where all the functions appearing here are continuous, nonnegative for $x, y \geqslant 0$.

In what follows, G is given by (2) and G_1 by

$$G_1(x, y) = G^{-1}[1 + a(x, y) \int_0^x \int_0^y b(s, t) ds\, dt]. \tag{23}$$

Denote also

$$E(x, y) = \exp[a_1(x, y) G_1(x, y) \int_0^x \int_0^y b_1(s, t) ds\, dt]. \tag{24}$$

Theorem 3. *Assume that in* (22) *the hypotheses* (i), (iii), (iv) *of Theorem* 1 *and* (ii)', (v)' *of Theorem* 2 *are satisfied, the functions a_1 and b_1 being continuous nonnegative for $x, y \geqslant 0$. Assume also that a_1 is nondecreasing with respect to their variables. Then it follows*

(a) *if $G(\infty) = \infty$, we have for $x, y \geqslant 0$*

$$u(x, y) \leqslant f(x, y) G_1(x, y) E(x, y) \tag{25}$$

(b) *if $G(\infty) =$ finite and* (4) *holds for $0 \leqslant x < \alpha$, $0 \leqslant y < \beta$, then* (25) *also holds for $0 \leqslant x < \alpha$, $0 \leqslant y < \beta$.*

Proof. Denoting

$$F(x, y) = f(x, y) + a_1(x, y) \int_0^x \int_0^y b_1(s, t) u(s, t) ds\, dt \tag{26}$$

and taking into account (22), we can write the inequality

$$u(x, y) \leqslant F(x, y) + a(x, y) \int_0^x \int_0^y b(s, t) g[u(s, t)] ds\, dt \tag{27}$$

and we remark that all the conditions required by Theorem 2 are satisfied. Consequently, using the notation (23), we find $u(x, y) \leqslant F(x, y) G_1(x, y)$ in the domain of definition of G_1. From (26) and (28), using the notation (24) and Corollary 1 of the paper [2], we obtain the desired bound (25). We end with the remark that in part (b) of the conclusion, the case $\alpha = \beta = \infty$ is not excluded.

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ON ASYMPTOTIC PROPERTIES OF DISCRETE-TIME SYSTEMS WITH RETARDED ARGUMENT

BY

PAVEL TALPALARU

1. Introduction. In recent years difference equations have received much attention due, in part, to the importance of such equations in various branches of technology. Considerable literature on the study of difference equations has been published during the past few years. Some of the most important applications of discrete-time theory are found in the field of control systems, where the second method of Lyapunov to the study of the stability and asymptotic properties of solutions was used. It seems that few papers have appeared in the field of difference equations with retarded argument. However, some results concerning the different kinds of stability, boundedness properties and asymptotic behaviour of solutions of such equations were given by M. Slemrod and E. F. Infante [6], B. G. Pachpatte in [3] and together with S. G. Deo in [4]. Our aim in the present paper is to study the asymptotic relationship between solutions of two systems of difference equations, from which one of them is a linear system and the other is a nonlinear system with retarded argument. One of our results (Theorem 3.2) is related with a result of J. Futák [2] established for a nonlinear delay differential system.

2. Notations and definitions. We begin by stating some notations which will be used in our subsequent discussion. Denote $I(k_0) = \{k_0, k_0+1, \ldots\}$ where k_0 is a fixed integer. Also, for any positive integer p denote $X_p = R^n \times R^n \times \ldots \times R^n$, where R^n is a real or complex n-dimensional vector space with norm $|\,.\,|$. If $\bar{z} = (z_0, z_{-1}, \ldots, z_{-p+1})$, $z_j \in R^n$, for $_{-p+1} \leqslant j \leqslant 0$, is an element of X_p, define $\|\bar{z}\| = \sup\{|z_j|, -p+1 \leqslant j \leqslant 0\}$. Let $\Phi = \Phi(I, R^n)$ be the space of all functions from $I(k_0)$ into R^n, that is, for each $k \in I(k_0)$ the value of x at k is $x(k) \in R^n$. The topology of Φ is the topology of uniform convergence on every set $I_r(k_0) = \{k_0, k_0+1, \ldots, k_0+r\}$, $r = 0, 1, 2, \ldots$, that is, $x^i \to x$ as $i \to \infty$ in Φ if and only if $\lim\limits_{i\to\infty} |x^i(k) - x(k)| = 0$ uniformly on every set $I_r(k_0)$, $r = 0, 1, 2, \ldots$. Note also that Φ is a locally convex space [9, pp. 24—26] with the topology defined by the following family of seminorms $|x(k)|_r = \sup\{|x(k)|, k \in I_r(k_0), r = 0, 1, 2, \ldots\}$. Denote by $\Phi_g(I, R^n)$ the Banach space of all functions in Φ for which there exists positive numbers $g(k) < \infty$ and some constant $Q > 0$ such that $|x(k)| \leqslant Q g(k)$, $k \in I(k_0)$. The norm in Φ_g is defined by $|x|_{\Phi_g} = \sup\{|x(k)|/g(k), k \in I(k_0)\}$. When $g(k) = 1$ for $k \in I(k_0)$ one obtains the Banach space Φ_1 of all bounded functions from $I(k_0)$ into R^n. The norm in Φ_1 is defined by $|x|_{\Phi_1} = \sup\{|x(k)|, k \in I(k_0)\}$. The definitions of Φ, Φ_g and Φ_1 were considered by D. Petrovanu [5] in the study of discrete Hammerstein equation and by C. P. Tsokos and W. J. Padjett [8, Ch. V] in the study of random discrete Fredholm and Volterra equations. A discrete-time system with retarded argument is a system of the form

$$\text{(2.1)}\qquad \begin{cases} x(k+1)=f(k, x(k), x(k-1),\dots, x(k-m)), \ k\in I(k_0), \\ x(j)=c_j, \ k_0-m\leqslant j\leqslant k_0, \end{cases}$$

where each $x(j)\in R^n$, the c_j are given initial values, and $f: I(k_0)\times X_{m+1}\to R^n$. Denoting $\bar{c}=(c_{k_0}, c_{k_0-1},\dots, c_{k_0-m})$, a function $x(k\,;\,k_0, \bar{c})\in R^n$ is called a *solution* of the difference-retarded system (2.1) if it satisfies the following conditions:
a) $x(k\,;\,k_0, \bar{c})$ is defined for $k_0-m\leqslant k\leqslant k_0+k^*$ for some integer $k^*>0$;
b) $x(k\,;\,k_0, \bar{c})=c_k$ for some $k_0-m\leqslant k\leqslant k_0$ (the initial value);
c) $x(k\,;\,k_0, \bar{c})$ satisfies (2.1) for $k_0\leqslant k\leqslant k_0+k^*-1$.
For a solution $x(k\,;\,k_0, \bar{c})$ of (2.1) denote $\bar{x}(k\,;\,k_0, \bar{c})=(x(k\,;\,k_0, c), x(k-1\,;\,k_0, \bar{c}),\dots, x(k-m\,;\,k_0, \bar{c}))\in X_{m+1}$. We then see that $\bar{x}(k_0\,;\,k_0, \bar{c})=\bar{c}$. In the following it will be assumed that a solution to (2.1) exists and is unique for all $k\in I(k_0)$ we will consider two systems

$$\text{(2.2)}\qquad x(k+1)=A(k)x(k)$$

and

$$\text{(2.3)}\qquad y(k+1)=A(k)y(k)+f(k, y(k), y(k-1), \dots, y(k-m)),$$

where x, y are n-dimensional vectors, $A: I(k_0)\to M^n$, where M^n is the set of all non-singular matrices on $I(k_0)$ and $f: I(k_0)\times X_{m+1}\to R^n$ is, for any $k\in I(k_0)$, continuous as a function of $y_0, y_1,\dots, y_m$.

3. Basic results. Let $X(k)$ be the fundamental matrix of (2.2) such that $X(k_0)=E$, where E denotes the identity matrix. In this section we shall show, on one hand, that, under suitable conditions, for a given solution $x=x(k)\in\Phi_1$ of (2.2) there exists at least a solution $y=y(k)\in\Phi_1$ of (2.3) such that

$$\text{(3.1)}\qquad \lim_{k\to\infty} |y(k)-x(k)|=0,$$

and conversely. On the other hand, we shall show that, under suitable conditions, instead of (3.1), a condition of the form

$$\text{(3.2)}\qquad |y(k)-x(k)|\leqslant Qg(k), \ k\in I(k_0),$$

holds, with a certain function $g(k)$. If a condition of the form (3.1) or (3.2) is satisfied, then the two systems (2.2) and (2.3) may be studied by comparing their solutions *as regards to* the stability, boundedness and, generally, the asymptotic behaviour.

We now give a lemma which is a particular case of our Lemma in [7].

Lemma 3.1. *Let $X(k)$ be a non-singular matrix defined for $k\in I(k_0)$. Assume there exists a constant $K>1$ such that for $k\in I(k_0)$, the inequality*

$$\sum_{s=k_0}^{k-1} |X(k)X^{-1}(s+1)|^q\leqslant K^q, \quad q\geqslant 1,$$

holds. Then it follows $\lim\limits_{k\to\infty} |X(k)|=0$.

Before giving our main results we note that the technique used in the proof is a combination of the Hölder's well-known inequality and the Schauder-Tychonoff theorem.

Theorem 3.1. *Suppose that the following conditions are satisfied:*

i) *there exists a constant $K>1$ such that*

$$\text{(3.3)}\qquad \sum_{s=k_0}^{k-1} |X(k)X^{-1}(s+1)|^q\leqslant K^q, \textit{ for } k\in I(k_0+1) \textit{ and } 1<q<\infty;$$

ii) *there exists a scalar nonnegative function $\omega(k, u)$ defined on $I(k_0)\times[0, \infty)$, continuous with respect to u, nondecreasing in u for each fixed k, such that*

$$\text{(3.4)}\qquad |f(k, z_0, z_{-1},\dots, z_{-m})|\leqslant\omega(k, \|\bar{z}\|), \ k\in I(k_0)$$

and $\bar{z}=(z_0, z_{-1},\dots, z_{-m})\in X_{m+1}$;

iii). *for $p>1$ $(1/p+1/q=1)$ and every constant $0<\lambda<\infty$, we have*

$$\text{(3.5)}\qquad \sum_{s=k_0}^{\infty} \omega^p(s, \lambda)\leqslant(\rho/K)^p,$$

where ρ is defined below.

Then, corresponding to each solution $x=x(k)\in\Phi_1$ of (2.2), there exists at least a solution $y=y(k)\in\Phi_1$ of (2.3) such that (3.1) holds. Conversely, to each solution $y=y(k)\in\Phi_1$ of (2.3), there exists a solution $x=x(k)$ of (2.2) such that (3.1) is satisfied.

Proof. Let $x=x(k)$ be a solution of (2.2) such that $x(k)\in\Phi_1$, i.e., $|x(k)|\leqslant Q$, $k\in I(k_0)$. Because (2.2) is a homogeneous system, we may assume always $|x(k)|\leqslant\rho$, where $\rho=\max\{|c_k|\,;\ c_k=y(k),\ k_0-m\leqslant k\leqslant k_0\}$.

Define the ball

$$B_\rho=\{u\,;\ |u|_{\Phi_1}\equiv\sup(|u(k)|,\ k\in I(k_0-m)\leqslant\rho\}$$

which is a convex closed set. For $y\in B_{2\rho}$, define the operator T by

$$\text{(3.6)}\qquad \begin{cases} (Ty)(k)=c_k,\ k_0-m\leqslant k\leqslant k_0\ \ (c_k=y(k)), \\ (Ty)(k)=x(k)+\sum\limits_{s=k_0}^{k-1} X(k)X^{-1}(s+1)f(s, y(s), y(s-1), \dots, y(s-m)), \end{cases}$$
$$k\in I(k_0),$$

where $x(k)$ is a solution of (2.2) for which $|x(k)|\leqslant\rho$. We show that $T(B_{2\rho})\subset B_{2\rho}$. If $k_0-m\leqslant k\leqslant k_0$, then $|(Ty)(k)|=|c_k|\leqslant\rho$. If $k\in I(k_0)$; then

$$|(Ty)(k)|\leqslant|x(k)|+\sum_{s=k_0}^{k-1}|X(k)X^{-1}(s+1)|\omega(s, \|\bar{y}(s)\|)\leqslant$$
$$\rho+\Big[\sum_{s=k_0}^{k-1}|X(k)X^{-1}(s+1)|^q\Big]^{1/q}\Big[\sum_{s=k_0}^{k-1}\omega^p(s, 2\rho)\Big]^{1/p}\leqslant\rho+K\rho/K=2\rho.$$

Further, we show that T is continuous on $B_{2\rho}$. Let $\{y^i\}_{i=1}^{\infty}$, $y^i\in B_{2\rho}$ be a sequence converging uniformly to $y\in B_{2\rho}$ on every $I_r(k_0)$, $r=0,1,\dots$. Let $\varepsilon>0$ be given. We show that for $k\in I_r(k_0)$, we have $(Ty^i)(k)\rightrightarrows(Ty)(k)$. Since f is continuous and $y^i(k)\rightrightarrows y(k)$ on each compact set $I_r(k_0)$, there exists a natural number N such that for $i\geqslant N$ we have

$$\text{(3.7)}\qquad |f(k, y^i(k), y^i(k-1),\dots, y^i(k-m))-f(k, y(k), y(k-1),\dots, y(k-m))|<\varepsilon/Kr.$$

Using (3.6) and (3.7), for $i\geqslant N$, we get

$$|(Ty^i)(k)-(Ty)(k)|\leqslant K\Big[\sum_{s=k_0}^{k-1}|f(s, y^i(s), y^i(s-1),\dots, y^i(s-m))-$$
$$f(s, y(s), y(s-1),\dots, y(s-m))|^p\Big]1/p\leqslant K(\varepsilon/Kr)r=\varepsilon.$$

For $k_0-m\leqslant k\leqslant k_0$, $(Ty)(k)=c_k$ is continuous. The functions in the image space $T(B_{2\rho})$ are uniformly bounded for each k since $T(B_{2\rho})\subset B_{2\rho}$. The equicontinuity of the family $T(B_{2\rho})$ follows immediately because the functions in $T(B_{2\rho})$ are defined for a discrete variable k. Therefore, $\overline{T(B_{2\rho})}$ is a compact set. The Schauder-Tychonoff fixed point theorem now establishes that the mapping T has at least a fixed point in $B_{2\rho}$. This fixed point is a solution of (2.3). To verify that (3.1) holds; we note that from (3.6) we have

$$|y(k)-x(k)|\leqslant|X(k)|\sum_{s=k_0}^{k-1}|X^{-1}(s+1)f(s, y(s), y(s-1), \ldots, y(s-m))|\leqslant$$
$$\leqslant\Big[\sum_{s=k_1}^{k-1}|X(k)X^{-1}(s+1)|^q\Big]^{1/q}\Big[\sum_{s=k_1}^{k-1}\omega^p(s, 2\rho)\Big]^{1/p}.$$

For $\varepsilon>0$, by ii) we may choose $k_1\in I(k_0)$ such that $\sum_{s=k_1}^{\infty}\omega^p(s, 2\rho)<(\varepsilon/2K)^p$. If we take $k_2\in I(k_1)$, as a consequence of Lemma 3.1, we have

$$|X(k)|\sum_{s=k_0}^{k_1-1}|X^{-1}(s+1)f(s, y(s), y(s-1), \ldots, y(s-m))|<\varepsilon/2,\ k\in I(k_2).$$

Hence, for sufficiently large k, the relation (3.1) is satisfied. The last statement of the theorem follows immediately. Let $y=y(k)\in\Phi_1$ be a solution of (2.3) such that $y(k)=c_k$, $k_0-m\leqslant k\leqslant k_0$. Define

$$\text{(3.8)}\quad x(k)=y(k)-\sum_{s=k_0}^{k-1}X(k)X^{-1}(s+1)f(s, y(s), y(s-1), \ldots, y(s-m)),\ k\in I(k_0).$$

By the same argument, from (3.8) it follows that $x=x(k)$ is a solution of (2.2) which belongs to Φ_1 and satisfies (3.1).

Corollary 3.1. *When $p=1$, hence $q=\infty$, the condition (3.3) becomes*

$$\text{(3.9)}\qquad \sup\{|X(k)X^{-1}(s+1)\ ;\ k_0\leqslant s\leqslant k-1\}\leqslant K,$$

for $k\in I(k_0+1)$, and it follows that (3.2) holds with $g(k)\equiv 1$ and $Q=\rho$. If, in addition, we suppose that $\lim_{k\to\infty}|X(k)|=0$, which is not a consequence of our Lemma 3.1, then (3.1) holds.

We note that the condition (3.9) occurs when, for example, the linear system (2.2) is exponentially asymptotically stable, i.e., if there exist positive constants K and α such that $|X(k)X^{-1}(s)|\leqslant Ke^{-\alpha(k-s)}$, $k>s\geqslant 0$.

Remark 3.1. A result of this kind has been obtained by B. G. Pachpatte [3] when the retarded difference system (2.3) is of a particular form, namely, when $m=1$. In that case he assumed that $\omega(k, u)\equiv h(k)u$, with $\sum_{s=k_0}^{\infty}h(s)<\infty$ and all solutions of (2.2) are bounded.

The Theorem 3.2 below deals with a relation of the form (3.2) where $g(k)=|X(k)|$, $k\in I(k_0)$ and $g(k)=|E|$, $k_0-m\leqslant k\leqslant k_0$. We note that this theorem occurs only in a sense and only with an additional condition concerning the constant c from $x(k)=X(k)c$.

Theorem 3.2. *Let $x(k)=X(k)c$ a solution of (2.2) with $|c|<\rho$. Assume that there exists a scalar nonnegative function $\omega(k, u)$ defined on $I(k_0)\times[0, \infty)$, continuous with respect to u, nondecreasing in u for each fixed k such that*

$$\text{(3.10)}\qquad |f(k, z_0, z_{-1}, \ldots, z_{-m})|\leqslant\omega(k, \|\bar{z}\|),\ k\in I(k_0),\ \bar{z}\in X_{m+1},$$

and

$$\text{(3.11)}\qquad \sum_{s=k_0}^{\infty}|X^{-1}(s+1)|\omega(s, \rho g(s))<\rho-|c|.$$

Then, there exists at least a solution $y(k)$ of (2.3) such that (3.2) holds with $Q=\rho$ and $g(k)$ defined above.

Proof. Because the proof is similar to the proof of Theorem 3.1, we will give only an outline. Consider the set

$$F=\{u\ ;\ |u(k)|\leqslant\rho g(k),\ k\in I(k_0-m)\}\subset\Phi_g.$$

For $y\in F$, define the operator T by (3.6). It is evident that F is a convex closed set. If $k_0-m\leqslant k\leqslant k_0$, then $|(Ty)(k)|=|c_k|\leqslant\rho\leqslant\rho|E|=\rho g(k)$. If $k\in I(k_0)$ then, from (3.6), (3.10) and (3.11), it follows

$$|(Ty)(k)|\leqslant g(k)[|c|+\rho-|c|]=\rho g(k),\ \text{i.e.,}\ T(F)\subset F.$$

The continuity of T and therefore the compactness of $\overline{T(F)}$ can be shown in the same manner as in Theorem 3.1.

Example 3.1. R. K. Brayton and R. A. Willoughby [1] investigated a discrete analogue of the system

$$\text{(3.12)}\qquad \dot{x}(t)+A\dot{x}(t-r)+Bx(t)+Cx(t-r)=0,$$

where A, B, C are real $n\times n$ matrices, $r>0$, and x is an n-vector. The analogue has been obtained by making the replacements

$$\text{(3.13)}\quad\begin{cases}\dot{x}(t)\to(x(t+h)-x(t))/h, & \dot{x}(t-r)\to(x(t+h-r)-x(t))/h,\\ x(t)\to(\mu x(t+h)+(1-\mu)x(t)), & x(t-r)\to(\mu x(t+h-r)+(1-\mu)x(t-r)),\end{cases}$$

where $0\leqslant\mu\leqslant 1$. Choosing h so that $r=mh$ for some integer m and setting $t=kh$, we obtain from (3.12) the discrete difference equation

$$\text{(3.14)}\quad (E+\mu hB)x(k+1)=[E-(1-\mu)hB]x(k)-(A+\mu hC)x(k-m+1)+$$
$$+[A-(1-\mu)hC]x(k-m),$$

where $x(i)\equiv x(ih)$. In the following we will apply the Theorem 3.2 for a discrete analogue of the equation

$$\text{(3.15)}\qquad \dot{y}(t)=-by(t)+d[y(t-r)]^{\nu},$$

where $b>0$ and d and $\nu>1$ are constants, by using the same replacements (3.13). If we denote $y(i)=y(ih)$, then (3.15) reduces to

$$\text{(3.16)}\qquad y(k+1)=\beta y(k)+\gamma[\mu y(k-m+1)+(1-\mu)y(k-m)]^{\nu},$$

where $\beta=(1-bh+bh\mu)/(1+bh\mu)$, $\gamma=dh/(1+bh\mu)$. We may assume always that $h>0$ is small enough so that $\beta>0$. The fundamental „matrix" of the linear equation

$$\text{(3.17)}\qquad x(k+1)=\beta x(k),$$

is $X(k)=\beta^{k-k_0}$, and therefore, $X^{-1}(k)=\beta^{k_0-k}$. To see that a condition of the form (3.10) is satisfied, we note that $|f(k, z_0, z_{-1}, \ldots, z_{-m})|\equiv\gamma[\mu z_{-m+1}+(1-\mu)z_{-m}]$ and hence $|f(k, z_0, z_{-1}, \ldots, z_{-m})|\leqslant|\gamma|[\mu|z_{-m+1}|^{\nu}+(1-\mu)|z_{-m}|^{\nu}]\leqslant|\gamma|\|\bar{z}\|^{\nu}$, where we used the convexity of the function $\varphi(t)=t^{\nu}$, $\nu>1$, and $\|\bar{z}\|=\sup(|z_{-m+1}|, |z_{-m}|)$. From here it follows that $\omega(k, u)=|\gamma|u^{\nu}$ and because $g(k)=|X(k)|=\beta^{k-k_0}$, then (3.11) becomes

$$\sum_{s=k_0}^{\infty}|X^{-1}(s+1)|\omega(s, \rho g(s))=\sum_{s=k_0}^{\infty}\beta^{-s+k_0-1}|\gamma|\rho^{\nu}\beta^{\nu(s-k_0)}=|\gamma|\rho^{\nu}/(\beta-\beta^{\nu})<\rho-|c|,$$

which is satisfied if, for example, $|\gamma|<(\beta-\beta^{\nu})(\rho-|c|)/\rho^{\nu}$ and this is possible to realise if h is sufficiently small. The above limitation for $|\gamma|$ means in fact that (3.2) holds if the perturbation in (3.16) is small enough.

Example 3.2. Now we shall apply the Theorem 3.1 to the following perturbed system

(3.18) $y(k+1)=A(k)y(k)+f(k, y(k), y(k-1), \ldots, y(k-m)), \quad k\in I(1),$

where

$$y=\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad A(k)=\begin{pmatrix} 0 & (1+k)^{-1} \\ (3k^2+1)^{-1} & (k^4+1)^{-1} \end{pmatrix}, \quad f=\begin{pmatrix} f_1 \\ f_2 \end{pmatrix},$$

with $f_1=(\gamma_1/k)[y_1(k-m+1)]^{-\nu_1}$, $f_2=(\gamma_2/k)[y_2(k-m)]^{\nu_2}$, $\nu_1>0$, $\nu_2>0$ constants. Denote $\gamma=\max\{|\gamma_1|, |\gamma_2|\}>0$ and $\nu=\min(\nu_1, \nu_2)>0$. We note that in this case

$$y(k-m+1)=\begin{pmatrix} y_1(k-m+1) \\ 0 \end{pmatrix}, \quad y(k-m)=\begin{pmatrix} 0 \\ y(k-m) \end{pmatrix},$$

and

$$|f|=|f_1|+|f_2|\leqslant(\gamma/k)[|y(k-m+1)|^{-\nu}+|y(k-m)|^{-\nu}]\leqslant(2\gamma/k)\|z(k)\|,$$

and therefore, $\omega(k, u)=(2\gamma/k)u^{-\nu}$. If we take $\lambda\geqslant 1$ then the relation (3.5) becomes

$$\sum_{s=1}^{\infty}\omega^p(s, \lambda)=\frac{2^p\gamma^p}{\lambda^{p\nu}}\sum_{s=1}^{\infty}\frac{1}{s^p}=\frac{2^p\gamma^p}{\lambda^{p\nu}}\zeta(p),$$

where $\zeta(p)$ is Riemann's function. If. we choose γ such that $\gamma\leqslant(\rho/4)[\zeta(p)]^{-1/p}$, then (3.5) holds. This restriction concerning γ can be simplified for particular p. For example, because $\zeta(p)<1+1/2^p+2^{-\sigma}/(1+2^{-\sigma})$, where $p=1+\sigma$, when $p=2$, we have $\zeta(2)<9/4$ and (3.5) will be satisfied if $\gamma<\rho/6$. With regard to (3.3) we note before that for the fundamental matrix $X(k)$ we have $X(k)=A(k-1)A(k-2)\ldots A(1)$. Hence, $X(k)X^{-1}(s+1)=\prod_{\nu=s+2}^{k-1}A(\nu)$, and because $|A(\nu)|=\max_j\sum_{i=1}^{2}|a_{ij}|\leqslant 2/(v+1)$, we obtain

$$|X(k)X^{-1}(s+1)|([\leqslant 2/s+3)2/(s+4)] \ldots 2/k<2/k, \quad 1\leqslant s<k.$$

Therefore, for $k\in I(1)$,

$$\sum_{s=1}^{k-1}|X(k)X^{-1}(s+1)|^q\leqslant(k-1)2/k<2^q, \quad q>1,$$

and (3.3) holds with $K=2$.

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ON SECOND ORDER ITÔ TYPE STOCHASTIC INTEGRODIFFERENTIAL EQUATIONS

BY

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1. Introduction. During the last few years stimulating research works have been undertaken in the field of stochastic differential and integral equations. A good account on this subject may be found in several recent monographs, see Arnold [2], Bharucha-Reid [4], Friedman [8], Gihman and Skorohod [9], Ladde and Lakshmikantham [14], Tsokos and Padgett [16] to name a few. An interesting study of random operator equations have been made by Adomian [1], Bharucha-Reid [5], . Hans [11] and others (see, [4, chapter 3]). Recently Berger and Mizel [3], Kannan and Bharucha-Reid [12] and Pachpatte [15] have studied more general Itô type integro-differential equations by using various techniques. Although there exists a vast literature on first order Itô type stochastic differential and integral equations, the field of higher order Itô type stochastic differential equation awaits for its development. A physical phenomenon such as the simple harmonic oscillator driven by a Wiener process leads to the following second order stochastic differential equation

(1.1) $$d\dot{x}(t, \omega)+2a\dot{x}(t, \omega)+bx(t, \omega)=d\beta(t, \omega),$$

where $\dot{x}(t, \omega)$ is the derivative of $x(t, \omega)$ process and describes the position of the particle. The equation of the above form leads to the stochastic differential equation of second order of the form

(1.2) $$d\dot{x}(t, \omega)=f(t, x(t, \omega), \dot{x}(t, \omega))dt+g(t, x(t, \omega), \dot{x}(t, \omega))d\beta(t).$$

The study of existence and uniqueness of solutions of equation (1.2) have been developed by Borchers [6], Bharucha-Reid [4], Goldstein [10] and Kannan [13]. The object of the present paper is to study the existence and uniqueness of the solutions of a more general Itô type second order integrodifferential equation, which contains equation (1.2) as a special case, by using suitable version of the well known Banach fixed point theorem. The more general formulation of the problem considered in this paper is strongly inspired by the recent studies of Berger and Mizel [3], Kannan and Bharucha-Reid [12] and Pachpatte [15].