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A NONLINEAR INTEGRAL INEQUALITY IN TWO INDEPENDENT VARIABLES

BY

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We shall consider the following integral inequality of Gronwall-Bihari type

(1) $$u(x, y)\leqslant f(x, y)+a(x, y)\int_0^x\int_0^y b(s, t)g[u(s, t)]ds\, dt, \quad x, y\geqslant 0$$

where u, f, a, b are continuous nonnegative functions for $x, y\geqslant 0$ and $g=g(u)$ is continuous nondecreasing for $u\geqslant 0$, $g(u)>0$ for $u>0$. Inequalities similar to (1) were studied by many authors. Under the conditions $f(x, y)=\text{const.}$ and $a(x, y)=1$, the corresponding inequality was considered in ([4], p. 51–54). Under the assumption $a(x, y)=1$, bounds for the function u were obtained in the author's paper [3].

Denote by $G=G(u)$ the function defined for $u>0$, such that

(2) $$G'(u)=(1/g(u)) \text{ for } u>0, G(1)=1$$

and by $G(\infty)$ the limit (finite or infinite) of this function, when $u\to\infty$. We remark that G is strictly increasing on $(0, \infty)$ and we denote by G^{-1} the inverse of G.

Theorem 1. *Assume that in* (1), *the following conditions are fulfiled*

(i) *u, f, a, b are continuous nonnegative for $x, y\geqslant 0$,*

(ii) *f is nondecreasing and $0\leqslant f(x, y)<1$ for $x, y\geqslant 0$,*

(iii) *the function a has continuous nonnegative partial derivatives*

$$\partial a/\partial x, \quad \partial a/\partial y, \quad \partial^2 a/\partial x\partial y \text{ for } x, y\geqslant 0,$$

(iv) *g is continuous and nondecreasing for $u\geqslant 0$, $g(u)>0$ for $u>0$, $g'(u)$ exists and it is continuous for $u>0$,*

(v) *$\lambda g(u)\leqslant g(\lambda u)$ for $\lambda\geqslant 1$ and $u\geqslant 0$.*

Then it follows

(a) *if $G(\infty)=\infty$, for $x, y\geqslant 0$*

(3) $$u(x, y)\leqslant f(x, y)G^{-1}\left[1+a(x, y)\int_0^x\int_0^y b(s, t)ds\, dt\right];$$

(b) *if $G(\infty)\neq\infty$ we have*

(4) $$1+a(x, y)\int_0^x\int_0^y b(s, t)ds\, dt<G(\infty),$$

for $0\leqslant x<\alpha$, $0\leqslant y<\beta$,
then (3) *holds for $0\leqslant x<\alpha$, $0\leqslant y<\beta$.*

Proof. For fixed $x_0, y_0>0$, satisfying (4), we take $\varepsilon>0$ such that $f(x_0, y_0)+\varepsilon<1$ and consider for $0\leqslant x\leqslant x_0$, $0\leqslant y\leqslant y_0$ the function

$$v(x, y)=u(x, y)[f(x, y)+\varepsilon]^{-1}. \tag{5}$$

From (1), we obtain for $0\leqslant x\leqslant x_0$, $0\leqslant y\leqslant y_0$:

$$v(x, y)\leqslant 1+a(x, y)\int_0^x\int_0^y b(s, t)g[v(s, t)]\, ds\, dt=h(x, y). \tag{6}$$

Denoting

$$H(x, y)=G[h(x, y)], \tag{7}$$

and using the restrictions imposed on a and g, we get, after a simple computation that

$$H''_{xy}(x, y)\leqslant A''_{xy}(x, y), \text{ where} \tag{8}$$

$$A(x, y)=a(x, y)\int_0^x\int_0^y b(s, t)ds\, dt. \tag{9}$$

Integrating (8) on the rectangle $0\leqslant s\leqslant x$, $0\leqslant t\leqslant y$, we obtain

$$G[h(x, y)]\leqslant 1+a(x, y)\int_0^x\int_0^y b(s, t)ds\, dt, \tag{10}$$

from which, applying the function G^{-1}, we deduce

$$h(x, y)\leqslant G^{-1}[1+a(x, y)\int_0^x\int_0^y b(s, t)ds\, dt] \tag{11}$$

for $0\leqslant x\leqslant x_0$, $0\leqslant y\leqslant y_0$. Using the relations (5), (6) and (11), we conclude that

$$u(x, y)\leqslant [f(x, y)+\varepsilon]G^{-1}[1+a(x, y)\int_0^x\int_0^y b(s, t)ds\, dt]. \tag{12}$$

for $0\leqslant x\leqslant x_0$, $0\leqslant y\leqslant y_0$. Letting $\varepsilon\to 0$ in the preceding inequality, we obtain that (3) is true for $0\leqslant x\leqslant x_0$, $0\leqslant y\leqslant y_0$ and this concludes the proof, taking into account that (x_0, y_0) was arbitrarily choosen. We remark that in the part (b) of the conclusion, the case $\alpha=\beta=\infty$ is not excluded.

Corollary 1.1. *Assume that*

$$u(x, y)\leqslant f(x, y)+a(x, y)\int_0^x\int_0^y b(s, t)\ [u(s, t)]^p\, ds\, dt,\ x, y\geqslant 0. \tag{13}$$

where u, f, a, b satisfy the conditions of Theorem 1, and $p>1$ is a constant. Assume also that

$$a(x, y)\int_0^x\int_0^y b(s, t)ds\, dt<(p-1)^{-1},\ \ x, y\geqslant 0. \tag{14}$$

Then from (13) we obtain for $x, y\geqslant 0$ the bound

$$u(x, y)\leqslant f(x, y)\ [1-(p-1)a(x, y)\int_0^x\int_0^y b(s, y)\ \ b(s, t)ds\, dt]^{1(1-p)}. \tag{15}$$

Corollary 1.2. *Under the assumptions of the preceding corollary, if we suppose*

$$a(x, y)\int_0^x\int_0^y b(s, t)ds\, dt\leqslant K<(p-1)^{-1},\ \ x, y\geqslant 0, \tag{16}$$

we obtain from (13) the bound

$$u(x, y)\leqslant Mf(x, y),\ x, y\geqslant 0, \tag{17}$$

where $M=[1-(p-1)K]^{1/(p-1)}$ is a positive constant.

Considering again the inequality (1), we can state

Theorem 1. *Assume that in* (1), *the conditions* (i), (iii), (iv) *are satisfied and that*

(ii)′ *f is nondecreasing and $f(x, y)\geqslant 1$ for $x, y\geqslant 0$,*

(v)′ *$\lambda g(u)\leqslant g(\lambda u)$ for $0\leqslant\lambda\leqslant 1$, $u\geqslant 0$,*

Then the conclusions (a) *and* (b) *of Theorem 1 remain true.*

The proof is similar to that of Theorem 1 and we shall omit it.

Corollary 2.1. *Suppose that*

$$u(x, y)\leqslant f(x, y)+a(x, y)\int_0^x\int_0^y b(s, t)\ [u(s, t)]^p\, ds\, dt,\ \ x, y\geqslant 0, \tag{18}$$

where u, f, a, b satisfy the conditions of Theorem 2, p being a constant $0<p<1$. Then from (18), we obtain for $x, y\geqslant 0$ the bound

$$u(x, y)\leqslant f(x, y)\ [1+(1-p)a(x, y)\int_0^x\int_0^y b(s, t)ds\, dt]^{1/(1-p)}. \tag{19}$$

Example. Assume that

$$u(x, y)\leqslant f(x, y)+a(x, y)\int_0^x\int_0^y b(s, t)\sqrt{u(s, t)}\ \ ds\, dt,\ x, y\geqslant 0 \tag{20}$$

where u, f, a, b are continuous nonnegative for $x, y\geqslant 0$, $f(x, y)\geqslant 1$ being nondecreasing and a satisfying condition (iii) of Theorem 1. Then from (20), we find.

$$u(x, y)\leqslant (1/4)f(x, y)\ [2+a(x, y)\int_0^x\int_0^y b(s, t)ds\, dt]^2,\ \ x, y\geqslant 0. \tag{21}$$

Remark 1. If in Theorem 2 and Corollary 2.1, f do not satisfy the condition $f(x, y)\geqslant 1$, then in (3) and respectively in (19) it must be writen $1+f(x, y)$ instead of $f(x, y)$.

Remark 2. It is very interesting to know if we can remove the restriction $\partial^2 a/\partial x\partial y\geqslant 0$ or, more generally, if we can prove the bound (3) by imposing on $a=a(x, y)$ only the conditions to be continuous, nonnegative and monotone nondecreasing with respect to each variable.

Remark 3. The functions $a(x, y)=a_1(x)+a_2(y)$ or $a(x, y)=a_1(x)a_2(y)$, where $a_1(x)$ and $a_2(y)$ are nonnegative, $a_1'(x)$ and $a_2'(y)$ are continuous nonnegative, satisfy the condition (iii) of Theorem 1.

The method used in the proof of Theorem 1 and 2 may be applied also in the case of the nonlinear inequality

$$\begin{aligned} u(x, y)\leqslant f(x, y)+a(x, y)\int_0^x\int_0^y b(s, t)g[u(s, t)g[u(s, t)]ds\, dt+ \\ +a_1(x, y)\int_0^x\int_0^y b_1(s, t)u(s, t)ds\, dt. \end{aligned} \tag{22}$$

where all the functions appearing here are continuous, nonnegative for $x, y \geq 0$.

In what follows, G is given by (2) and G_1 by

$$G_1(x, y) = G^{-1}\left[1 + a(x, y)\int_0^x\int_0^y b(s, t)ds\, dt\right]. \tag{23}$$

Denote also

$$E(x, y) = \exp\left[a_1(x, y)G_1(x, y)\int_0^x\int_0^y b_1(s, t)ds\, dt\right]. \tag{24}$$

Theorem 3. *Assume that in (22) the hypotheses (i), (iii), (iv) of Theorem 1 and (ii)', (v)' of Theorem 2 are satisfied, the functions a_1 and b_1 being continuous nonnegative for $x, y \geq 0$. Assume also that a_1 is nondecreasing with respect to their variables. Then it follows*

(a) *if $G(\infty) = \infty$, we have for $x, y \geq 0$*

$$u(x, y) \leq f(x, y)G_1(x, y)E(x, y) \tag{25}$$

(b) *if $G(\infty) =$ finite and (4) holds for $0 \leq x < \alpha$, $0 \leq y < \beta$, then (25) also holds for $0 \leq x < \alpha$, $0 \leq y < \beta$.*

Proof. Denoting

$$F(x, y) = f(x, y) + a_1(x, y)\int_0^x\int_0^y b_1(s, t)u(s, t)ds\, dt \tag{26}$$

and taking into account (22), we can write the inequality

$$u(x, y) \leq F(x, y) + a(x, y)\int_0^x\int_0^y b(s, t)g[u(s, t)]ds\, dt \tag{27}$$

and we remark that all the conditions required by Theorem 2 are satisfied. Consequently, using the notation (23), we find $u(x, y) \leq F(x, y)G_1(x, y)$ in the domain of definition of G_1. From (26) and (28), using the notation (24) and Corollary 1 of the paper [2], we obtain the desired bound (25). We end with the remark that in part (b) of the conclusion, the case $\alpha = \beta = \infty$ is not excluded.

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ON ASYMPTOTIC PROPERTIES OF DISCRETE-TIME SYSTEMS WITH RETARDED ARGUMENT

BY

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1. Introduction. In recent years difference equations have received much attention due, in part, to the importance of such equations in various branches of technology. Considerable literature on the study of difference equations has been published during the past few years. Some of the most important applications of discrete-time theory are found in the field of control systems, where the second method of Lyapunov to the study of the stability and asymptotic properties of solutions was used. It seems that few papers have appeared in the field of difference equations with retarded argument. However, some results concerning the different kinds of stability, boundedness properties and asymptotic behaviour of solutions of such equations were given by M. Slemrod and E. F. Infante [6], B. G. Pachpatte in [3] and together with S. G. Deo in [4]. Our aim in the present paper is to study the asymptotic relationship between solutions of two systems of difference equations, from which one of them is a linear system and the other is a nonlinear system with retarded argument. One of our results (Theorem 3.2) is related with a result of J. Futák [2] established for a nonlinear delay differential system.

2. Notations and definitions. We begin by stating some notations which will be used in our subsequent discussion. Denote $I(k_0) = \{k_0, k_0+1, \ldots\}$ where k_0 is a fixed integer. Also, for any positive integer p denote $X_p = R^n \times \times R^n \times \ldots \times R^n$, where R^n is a real or complex n-dimensional vector space with norm $|\,.\,|$. If $z = (z_0, z_{-1}, \ldots, z_{-p+1})$, $z_j \in R^n$, for $_{-p+1} \leq j \leq 0$, is an element of X_p, define $\|z\| = \sup\{|z_j|, -p+1 \leq j \leq 0\}$. Let $\Phi = \Phi(I, R^n)$ be the space of all functions from $I(k_0)$ into R^n, that is, for each $k \in I(k_0)$ the value of x at k is $x(k) \in R^n$. The topology of Φ is the topology of uniform convergence on every set $I_r(k_0) = \{k_0, k_0+1, \ldots, k_0+r\}$, $r = 0, 1, 2, \ldots$, that is, $x^i \to x$ as $i \to \infty$ in Φ if and only if $\lim_{i\to\infty} |x^i(k) - x(k)| = 0$ uniformly on every set $I_r(k_0)$, $r = 0, 1, 2, \ldots$. Note also that Φ is a locally convex space [9, pp. 24—26] with the topology defined by the following family of seminorms $|x(k)|_r = \sup\{|x(k)|, k \in I_r(k_0), r = 0, 1, 2, \ldots\}$. Denote by $\Phi_g(I, R^n)$ the Banach space of all functions in Φ for which there exists positive numbers $g(k) < \infty$ and some constant $Q > 0$ such that $|x(k)| \leq Qg(k)$, $k \in I(k_0)$. The norm in Φ_g is defined by $|x|_{\Phi_g} = \sup\{|x(k)|/g(k), k \in I(k_0)\}$. When $g(k) = 1$ for $k \in I(k_0)$ one obtains the Banach space Φ_1 of all bounded functions from $I(k_0)$ into R^n. The norm in Φ_1 is defined by $|x|_{\Phi_1} = \sup\{|x(k)|, k \in I(k_0)\}$. The definitions of Φ, Φ_g and Φ_1 were considered by D. Petrovanu [5] in the study of discrete Hammerstein equation and by C. P. Tsokos and W. J. Padgett [8, Ch. V] in the study of random discrete Fredholm and Volterra equations. A discrete-time system with retarded argument is a system of the form