixed values of the other argument, and* $\mathbf{G}(T, S)\equiv\equiv(\mathbf{G}(S, T))^*$ on $\square$.

Proof. By virtue of (2.6) and (4.1) it results that the problem $(\mathcal{B}_0)$ is self-adjoint (Lemma 2) and has only the identically vanishing solution (Remark 2). Then the proof results from Theorem 8.2 [3 p. 371].

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ON STABILITY OF SOLUTION IN LINEAR VISCOELASTICITY

BY
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0. Sufficient stability conditions are given for solution of linear viscoelasticity in the case of prescribed surface traction and displacement. The obtained results are extending to linear viscoelasticity of the results of K n o p s and W i l k e s [5] concerning linear elasticity. Sufficient stability conditions of equilibrium solution are given in author's papers [1], [2].

1. Let us consider a linear viscoelastic solid with constitutive equations

(1.1) $$\sigma_{ij}(\mathbf{x},t)=C_{ijkl}(\mathbf{x}, t)\, u_{k,l}(\mathbf{x}, t)+\int_0^t G_{ijkl}(\mathbf{x}, t, \tau)u_{k,l}(\mathbf{x}, \tau)\, d\tau,$$

where $\sigma_{ij}(\mathbf{x}, t)$ are Cartesian components of the stress tensor and $\mathbf{u}(\mathbf{x}, t)= =(u_1(\mathbf{x}, t), u_2(\mathbf{x}, t), u_3(\mathbf{x}, t))$ is the displacement vector at the point $\mathbf{x}$ and instant t. All indices range over values 1, 2, 3, and usual convention of summation over repeated indices is adopted.

The viscoelastic body accupies the regular region [4, p. 13] $\mathbf{B}(t)$ at instant $t\in I=[0, T]$, $0<T<\infty$. $\partial\mathbf{B}(t)$ is the boundary of $\mathbf{B}(t)$ and $\mathbf{n}=(n_1, n_2, n_3)$ is its unit normal.

Here we consider the following initial boundary value problem of linear dynamic viscoelasticity.

Find the solution of the system

(1.2) $$\sigma_{ij,j}(x, t)+\rho(\mathbf{x})\, F_i(\mathbf{x}, t)=\rho(\mathbf{x})\, \ddot{u}_i(\mathbf{x}, t),\ (\mathbf{x}, t)\in\mathbf{B}(t)\times I$$

satisfying boundary conditions

(1.3) $$\begin{aligned} &\mathbf{u}(\mathbf{x}, t)=0,\ (\mathbf{x}, t)\in\partial\mathbf{B}_1(t)\times I,\\ &\sigma_{ij}(\mathbf{x}, t)\, n_j=s_i(\mathbf{x}, t),\ (\mathbf{x}, t)\in\partial\mathbf{B}_2(t)\times I. \end{aligned}$$

$\partial\mathbf{B}_1(t)\cup\mathbf{B}_2(t)=\partial\mathbf{B}(t)$, and initial conditions.

(1.4) $$\mathbf{u}(\mathbf{x}, 0)=\mathbf{u}_0(\mathbf{x}),\ \dot{\mathbf{u}}(\mathbf{x}, 0)=\mathbf{V}_0(\mathbf{x}),\ \mathbf{x}\in\mathbf{B}(0).$$

2. Let. $\mathbf{C}_0^\infty(\mathbf{B})$ be the set of vector functions with compact support in $\mathbf{B}$, and let $\mathbf{H}_0$, $\mathbf{H}_+$ be Hilbert spaces obtained by completion of $\mathbf{C}_0^\infty(\mathbf{B})$ under the norm $\|\cdot\|_0$, $\|\cdot\|_+$ induced by inner products

$$<\mathbf{u}, \mathbf{v}>_{\mathbf{H}_0}=\int_{\mathbf{B}} u_i v_i dx \quad <\mathbf{u}, \mathbf{v}>_{\mathbf{H}_+}=\int_{\mathbf{B}} u_{i,j} v_{i,j}\, dx,$$

respectively, and let $\mathbf{H}_-$ be the completion of $\mathbf{C}_0^\infty(\mathbf{B})$ via the norm