$$w(a)=w_a,\ w(b)=w_b \text{ or } w(a)=w_a,\ p(b)=p_b, \tag{4.2}$$

has a solution, it is unique.

Proof. Let $\mathbf{w}$ denotes the difference of two solutions $\mathbf{w}=\mathbf{w}_1-\mathbf{w}_2$. The vector $\mathbf{w}$ satisfies the zero boundary conditions

$$\mathbf{w}(a)=\mathbf{w}(b)=0 \quad \text{or} \quad \mathbf{w}(a)=\mathbf{p}(b)=0, \tag{4.3}$$

and the equations (2.7) for $\mathbf{f}=0$. We multiply the equation $(2.7)_1$ by u_i and $(2.7)_2$ by $\bar{\delta}_{\alpha i}$. Then we add all these equations and we integrate. We obtain

$$2\int_a^b \rho_0 \psi^* dS=(\mathbf{nu}+\mathbf{p}\bar{\delta}_\alpha)\Big|_a^b. \tag{4.4}$$

In virtue of (4.1) and (4.3) it results that $\gamma_{\alpha i}=k_{\alpha i}=0$. From (2.3) we obtain $u_1=bS+c$, $u_2=dS+e$, $u_3=a$, $\bar{\delta}_{11}=0$, $\bar{\delta}_{22}=0$, $\bar{\delta}_{12}=fD_{(1)}$, $\bar{\delta}_{21}=gD_{(2)}$, $\bar{\delta}_{13}=-D_1 b$, $\bar{\delta}_{23}=-D_{(2)}d$, where a, b, c, d, e, f, g are constants. Hence, from (4.3) it results $\mathbf{w}=\mathbf{0}$, $S\in[a, b]$, and the theorem is proved.

Remark 2. In conformity with this theorem, the problem $(\mathcal{B}_0-a)$ and (3.3) (3.4) has only the null solution.

Theorem 2. (*Existence and uniqueness theorem*) *If* (2.6), (4.1) *hold* E *and* ρ_0, $D_{(1)}$, $D_{(2)}\in C^1[a, b]$, A_{ijkl}, B_{ijkl}, $C_{ijkl}\in\mathcal{L}^\infty[a, b]$ *and* f^i, $l^{\alpha i}\in\mathcal{L}^2[a, b]$ *then there exists a* 9×9 *matrix function* (Green *function*) *so that the problem* $(\mathcal{B}-a)$ *and* (3.3), (3.4) *has a unique solution given by*

$$\mathbf{w}(S)=\int_a^b \mathbf{G}(S, T)\mathbf{f}(T)dT.$$

$\mathbf{G}(T, S)$ *is continuous in* (T, S) *on* $\square=[a, b]\times[a, b]$, *is absolutely continuous in each argument on* $[a, b]$ *for fixed values of the other argument, and* $\mathbf{G}(T, S)\equiv\equiv(\mathbf{G}(S, T))^*$ on $\square$.

Proof. By virtue of (2.6) and (4.1) it results that the problem $(\mathcal{B}_0)$ is self-adjoint (Lemma 2) and has only the identically vanishing solution (Remark 2). Then the proof results from Theorem 8.2 [3 p. 371].

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ON STABILITY OF SOLUTION IN LINEAR VISCOELASTICITY

BY

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0. Sufficient stability conditions are given for solution of linear viscoelasticity in the case of prescribed surface traction and displacement. The obtained results are extending to linear viscoelasticity of the results of Knops and Wilkes [5] concerning linear elasticity. Sufficient stability conditions of equilibrium solution are given in author's papers [1], [2].

1. Let us consider a linear viscoelastic solid with constitutive equations

$$\sigma_{ij}(\mathbf{x},t)=C_{ijkl}(\mathbf{x},t)\,u_{k,l}(\mathbf{x},t)+\int_0^t G_{ijkl}(\mathbf{x},t,\tau)u_{k,l}(\mathbf{x},\tau)\,d\tau, \tag{1.1}$$

where $\sigma_{ij}(\mathbf{x}, t)$ are Cartesian components of the stress tensor and $\mathbf{u}(\mathbf{x}, t)==(u_1(\mathbf{x}, t), u_2(\mathbf{x}, t), u_3(\mathbf{x}, t))$ is the displacement vector at the point $\mathbf{x}$ and instant t. All indices range over values 1, 2, 3, and usual convention of summation over repeated indices is adopted.

The viscoelastic body accupies the regular region [4, p. 13] $\mathbf{B}(t)$ at instant $t\in I=[0, T]$, $0<T<\infty$. $\partial\mathbf{B}(t)$ is the boundary of $\mathbf{B}(t)$ and $\mathbf{n}=(n_1, n_2, n_3)$ is its unit normal.

Here we consider the following initial boundary value problem of linear dynamic viscoelasticity.

Find the solution of the system

$$\sigma_{ij,j}(x, t)+\rho(\mathbf{x})F_i(\mathbf{x}, t)=\rho(\mathbf{x})\ddot{u}_i(\mathbf{x}, t),\ (\mathbf{x}, t)\in\mathbf{B}(t)\times I \tag{1.2}$$

satisfying boundary conditions

$$\begin{gathered}\mathbf{u}(\mathbf{x}, t)=0,\ (\mathbf{x}, t)\in\partial\mathbf{B}_1(t)\times I,\\ \sigma_{ij}(\mathbf{x}, t)\,n_j=s_i(\mathbf{x}, t),\ (\mathbf{x}, t)\in\partial\mathbf{B}_2(t)\times I.\end{gathered} \tag{1.3}$$

$\partial\mathbf{B}_1(t)\cup\mathbf{B}_2(t)=\partial\mathbf{B}(t)$, and initial conditions.

$$\mathbf{u}(\mathbf{x}, 0)=\mathbf{u}_0(\mathbf{x}),\ \dot{\mathbf{u}}(\mathbf{x}, 0)=\mathbf{V}_0(\mathbf{x}),\ \mathbf{x}\in\mathbf{B}(0). \tag{1.4}$$

2. Let. $C_0^\infty(\mathbf{B})$ be the set of vector functions with compact support in $\mathbf{B}$, and let $\mathbf{H}_0$, $\mathbf{H}_+$ be Hilbert spaces obtained by completion of $C_0^\infty(\mathbf{B})$ under the norm $\|\cdot\|_0$, $\|\cdot\|_+$ induced by inner products

$$<\mathbf{u}, \mathbf{v}>_{\mathbf{H}_0}=\int_{\mathbf{B}} u_i v_i d\mathbf{x} \quad <\mathbf{u}, \mathbf{v}>_{\mathbf{H}_+}=\int_{\mathbf{B}} u_{i,j} v_{i,j}\, d\mathbf{x},$$

respectively, and let $\mathbf{H}_-$ be the completion of $C_0^\infty(\mathbf{B})$ via the norm

$$\| \mathbf{u} \|_{-} = \sup_{\mathbf{v} \in \mathbf{H}_{+}} \frac{| < \mathbf{u}, \mathbf{v} >_{\mathbf{H}_0} |}{\| \mathbf{v} \|_{+}}, [3].$$

A generalized solution of problem (1. 2) -(1.4) is a function $\mathbf{u} \in \mathbf{L}^2(I ; \mathbf{H}_{+})$ with $\dot{\mathbf{u}} \in \mathbf{L}^2(I ; \mathbf{H}_0)$, $\rho \mathbf{u} \in \mathbf{L}^2(I ; \mathbf{H}_{-})$ satisfying a.e. the system (1.2), and conditions (1.4) in $[\mathbf{H}_{+}, \mathbf{H}_0]_{1/2}$, $[\mathbf{H}_0, \mathbf{H}_{-}]_{1/2}$ respectively [3].

The following conditions are sufficient for the existence and uniqueness of the generalized solution [3].

a) The elasticities C_{ijkl} and their derivatives $\dot{C}_{ijkl}$ with respect to t are Lebesgues integrable and essentialy bounded functions on $\mathbf{B}$ for any fixed $t \in I$.

b) The relaxation moduli G_{ijkl} and their derivatives $\partial/\partial\tau(G_{ijkl})$ are Lebesgues measurable and essentialy bounded functions on $\mathbf{B}$ for any fixed $(t, \tau) \in I \times I$.

c) There exists the constant $c > 0$ such that for every $\mathbf{u}(\mathbf{x}, t) \in \mathbf{H}_{+}$

$$\int_{\mathbf{B}(t)} C_{ijkl}(\mathbf{x}, t)\, u_{i,j}(\mathbf{x}, t)\, u_{k,l}(\mathbf{x}, t)\, d\mathbf{x} \geqslant c \| \mathbf{u}(\mathbf{x}, t) \|_{+}^{2}, \tag{2.1}$$

uniformly with respect to $t \in I$.

d) There exists $\rho_0 > 0$ such that

$$\operatorname*{ess\,inf}_{\mathbf{x} \in \mathbf{B}} \rho(\mathbf{x}) \geqslant \rho_0. \tag{2.2}$$

e) $\mathbf{F}(\mathbf{x}, t) \in L^2(I ; \mathbf{H}_{-})$, f) $\mathbf{u}_0 \in \mathbf{H}_{+}$, $\mathbf{V}_0 \in \mathbf{H}_0$.

3. Multiplying (1.2) by $\dot{u}_i(\mathbf{x}, t)$, summing with respect to i, integrating over $[0, t] \times \mathbf{B}(t)$, applying the divergence theorem, and taking into account (1.3) we obtain

$$\frac{dE}{dt} = \frac{1}{2} \int_{\mathbf{B}(t)} \dot{C}_{ijkl}(\mathbf{x}, t)\, u_{i,j}(\mathbf{x}, t)\, u_{k,l}(\mathbf{x}, t)\, d\mathbf{x} - \int_{\mathbf{B}(t)} \sigma_{ij}^{(v)}(\mathbf{x}, t)\, \dot{u}_{i,j}(\mathbf{x}, t)\, d\mathbf{x} + \mathsf{P}(t), \tag{3.1}$$

where

$$E(t) = \frac{1}{2} \int_{\mathbf{B}(t)} [C_{ijkl}(\mathbf{x}, t)\, u_{i,j}(\mathbf{x}, t) u_{k,l}(\mathbf{x}, t) + \rho(\mathbf{x})\, \dot{\mathbf{u}}^2(\mathbf{x}, t)]\, d\mathbf{x} \tag{3.2}$$

is the sum of *elastic energy* of deformation and kinetic energy,

$$\sigma_{ij}^{(v)}(\mathbf{x}, t) = \int_0^t G_{ijkl}(\mathbf{x}, t, \tau)\, \mathbf{u}_{k,l}(\mathbf{x}, \tau)\, d\tau \tag{3.3}$$

are the components of the *viscous part* of stress tensor, and

$$\mathsf{P}(t) = \int_{\partial \mathbf{B}_2(t)} s_i(\mathbf{x}, t)\, \dot{u}_i(\mathbf{x}, t) da + \int_{\mathbf{B}(t)} \rho(\mathbf{x})\, F_i(\mathbf{x}, t)\, \dot{u}_i(\mathbf{x}, t)\, d\mathbf{x} \tag{3.4}$$

is the work done by surface traction and body forces.

An integration of (3.1) on $[0, t]$ gives

$$\begin{aligned} E(t) - E(0) = {} & \frac{1}{2} \int [\int_{\mathbf{B}(\eta)} \dot{C}_{ijkl}(\mathbf{x}, \eta)\, u_{i,j}(\mathbf{x}, \eta)\, u_{k,l}(\mathbf{x}, \eta)\, d\mathbf{x}]\, d\eta - \\ & - \int_0^t [\int_{\mathbf{B}(\eta)} \sigma_{ij}^{(v)}(\mathbf{x}, \eta)\, \dot{u}_{i,j}(\mathbf{x}, \eta)\, d\mathbf{x}]\, d\eta + \int_0^t \mathsf{P}(\eta)\, d\eta. \end{aligned} \tag{3.5}$$

In addition to assumptions a) – f) of Sec. 2, we require the time derivatives of elasticities and relaxation moduli to satisfy

$$-\int_{\mathbf{B}(t)} \dot{C}_{ijkl}(\mathbf{x}, t)\, u_{i,j}(\mathbf{x}, t)\, u_{k,l}(\mathbf{x}, t)\, d\mathbf{x} \geqslant \bar{c} \| \mathbf{u} \|_{+}^{2}, \quad \bar{c} > 0, \tag{3.6}$$

for every $\mathbf{u} \in \mathbf{H}_{+}$, uniformly with respect to $t \in I$, and

$$\int_0^t [\int_{\mathbf{B}(\eta)} \sigma_{ij}^{(v)}(\mathbf{x}, \eta)\, \dot{u}_{i,j}(\mathbf{x}, \eta)\, d\mathbf{x}]\, d\eta \geqslant 0, \quad t \in I. \tag{3.7}$$

The condition (3.6) is essential in the study of elastic stability when elasticities are time dependent [5]. The condition (3.7) is expressing that the total power of viscoelastic part of the stress on time internal $[0, t]$ is non-negative.

4. If we suppose that the body force is satisfying the condition

$$\int_{\mathbf{B}(t)} \rho(\mathbf{x})\, F_i(\mathbf{x}, t)\, \dot{u}_i(\mathbf{x}, t)\, d\mathbf{x} = 0, \quad t \in I, \tag{4.1}$$

the case of zero body force is included, from (3.5) and (3.7) it results

$$\begin{aligned} E(t) - \frac{1}{2} \int_0^t [\int_{\mathbf{B}(\eta)} \dot{C}_{ijkl}(\mathbf{x}, \eta)\, u_{i,j}(\mathbf{x}, \eta)\, u_{k,l}(\mathbf{x}, \eta)\, d\mathbf{x}]\, d\eta \leqslant \\ \leqslant E(0) + \int_{\partial \mathbf{B}(t)} s_i(\mathbf{x}, t)\, u_i(\mathbf{x}, t)\, da. \end{aligned} \tag{4.2}$$

Assuming that the order of integration may be reversed and applying Schwarz's inequality we get

$$\begin{aligned} & \int_0^t [\int_{\partial \mathbf{B}_2(\eta)} s_i(\mathbf{x}, \eta)\, \dot{u}_i(\mathbf{x}, \eta)\, da]\, d\eta = \int_{\partial \mathbf{B}_2(t)} s_i(\mathbf{x}, t)\, u_i(\mathbf{x}, t)\, da - \\ & - \int_0^t [\int_{\partial \mathbf{B}_2(\eta)} \dot{s}_i(\mathbf{x}, \eta)\, u_i(\mathbf{x}, \eta)\, da]\, d\eta \leqslant (\int_{\partial \mathbf{B}_2(t)} u_i(\mathbf{x}, t)\, u_i(\mathbf{x}, t)\, da)^{1/2} \\ & (\int_{\partial \mathbf{B}_2(t)} s_i(\mathbf{x}, t)\, s_i(\mathbf{x}, t)\, da)^{1/2} + \int_0^t ([\int_{\partial \mathbf{B}_2(\eta)} \dot{s}_i(\mathbf{x}, \eta)\, \dot{s}\, (\mathbf{x}, \eta)\, da)^{1/2} \\ & \quad (\int_{\partial \mathbf{B}_2(\eta)} u_i(\mathbf{x}, \eta)\, u_i(\mathbf{x}, \eta)\, da)^{1/2}]\, d\eta. \end{aligned} \tag{4.3}$$

Taking into account (2.1) and Stekloff inequality (see (70.9) and (70.10) of [5]) we have :

$$\alpha c \int_{\partial \mathbf{B}_2(t)} \mathbf{u}^2(\mathbf{x}, t)\, dt \leqslant \int_{\mathbf{B}(t)} C_{ijkl}(\mathbf{x}, t)\, u_{i,j}(\mathbf{x}, t)\, u_{k,l}(\mathbf{x}, t)\, d\mathbf{x} \leqslant 2E(t) \leqslant 2\tilde{E}(T) \tag{4.4}$$

where $\alpha > 0$ is a constant and

$$\tilde{E}(T) = \sup_{t \in I} E(t). \tag{4.5}$$

We assume that $\tilde{E}(T) < \infty$. Thus from (4.2)—(4.4) it results

$$E(t)-\frac{1}{2}\int_0^t\Big[\int_{B(\eta)} C_{ijkl}(\mathbf{x},\eta)\, u_{i,j}(\mathbf{x},\eta)\, u_{k,l}(\mathbf{x},\eta)\, d\mathbf{x}\Big]\, d\eta \leqslant$$

$$\leqslant E(0)+\left(\frac{2\tilde{E}(T)}{c\alpha}\right)^{1/2}\Big(\int_{\partial B_2(t)} s_i(\mathbf{x},t)\, s_i(\mathbf{x},t)\, da\Big)^{1/2}+ \tag{4.6}$$

$$+\int_0^t\Big(\int_{\partial B_2(\eta)} \dot{s}_i(\mathbf{x},\eta)\, \dot{s}_i(\mathbf{x},\eta)\, da\Big)^{1/2} d\eta.$$

From the apparent relations

$$\frac{d}{dt}\int_{\partial B_2(t)} s_i(\mathbf{x},t)\, s_i(\mathbf{x},t)\, da=2\int_{\partial B_2(t)} s_i(\mathbf{x},t)\, \dot{s}_i(\mathbf{x};t)\, da\leqslant$$

$$2\Big(\int_{\partial B_2(t)} s_i(\mathbf{x},t)\, s_i(\mathbf{x},t)\, da\Big)^{1/2}\Big(\int_{\partial B_2(t)} \dot{s}_i(\mathbf{x},t)\, \dot{s}_i(\mathbf{x},t)\, da\Big)^{1/2}$$

we get

$$\Big(\int_{\partial B_2(t)} s_i(\mathbf{x},t)\, s_i(\mathbf{x},t)\, da\Big)^{1/2}\leqslant \int_0^t\Big(\int_{\partial B_2(\eta)} \dot{s}_i(\mathbf{x},\eta)\, \dot{s}_i(\mathbf{x},\eta)\, da\Big)^{1/2} d\eta, \tag{4.7}$$

so that from (4.6) it results

$$E(t)-\frac{1}{2}\int_0^t\Big[\int_{B(\eta)} C_{ijkl}(\mathbf{x},\eta)\, \mathbf{u}_{i,j}(\mathbf{x},\eta)\, u_{k,l}(\mathbf{x},\eta)\, d\mathbf{x}\Big]\, d\eta\leqslant$$

$$\leqslant E(0)+\left(\frac{8E(T)}{c\alpha}\right)^{1/2}\int_0^t\Big[\int_{\partial B_2(\eta)} \dot{s}_i(\mathbf{x},\eta)\, \dot{s}_i(\mathbf{x},\eta)\, da\Big]^{1/2} d\eta. \tag{4.8}$$

Finally from (4.8), (2.1), (2.2), (3.6), and from the Poincaré inequality

$$\frac{1}{c_1}\int_{B(t)} \mathbf{u}^2(\mathbf{x},t)\, d\mathbf{x}\leqslant \int_{B(t)} u_{i,j}(\mathbf{x},t)\, u_{i,j}(\mathbf{x},t)\, d\mathbf{x},$$

(see (35.13)), [5] it leads to

$$\rho_0\int_{B(t)}[\mathbf{u}^2(\mathbf{x},t)+\dot{\mathbf{u}}^2(\mathbf{x},t)]\, d\mathbf{x}+\int_0^t\Big(\int_{B(\eta)} \mathbf{u}_2(\mathbf{x},\eta)\, d\mathbf{x}\Big)\, d\eta\leqslant$$

$$\leqslant E(0)+\frac{1}{d}\left(\frac{8\tilde{E}(T)}{c\alpha}\right)^{1/2}\int_0^t\Big(\int_{\partial B_2(\eta)} \dot{s}_i(\mathbf{x},\eta)\, \dot{s}_i(\mathbf{x},\eta)\, da\Big)^{1/2} d\eta, \tag{4.9}$$

where $d=\min\{1, c/c_1, \bar{c}/c_1\}$.

5. In conditions of the previous sections, from (4.9), we have

Theorem 1. *If*

$$V(t)=\int_0^t\Big(\int_{\partial B_2(\eta)} \dot{s}_i(\mathbf{x},\eta)\dot{s}_i(\mathbf{x},\eta)\, da\Big)^{1/2} d\eta \tag{5.1}$$

is sufficiently small, the solution of the problem (1.2)—(1.8) is stable vith respect to the measures

$$\mu(\mathbf{u})=\int_{B(t)}[\mathbf{u}^2(\mathbf{x},t)+\dot{\mathbf{u}}^2(\mathbf{x},t)]\, d\mathbf{x}+\int_0^t\Big(\int_{B(\eta)} \mathbf{u}^2(\mathbf{x},\eta)\, d\mathbf{x}\Big)\, d\eta, \tag{5.2}$$

$$\mu_0(\mathbf{u})=E(0).$$

Remark 1. (4.9) implies

$$\rho_0\int_{B(t)} \mathbf{u}^2(\mathbf{x},t)\, d\mathbf{x}+\int_0^t\Big(\int_{B(\eta)} \mathbf{u}^2(\mathbf{x},\eta)\, d\mathbf{x}\Big)\, d\eta< E(0) \tag{5.3}$$

and therefore we obtain

$$\int_0^t\Big(\int_{B(\eta)} \mathbf{u}^2(\mathbf{x},\eta)\, d\mathbf{x}\Big)\, d\eta\leqslant(1-e^{-t/\rho_0})\, E(0)\leqslant E(0). \tag{5.4}$$

Remark 2. Under the null initial condition $E(0)=0$ and the solution of the problem (1.2)—(1.3) is continuously dependent of surface traction with respect to $\mu(\mathbf{u})$.

Indeed, if $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ are the solution of (1.2), (1.3) corresponding to surface tractions $s_i^{(1)}$ and $s_i^{(2)}$, then $\mathbf{w}=\mathbf{u}^{(1)}-\mathbf{u}^{(2)}$ is the solution of (1.2)—(1.3) corresponding to surface traction $\bar{s}_i=s_i^{(1)}s_i^{(2)}$ and it holds the inequality

$$\mu(\mathbf{u})\leqslant\frac{1}{d}\left(\frac{8\tilde{E}_w(T)}{c\alpha}\right)^{1/2}\int_0^t\Big(\int_{\partial B_2(\eta)} s_i(\mathbf{x},\eta)\, s_i(\mathbf{x},\eta)\, d\mathbf{x}\Big)^{1/2} d\eta. \tag{5.5}$$

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