$$y(k+1)=A(k)y(k)+f(k, y(k), y(k-1), \dots, y(k-m)), \quad k\in I(1), \tag{3.18}$$

where

$$y=\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad A(k)=\begin{pmatrix} 0 & (1+k)^{-1} \\ (3k^2+1)^{-1} & (k^4+1)^{-1} \end{pmatrix} f=\begin{pmatrix} f_1 \\ f_2 \end{pmatrix},$$

with $f_1=(\gamma_1/k)[y_1(k-m+1)]^{-\nu_1}$, $f_2=(\gamma_2/k)[y_2(k-m)]^{\nu_2}$, $\nu_1>0$, $\nu_2>0$ constants. Denote $\gamma=\max\{|\gamma_1|, |\gamma_2|\}>0$ and $\nu=\min(\nu_1, \nu_2)>0$. We note that in this case

$$y(k-m+1)=\begin{pmatrix} y_1(k-m+1) \\ 0 \end{pmatrix}, \quad y(k-m)=\begin{pmatrix} 0 \\ y(k-m) \end{pmatrix},$$

and

$$|f|=|f_1|+|f_2|\leqslant(\gamma/k)[|y(k-m+1)|^{\nu}+|y(k-m)|^{-\nu}]\leqslant(2\gamma/k)\|z(k)\|,$$

and therefore, $\omega(k, u)=(2\gamma/k)u^{-\nu}$. If we take $\lambda\geqslant1$ then the relation (3.5) becomes

$$\sum_{s=1}^{\infty}\omega^p(s, \lambda)=\frac{2^p\gamma^p}{\lambda^{p\nu}}\sum_{s=1}^{\infty}\frac{1}{s^p}=\frac{2^p\gamma^p}{\lambda^{p\nu}}\zeta(p),$$

where $\zeta(p)$ is Riemann's function. If. we choose γ such that $\gamma\leqslant(\rho/4)[\zeta(p)]^{-1/p}$, then (3.5) holds. This restriction concerning γ can be simplified for particular p. For example, because $\zeta(p)<1+1/2^p+2^{-\sigma}/(1+2^{-\sigma})$, where $p=1+\sigma$, when $p=2$, we have $\zeta(2)<9/4$ and (3.5) will be satisfied if $\gamma<\rho/6$. With regard to (3.3) we note before that for the fundamental matrix $X(k)$ we have $X(k)=$ $=A(k-1)A(k-2)\dots A(1)$. Hence, $X(k)X^{-1}(s+1)=\prod_{\nu=s+2}^{k-1}A(\nu)$, and because $|A(\nu)|=\max_j\sum_{i=1}^{2}|a_{ij}|\leqslant2/(\nu+1)$, we obtain

$$|X(k)X^{-1}(s+1)|([\leqslant 2/s+3)2/(s+4)]\ \dots\ 2/k<2/k, \quad 1\leqslant s<k.$$

Therefore, for $k\in I(1)$,

$$\sum_{s=1}^{k-1}|X(k)X^{-1}(s+1)|^q\leqslant(k-1)2/k<2^q, \quad q>1,$$

and (3.3) holds with $K=2$.

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ON SECOND ORDER ITÔ TYPE STOCHASTIC INTEGRODIFFERENTIAL EQUATIONS*

BY

M.G. MURGE and B.G. PACHPATTE

1. Introduction. During the last few years stimulating research works have been undertaken in the field of stochastic differential and integral equations. A good account on this subject may be found in several recent monographs, see Arnold [2], Bharucha-Reid [4], Friedman [8], Gihman and Skorohod [9], Ladde and Lakshmikantham [14], Tsokos and Padgett [16] to name a few. An interesting study of random operator equations have been made by Adomian [1], Bharucha-Reid [5], , Hans [11] and others (see, [4, chapter 3]). Recently Berger and Mizel [3], Kannan and Bharucha-Reid [12] and Pachpatte [15] have studied more general Itô type integro-differential equations by using various techniques. Although there exists a vast literature on first order Itô type stochastic differential and integral equations, the field of higher order Itô type stochastic differential equation awaits for its development. A physical phenomenon such as the simple harmonic oscillator driven by a Wiener process leads to the following second order stochastic differential equation

$$d\dot{x}(t, \omega)+2a\dot{x}(t, \omega)+bx(t, \omega)=d\beta(t, \omega), \tag{1.1}$$

where $\dot{x}(t, \omega)$ is the derivative of $x(t, \omega)$ process and describes the position of the particle. The equation of the above form leads to the stochastic differential equation of second order of the form

$$d\dot{x}(t, \omega)=f(t, x(t, \omega), \dot{x}(t, \omega))dt+g(t, x(t, \omega), \dot{x}(t, \omega))d\beta(t). \tag{1.2}$$

The study of existence and uniqueness of solutions of equation (1.2) have been developed by Borchers [6], Bharucha-Reid [4], Goldstein [10] and Kannan [13]. The object of the present paper is to study the existence and uniqueness of the solutions of a more general Itô type second order integrodifferential equation, which contains equation (1.2) as a special case, by using suitable version of the well known Banach fixed point theorem. The more general formulation of the problem considered in this paper is strongly inspired by the recent studies of Berger and Mizel [3], Kannan and Bharucha-Reid [12] and Pachpatte [15].

2. Preliminaries. Consider the following second order stochastic integrodifferential equation of the form

$$(2.1)\qquad \begin{aligned} d\dot{x}(t,\omega)&=f(t,x(t,\omega),\dot{x}(t,\omega),Ax(t,\omega),\omega)dt+\\ &+g(t,x(t,\omega),\dot{x}(t,\omega),Bx(t,\omega),\omega)d\beta(t),\end{aligned}$$

with initial values

$$(2.2)\qquad x(0,\omega)=\xi_0(\omega),\ \dot{x}(0,\omega)=\eta_0(\omega),$$

where $\omega\in\Omega$, the supporting set of complete probability measure space $(\Omega,\mathcal{A},P)$ and

$$(2.3)\qquad Ax(t,\omega)=\int_0^t h(t,s,x(s,\omega),\dot{x}(s,\omega),\omega)d\beta(s),$$

$$(2.4)\qquad Bx(t,\omega)=\int_0^t k(t,s,x(s,\omega),\dot{x}(s,\omega),\omega)d\beta(s).$$

In particular, if we impose on f, g, h and k various meanings, it is apparent that equation (2.1) has a great diversity. For example if $h\equiv 0$, $k\equiv 0$, then equation (2.1) reduces to the second order Itô type stochastic equation studied by the authors in [4, 6, 10, 13]. The equation (2.1)—(2.2) is equivalent to the stochastic integrodifferential equation

$$(2.5)\qquad \begin{aligned}\dot{x}(t,\omega)&=\eta_0(\omega)+\int_0^t f(s,x(s,\omega),\dot{x}(s,\omega),Ax(s,\omega),\omega)ds+\\ &+\int_0^t g(s,x(s,\omega),\dot{x}(s,\omega),Bx(s,\omega),\omega)d\beta(s).\end{aligned}$$

Equivalently we consider either the random integrodifferential equation (2.5) or the equation

$$(2.6)\qquad \begin{aligned}x(t,\omega)&=\xi_0(\omega)+\eta_0(\omega)t+\int_0^t\Big[\int_0^s f(u,x(u,\omega),\dot{x}(u,\omega),Ax(u,\omega),\omega)du+\\ &+\int_0^s g(u,x(u,\omega),\dot{x}(u,\omega),Bx(u,\omega),\omega)d\beta(u)\Big]ds.\end{aligned}$$

We first list the following assumptions used in our subsequent discussion.

(H_1) Let $\beta(t)$, $t\geqslant 0$ be a continuous L_2-martingale with independent increments such that there exists a continuous, nondecreasing function $F(t)$ on R^+ with the property that $E\{[\beta(t)-\beta(s)]^2|\mathcal{A}_s\}=F(t)-F(s)$, almost surely (*a.s.*), $(s<t)$, where $\mathcal{A}_s$ denotes an increasing family of σ-fields generated by $\xi_0(\omega)$, $\eta_0(\omega)$ and $\beta(v)-\beta(u)$, $0\leqslant u\leqslant v\leqslant s$, $\xi_0(\omega)$, $\eta_0(\omega)\in L^2(\Omega)$ and $\xi_0(\omega)$, $\eta_0(\omega)$ are independent of the increments $\{\beta(v)-\beta(u)\}$, $0\leqslant u\leqslant v$.

(H_2) Let h and k be Borel measurable functions defined on $R^2\times R^2\times\Omega$ into R and f and g be Borel measurable functions defined on $R^+\times R^3\times\Omega$ into R and $\omega\to f(t,x,y,z,\omega)$, $\omega\to g(t,x,y,z,\omega)$, $\omega\to h(\tau,t,x,y,\omega)$, $\tau\geqslant t$; $\omega\to k(\tau,t,x,y,\omega)$, $\tau\geqslant t$; $x,y,z\in R$, are $\mathcal{A}_t$-measurable functions (where $R^+=$

so that $\leqslant$ is interval-asymptotic and *semi-closed* in the sense of N a c h b i n [12, Appendix] (hence interval-closed). Let $a\in X$. By a *b-segment* containing a we mean the set $C(a,b)$ where C is any maximal chain containing a. (Of course this definition is meaningful, by virtue of the fact that b is the only maximal element of $(X,\leqslant)$ (and hence it must be contained in any maximal chain)). To discuss the relationships between the notion we just introduced and that of *metric segment* in the sense of Menger [2, ch. VI, §6.3] put $s=d(a,b)$ and let f denote the function between $C(a,b)$ and the interval $[0,s]$ introduced as

$$(3)\qquad f(x)=d(x,a),\quad x\in C(a,b).$$

By (2), it immediately follows

$$a\leqslant x\leqslant y\leqslant b \text{ implies } f(y)-f(x)=d(x,y)$$

so that f is a strictly increasing isometry between $C(a,b)$ and its image in $[0,s]$. The main point is to find sufficient conditions under which $f(C(a,b))=[0,s]$. To do this, let us call the ordering $\leqslant$, *dense* when for each couple $u,v\in X$; $u<v$ there exists $z\in X$ with $u<z<v$, and the ambient metric space (X,d), *b-complete* provided any monotone Cauchy sequence for which there is an order-interval containing it, converges. Now, as an useful answer to this question, we have

Theorem 1. *Assume $\leqslant$ is dense and (X,d) is b-complete. Then $f(C(a,b))=[0,s]$.*

Proof. Suppose by contradiction there exists r in $[0,s]$ not belonging to $f(C(a,b))$ (hence $0<r<s$), and put $A=\{x\in C(a,b);f(x)\leqslant r\}$, $B=\{x\in C(a,b);f(x)\geqslant r\}$. Clearly, A and B are closed chains (since $C(a,b)$ is closed and f is continuous) and, in addition, $A\cup B=C(a,b)$, $A\cap B=\varnothing$. As (X,d) is interval-complete, it follows by Lemma 3 that the element $a'=$l.u.b. (A) and $b'=$g.l.b.(B) exist, belong to A and B respectively, and $a\leqslant a'<b'\leqslant b$. Since $\leqslant$ is dense, there exists $c\in X$ with $a'<c<b'$ (so that c does not belong to $C(a,b)$). On the other hand, c is comparable with any $x\in C$ and therefore (as C is maximal) $c\in C$ or, equivalently, $c\in C(a,b)$. This contradiction ends our argument. q.e.d.

From this result it follows that the notions of b-segment and metric segment with endpoint b are identical. Note at this moment the density of the ambient ordering means : for each couple $u,v\in X$ with $u\neq v$, $d(u,v)+d(v,b)=d(u,b)$, there exists $z\in X$ distinct from u and v, with $d(u,z)+d(z,v)=d(u,v)$ and, for this reason, it is natural to call the ambient space (X,d), *b-convex* in case such a property holds. Particularly, when (X,d) is (*metrically*) *convex* in the usual sense (cf. the above reference) it is surely b-convex for all $b\in X$ so that, as an immediate consequence of the above result we get the following classical statement (see, e.g., Kirk [8] for a modern treatment).

Corollary. *Let (X,d) be complete and convex. Then for any $a,b\in X$ there exists a metric segment with endpoints a and b.*

As another consequence of Theorem 1, note that, putting $t=\sup\{d(a,b);a\in C\}$ it necessarily follows the chain C itself is either an isometric and isotone image of $[0,t]$ in case it is a compact one, or as isometric and isotone image of $(-t, 0]$ (under the function $x \to -d(x, b)$) in case it is not compact. Some topological aspects of the problem can be found in Koch [9] (see also Ward Jr. [17]).

3. Mean value theorems. Let (X, d) be a metric space, and $b_0 \in X$, a given element. Letting $\leqslant$ be the ordering on X defined by (1), suppose C is a fixed maximal chain of $(X, \leqslant)$ and, under the general hypotheses X is both b_0-convex and b_0-complete, let C_0 denote either C itself or the b_0-segment $C(a_0, b_0)$ with $a_0 \in C$, $a_0 \neq b_0$, arbitrary fixed. Given $x \in C_0$, $x \neq b_0$, denote by $\limsup_{y \to x+}$ (inf) $g(y)$ (for each function $y \to g(y)$ from $C_0(x, <)$ into R) the infimum (supremum) over $\varepsilon > 0$ of the expressions sup (inf) $\{g(y) ; y \in C_0(x, <), d(x, y) < \varepsilon\}$, and by $\lim_{y \to x+} g(y)$ the element of $\bar{R}$ defined as the common limit of th sequences $(g(y_n); n \in N)$ with $(y_n; n \in N)$ decreasing in $C_0(x, <)$ and $y_n \to x$; in this context, given any function $f : C_0 \to \to R$, let $D^+(D_+)f(x)$ indicate (for each $x \in C_0$, $x \neq b_0$) lim sup (inf) as $y \to x+$ of the function $y \to (f(y) - f(x))/d(y, x)$ (from $C_0(x, <)$ into R) and $f(x+)$ the extended real number $\lim_{y \to x+} f(y)$ (remark at this moment that, when f is increasing, $f(x+)$ always exists (as an element of R) and $f(x) \leqslant f(x+)$). Now, let $(Y, \|\cdot\|)$ be a normed space. Given the convex subset K of Y, let dist $(., K)$ denote the usual distance from the points of Y to K; note the function $v \to \operatorname{dist}(v, K)$ is nonexpansive and

$$\operatorname{dist}(v + w, (\alpha + \beta)K) \leqslant \operatorname{dist}(v, \alpha K) + \operatorname{dist}(w, \beta K), \quad v, w \in Y, \ \alpha, \beta \geqslant 0 \tag{4}$$

as it can be readily verified. We shall assume further $\mathfrak{K} = (K_{ab} ; a, b \in C_0, a < b)$ is a family of such subsets of Y. Given the mapping $T : C_0 \to Y$ we shall say it possesses a $(b_0, C_0, \mathfrak{K})$-*mean value property* with respect to the function $f : C_0 \to R$, provided that

$$\operatorname{dist}\ (Tb - Ta)/d(b, a), K_{ab}) \leqslant (f(b) - f(a))/d(b, a), \quad a, b \in C_0, \ a < b. \tag{5}$$

It is our main aim in what follows to find a number of local conditions (involving operators D^+, D_+ we just introduced) from which the property (5) be obtained. To this end, given $T : C_0 \to Y$, let us term it *continuous at the left* when $(x_n ; n \in N)$ increasing in C_0 and $x_n \to x$ (hence $x_n \leqslant x$, $n \in N$) imply $Tx_n \to Tx$. Under these conventions the following answer to the above stated question may be given.

Theorem 2. *Letting C_0 and $\mathfrak{K}$ be as before, assume $T : C_0 \to Y$ is continuous at the left, $f : C_0 \to R$ is increasing and, in addition, that a denumerable subset A of C_0 may be found with the following properties at each $a, b \in C_0$, $a < b$.*

(i) *for any $x \in C_0(a, b) - A$ distinct from b,*

$$+\infty \neq \limsup_{y \to x+} (\inf) \operatorname{dist}((Ty - Tx)/d(y, x), K_{ab}) \leqslant D^+(D_+)f(x)$$

(ii) *for any $x \in C_0(a, b) \cap A$ distinct from b,*

$$\liminf_{y \to x+} \operatorname{dist}(Ty - Tx, d(y, x)K_{ab}) \leqslant f(x+) - f(x).$$

Then T possesses a $(b_0, C_0, \mathfrak{K})$-mean value property with respect to f, in the sense of (5).

$=[0, \infty)$, $R = (-\infty, +\infty)$) satisfying the following conditions: for each compact interval $I \subset R^+$ there is a positive constant $M = M(I)$ such that for all $t, s \in I$, $\omega \in \Omega$,

(a) (uniform Lipschitz condition). For all $x_i, y_i, z_i \in R$, $i = 1, 2$,

$$|h(t, s, x_1, y_1) - h(t, s, x_2, y_2)| + |k(t, s, x_1, y_1) - k(t, s, x_2, y_2)| \leqslant$$
$$\leqslant M(|x_1 - x_2|^2 + |y_1 - y_2)|^2)^{1/2},$$
$$|f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2)| + |g(t, x_1, y_1, z_1) - g(t, x_2, y_2, z_2)| \leqslant$$
$$\leqslant M(|x_1 - x_2|^2 + |y_1 - y_2|^2 + |z_1 - z_2|^2)^{1/2}.$$

(b) (Growth condition). For all $x, y, z \in R$,

$$|h(t, s, x, y)|^2 + |k(t, s, x, y)|^2 \leqslant M^2(1 + |x|^2 + |y|^2), \quad |f(t, x, y, z)|^2 +$$
$$+ |g(t, x, y, z)|^2 \leqslant M^2(1 + |x|^2 + |y|^2 + |z|^2).$$

We denote by $L^2(\Omega ; C([0, T_0]))$ the Banach space of the stochastic processes $m : \Omega X [0, T_0] \to R$ which are measurable in ω, continuous in t and $\|m\|_0^2 = E[\sup_{t \in [0, T_0]} |m(t)|^2] < \infty$ and by $L_{\mathfrak{F}}^2(\Omega ; C([0, T_0]))$ the closed subspace of non-anticipative processes. Let us denote

$$\|y\|_t^2 \equiv E[\sup_{0 \leqslant s \leqslant t} |y(s)|^2], \quad \text{for } y \in L^2(\Omega ; C([0, T_0])) ; \text{ then } \|\cdot\|_0 = \|\cdot\|_{T_0}.$$

Throughout, we assume in addition to (H_1) and (H_2) that the adapted family $\{x(t, \omega), \mathcal{A}_t, t \geqslant 0\}$ i.e. $x(t, \omega)$ is $\mathcal{A}_t$-measurable for $t \geqslant 0$ is an absolutely continuous process satisfying the following conditions:

(a) $\dot{x}(t, \omega)$ exists for every $t \geqslant 0$, and (a$_2$) $\int_0^t E\{|\dot{x}(s, \omega)|^2\} dG(s) < \infty$, $t \geqslant 0$

where $G(t) = t + F(t)$. Then, the (t, ω)-function $x^*(t, \omega)$ defined by right side of (2.5) is well defined.

By using the fact that

$$x(t, \omega) = \xi_0(\omega) + \int_0^t \dot{x}(s, \omega) ds \tag{2.7}$$

for $0 \leqslant t \leqslant T_0$, Doob's martingale inequality (see [8], Chapter 4, Theorem 3.6), Schwarz's inequality, the growth condition and condition (a$_2$) one can easily prove that the integrals

$$I(s, \omega) = \int_0^t f(u, x(u, \omega), \dot{x}(u, \omega), Ax(u, \omega), \omega) du$$

and

$$J(s, \omega) = \int_0^s g(u, x(u, \omega), \dot{x}(u, \omega), Bx(u, \omega), \omega) d\beta(u)$$

involved in the right side of (2.5) are stochastic processes from $L_{\mathfrak{F}}^2(\Omega ; C([0, T_0]))$ and J can be taken as a continuous martingale (see, Doob [7]).

We need the following lemmas in our subsequent discussion:

Lemma 1. *Let X be a Banach space. Let D be an operator which maps the elements of X into itself for which D^r is a contraction, where r is a positive integer. Then D has a unique fixed point in X.*

Lemma 2. *Suppose that the assumptions (H_1) and (H_2) (b) hold, then*

$$E[\sup_{0\leqslant t\leqslant T_0}|\dot{x}(t,\omega)|^2]\leqslant C(1+E|\xi_0(\omega)|^2+E|\eta_0(\omega)|^2) \tag{2.8}$$

and

$$E[\sup_{0\leqslant t\leqslant T_0}|x(t,\omega)|^2]\leqslant C^*(1+E|\xi_0(\omega)|^2+E|\eta_0(\omega)|^2) \tag{2.9}$$

where C and C^* are some constants depending on M, T_0 and $\Delta G(T_0)=G(T_0)-G(0)$.

Lemma 1 is a suitable version of the well known Banach fixed point theorem. Lemma 2 follows from (2.5) using $(a+b+c)^2\leqslant 3(a^2+b^2+c^2)$, Schwarz's inequality, Doob's martingale inequality, hypothesis (H_2) (b) definition of norm in $L^2_\beta(\Omega\,;C([0,T_0]))$ and (2.7).

3. Existence and Uniqueness of Solutions. In this section we state and prove our main result on the existence and uniqueness of solutions of equation (2.1)—(2.2).

Theorem 1. *Suppose that the hypotheses (H_1) and (H_2) hold. Then, there exists a unique process $\{x(t,\omega), t\geqslant 0\}$ with absolutely continuous sample functions such that*

(i) *$\{x(t,\omega)\}$ solves equation* (2.6) a.s.,

(ii) *$\dot{x}(t,\omega)$ exists also as L^2-derivative,*

(iii) $x(0,\omega)=\xi_0(\omega)$, $\dot{x}(0,\omega)=\eta_0(\omega)$ a.s.,

(iv) $x, \dot{x}\in L^2_\beta(\Omega\,;\, C([0, T_0]))$, *and*

(v) *$\dot{x}(t,\omega)$ solves a.s. equation* (2.5).

Proof. For $y\in L^2_\beta(\Omega\,;C([0,T_0]))$, we define the operator $D: L^2_\beta(\Omega\,; C([0,T_0]))\to L^2_\beta(\Omega, C([0,T_0]))$ by

$$\begin{aligned}(Dy)(t,\omega)=\eta_0(\omega)&+\int_0^t f(s,(Ly)(s,\omega),y(s,\omega),Ay(s,\omega),\omega)ds+\\&+\int_0^t g(s,(Ly)(s,\omega),y(s,\omega),By(s,\omega),\omega)d\beta(s).\end{aligned} \tag{3.1}$$

where

$$(Ly)(t,\omega)=\xi_0(\omega)+\int_0^t y(s,\omega)ds, \tag{3.2}$$

$$Ay(t,\omega)=\int_0^t h(t,s,(Ly)(s,\omega),y(s,\omega),\omega)d\beta(s), \tag{3.3}$$

$$By(t,\omega)=\int_0^t k(t,s,(Ly)(s,\omega),y(s,\omega),\omega)d\beta(s). \tag{3.4}$$

Let y, $z\in L^2_\beta(\Omega\,;C([0,T_0]))$. From (3.1) and using $(a+b)^2\leqslant 2(a^2+b^2)$, Schwarz's inequality, Doob's martingale inequality, hypothesis $(H_2)(a)$, $G(t)=t+F(t)$ and (3.2)—(3.4), we get

$$\begin{aligned}\|Dy-Dz\|_t=E[\sup_{0\leqslant s\leqslant t}|(Dy)(s,\omega)-(Dz)(s,\omega)|^2\leqslant C_1\int_0^t\|y-z\|_s dG(s)\leqslant\\ \leqslant C_1\|y-z\|_t\Delta G(t)\end{aligned}$$

where

$$C_1=2M^2(T_0+1)[\{T_0\Delta G(T_0)+1\}+M^2\Delta G(T_0)\{T_0\Delta G(T_0)/2+1\}] \tag{3.5}$$

and $\Delta G(t)=G(t)-G(0)$. By iteration it follows that $\|D^n y-D^n z\|_t\leqslant C_1^n\|y-z\|_t[\Delta G(t)]^n/n$.

Thus, there exists a positive integer n such that D^n is a contraction in $L^2_\beta(\Omega\,;C([0,T_0]))$. It follows from Lemma 1 that the operator D has a unique fixed point in $L^2_\beta(\Omega\,;C([0,T_0]))$. The fixed point $\dot{x}=y$, $x(t,\omega)=\xi_0(\omega)+\int_0^t y(s,\omega)ds$ of D is a solution of equation (2.5). The conclusion (ii) follows from the inequality (2.8) in Lemma 2. This completes the proof of the theorem.

4. Stronger Uniqueness and Existence. We establish next the following theorem which shows that, if two stochastic second order integro-differential equations have the same coefficients on the domain $[0, T_0]\times S$, $S\subset R$ and if the initial conditions coincide in S, then the corresponding solutions agree untill the first time they both leave S. Consider the equation (2.1)—(2.2) alongwith the following equation

$$\begin{aligned}d\dot{y}(t,\omega)=f^*(t,y(t,\omega),\dot{y}(t,\omega),A^*y(t,\omega),\omega)dt+\\+g^*(t,y(t,\omega),\dot{y}(t,\omega),B^*y(t,\omega),\omega)d\beta(t),\end{aligned} \tag{4.1}$$

with initial values

$$y(0,\omega)=\xi_0^*(\omega),\quad \dot{y}(0,\omega)=\eta_0^*(\omega), \tag{4.2}$$

where

$$A^*y(t,\omega)=\int_0^t h^*(t,s,y(s,\omega),\dot{y}(s,\omega),\omega)d\beta(s),$$

$$B^*y(t,\omega)=\int_0^t k^*(t,s,y(s,\omega),\dot{y}(s,\omega),\omega)d\beta(s),$$

and the functions f^*, g^*, h^*, k^* are as defined to the corresponding functions f, g, h, k respectively.

Theorem 2. *Suppose that the hypotheses (H_1) and (H_2) hold. Assume that the hypothesis (H_2) holds when f, g, h, k are replaced by f^*, g^*, h^*, k^* respectively and the hypothesis (H_1) holds when $\xi_0(\omega)$, $\eta_0(\omega)$ are replaced by $\xi_0^*(\omega)$, $\eta_0^*(\omega)$ respectively. Let $x(t,\omega)$ and $y(t,\omega)$ be solutions of equations (2.1)—(2.2) and (4.1)—(4.2) respectively. Let S be a domain in R and suppose that for $x(t,\omega)\in S$, $0\leqslant t\leqslant T_0$*

$$Ax(t,\omega)=A^*x(t,\omega),\qquad Bx(t,\omega)=B^*x(t,\omega),$$

$$f(t,x(t,\omega),\dot{x}(t,\omega),Ax(t,\omega),\omega)=f^*(t,x(t,\omega),\dot{x}(t,\omega),A^*x(t,\omega),\omega)$$

$$g(t, x(t, \omega), \dot{x}(t, \omega), Bx(t, \omega), \omega)=g^*(t, x(t, \omega), \dot{x}(t, \omega), B^*x(t, \omega), \omega).$$

Further assume that $x(0, \omega)=y(0, \omega)$, $\dot{x}(0, \omega)=\dot{y}(0, \omega)$ for almost all ω for which $x(0, \omega)\in S$ or $y(0, \omega)\in S$. Denote by τ the first time (after $t=0$) that the process $x(t, \omega)$ intersects $R|S$ if such time $t\in[0, T_0]$ exists, and $\tau=T_0$ otherwise. Denote by τ^* the analogous stopping time for the process $y(t, \omega)$. Then $P(\tau=\tau^*)=1$, and

$$P\{\sup_{0\leqslant t\leqslant\tau}|x(t, \omega)-y(t, \omega)|=0\}=1.$$

Proof. Let $\Phi(t, \omega)=1$ if $x(s, \omega)\in S$, $0\leqslant s\leqslant t$, and $\Phi(t, \omega)=0$ otherwise. Then $\Phi(t, \omega)$ is nonanticipating. We note that $\Phi(t, \omega)=\Phi(t, \omega)\Phi(s, \omega)$, $s\leqslant t$. From (2.5), equivalent integrodifferential equation form of (4.1)—(4.2) and using $a-c=(a-b)+(b-c)$ and $(a+b)^2\leqslant 2a^2+2b^2$, for $x(s, \omega)\in S$ we observe that

$$E[\Phi(t, \omega)|\dot{x}(t, \omega)-\dot{y}(t, \omega)|^2]\leqslant$$

$$\leqslant C_1\int_0^t E[\Phi(s, \omega)|\dot{x}(s, \omega)-\dot{y}(s, \omega)|^2]dG(s),\ 0\leqslant t\leqslant T_0 \tag{4.3}$$

where C_1 is as defined in (3.5). Now define $\Psi(t, \omega)=E[\Phi(t, \omega)|\dot{x}(t, \omega)-\dot{y}(t, \omega)|^2]$.

Then $\Psi(t, \omega)\geqslant 0$ and as $x(t, \omega)$ and $y(t, \omega)$ are solution processes of (2.1)—(2.2) and (4.1)—(4.2) respectively, we have

$$\Psi(t, \omega)\leqslant 2E[\sup_{0\leqslant t\leqslant T_0}\Phi(t, \omega)|\dot{x}(t, \omega)|^2+\sup_{0\leqslant t\leqslant T_0}\Phi(t, \omega)|\dot{y}(t, \omega)|^2]\leqslant M^*<\infty,$$

where M^* is a constant depending on M, T_0 and $\Delta G(T_0)$. Hence by iteration

$$0\leqslant\Psi(t, \omega)\leqslant C_1\int_0^t\Psi(s, \omega)dG(s)\leqslant(C_1)^nM^*[G(t)-G(0)]^n/n!.$$

Since $G(t)\leqslant G(T_0)<\infty$, letting $n\to\infty$, we have

$$E[\Phi(t, \omega)|\dot{x}(t, \omega)-\dot{y}(t, \omega)|^2]=0,\qquad 0\leqslant t\leqslant T_0.$$

Hence by the continuity of $\dot{x}(t, \omega)$ and $\dot{y}(t, \omega)$, $\sup_{0\leqslant t\leqslant T_0}\Phi(t, \omega)|\dot{x}(t, \omega)-\dot{y}(t, \omega)|=0$, a.s. It follows that $\dot{x}(t, \omega)=\dot{y}(t, \omega)$ a.s. if $0\leqslant t\leqslant\tau$ and consequently

$$x(t, \omega)=\xi_0(\omega)+\int_0^t\dot{x}(s, \omega)ds=\xi_0^*(\omega)+\int_0^t\dot{y}(s, \omega)ds=y(t, \omega)$$

a.s. if $0\leqslant t\leqslant\tau$. Therefore, by the continuity of $x(t, \omega)$ and $y(t, \omega)$,

$$P\{\sup_{0\leqslant t\leqslant\tau}|x(t, \omega)-y(t, \omega)|=0\}=1.$$

Also, it follows that $P(\tau^*\geqslant\tau)=1$.

A similar argument shows that $P(\tau\geqslant\tau^*)=1$. This completes the proof.

Now we shall use Theorem 2 to establish the stronger version of our Theorem 1.

Theorem 3. *Let assumption (H_1) hold. Suppose that the functions h, k, f and g in equation (2.1)—(2.2) satisfy the condition (b) of hypothesis (H_2). Suppose further that for each $N>0$, there exists a constant $M_N>0$, such that for all t, $s\in[0, T_0]$ and for all u, $\dot{u}\in R$ with $|u|+|\dot{u}|\leqslant N$, the functions h, k, f and g in equation (2.1)—(2.2) satisfy the condition (a) of hypothesis (H_2) by replacing M by M_N. Then there exists a unique a.s. continuous solution $x(t, \omega)$ of (2.1)—(2.2) on $[0, T_0]$ satisfying conclusions (i)—(v) of Theorem 1. Furthermore,*

$$E[\sup_{0\leqslant t\leqslant T_0}|x(t, \omega)|^2]\leqslant C^*[1+E|\xi_0(\omega)|^2+E|\eta_0(\omega)|^2], \tag{4.4}$$

where C^* is some constant depending only on M, T_0 and $\Delta G(T_0)$.

Proof. Let $\xi_{N_0}(\omega)=\xi_0(\omega)$ if $|\xi_0(\omega)|\leqslant N$ and zero otherwise, and for $|\eta_0(\omega)|\leqslant N$, $\eta_{N_0}(\omega)=\eta_0(\omega)$ and for $|\eta_0(\omega)|>N$, $\eta_{N_0}(\omega)=0$,

$$h_N(t, s, x, \dot{x}, \omega)=\begin{cases}h(t, s, x, \dot{x}, \omega), & \text{for } |x|+|\dot{x}|\leqslant N,\\ h(t, s, x, \dot{x}, \omega)[2-[(|x|+|\dot{x}|)/N]], & \text{for } N<|x|+|\dot{x}|\leqslant 2N,\\ 0, & \text{for } |x|+|\dot{x}|>2N,\end{cases}$$

$$A_Nx(t, \omega)=\int_0^t h_N(t, s, x, \dot{x}, \omega)d\beta(s),$$

and

$$f_N(t, x, \dot{x}, A_N, x, \omega)=\begin{cases}f(t, x, \dot{x}, Ax, \omega), & \text{for } |x|+|\dot{x}|\leqslant N,\\ f(t, x, \dot{x}, Ax, \omega)[2-[(|x|+|\dot{x}|)/N]], & \text{for } N<|x|+|\dot{x}|\leqslant 2N,\\ 0, & \text{for } |x|+|\dot{x}|>2N.\end{cases}$$

Similarly, $k_N(t, s, x, \dot{x}, \omega)$, $B_Nx(t, \omega)$, $g_N(t, x, \dot{x}, B_Nx, \omega)$ can be defined. Then $f_N(t, x(t, \omega), \dot{x}(t, \omega), A_Nx(t, \omega), \omega)$ and $g_N(t, x(t, \omega), \dot{x}(t, \omega), B_Nx(t, \omega), \omega)$ satisfy all the assumptions of Theorem 1. Therefore, Theorem 1 asserts that there exists a unique a.s. continuous solution $x_N(t, \omega)$ in $L_2^{\mathfrak{z}}(\Omega, C([0, T_0]))$ and consequently $\dot{x}_N(t, \omega)$ in $L_2^{\mathfrak{z}}(\Omega, C([0, T_0]))$ of the equation

$$\begin{aligned}d\dot{x}_N(t, \omega)=f_N(t, x_N(t, \omega), \dot{x}_N(t, \omega), A_Nx_N(t, \omega), \omega)dt+g_N(t, x_N(t, \omega),\\ \dot{x}_N(t, \omega), B_Nx_N(t, \omega), \omega)d\beta(t)\end{aligned} \tag{4.5}$$

with initial values

$$x_N(0, \omega)=\xi_{N_0}(\omega),\ \dot{x}_N(0, \omega)=\eta_{N_0}(\omega). \tag{4.6}$$

For definiteness we take $\mathcal{A}_t$ to be the σ-field generated by $\xi_0(\omega)$, $\eta_0(\omega)$ and $\beta(s)$, $0\leqslant s\leqslant t$. Let τ_N be the first time $|x_N(t, \omega)|+|\dot{x}_N(t, \omega)|\geqslant N$ if such a time $t\leqslant T_0$ exists and $\tau_N=T_0$ otherwise. If $N'>N$, then by Theorem 2, it follows that $x_N(t, \omega)=x_{N'}(t, \omega)$ and consequently $\dot{x}_N(t, \omega)=\dot{x}_{N'}(t, \omega)$ a.s. if $0\leqslant t\leqslant\tau_N$. Therefore, by using Chebyshev's inequality and Lemma 2, we get,

$$P[\sup_{0\leqslant t\leqslant T_0}|\{x_N(t, \omega)+\dot{x}_N(t, \omega)\}-\{x_{N'}(t, \omega)+\dot{x}_{N'}(t, \omega)\}|>0]\leqslant$$

$$\leqslant P[\sup_{0\leqslant t\leqslant T_0}\{|x_N(t, \omega)|+|\dot{x}_N(t, \omega)|\}>N]\leqslant P[\sup_{0\leqslant t\leqslant T_0}|x_N(t, \omega)|>N/2]+$$

$$+P[\sup_{0\leq t\leq T_0}|\dot{x}_N(t,\omega)|>N/2]\leq(4/N^2)E[\sup_{0\leq t\leq T_0}|x_N(t,\omega)|^2]+$$

$$+(4/N^2)E[\sup_{0\leq t\leq T_0}|\dot{x}_N(t,\omega)|^2](\leq 4/N^2)(C+C^*)(1+E|\xi_0(\omega)|^2+E|\eta_0(\omega)|^2).$$

It is clear that the set $\Omega_N=\{\sup_{0\leq t\leq T_0}|x_N(t,\omega)|+\sup_{0\leq t\leq T_0}|\dot{x}_N(t,\omega)|]>N\}$ decreases as N increases and $P(\Omega_N)\downarrow 0$ if $N\uparrow\infty$. Define

$$x(t,\omega)=\lim_{N\to\infty}x_N(t,\omega),\ \dot{x}(t,\omega)=\lim_{N\to\infty}\dot{x}_N(t,\omega),\ \text{if } \omega\in\lim\Omega_N.$$

Then $x(t,\omega)=x_N(t,\omega)$ and $\dot{x}(t,\omega)=\dot{x}_N(t,\omega)$ if $\omega\downarrow\Omega_N$ and $x(t,\omega)$, $\dot{x}(t,\omega)$ are in $L^2_\beta(\Omega, C([0,T_0]))$. By Lemma 4.2.11 (see [8, p. 66]), we get

$$\int_0^t g(s,x(s,\omega),\dot{x}(s,\omega),Bx(s,\omega),\omega)d\beta(s)=$$
$$(4.7)\qquad =\int_0^t g_N(s,x_N(s,\omega),\dot{x}_N(s,\omega),\ B_n x_N(s,\omega),\omega)d\beta(s)$$

a.s. if $\omega\in\Omega_N$. It is clear that

$$\int_0^t f(s,x(s,\omega),\dot{x}(s,\omega),Ax(s,\omega),\omega)ds=$$
$$(4.8)\qquad =\int_0^t f_N(s,x_N(s,\omega),\dot{x}_N(s,\omega),A_N x_N(s,\omega),\omega)ds$$

a.s. if $\omega\in\Omega_N$. From (4.7), (4.8) and (4.5)—(4.6) we observe that

$$x(t,\omega)=\xi_0(\omega)+\eta_0(\omega)t+\int_0^t[\int_0^s f(u,x(u,\omega),\dot{x}(u,\omega),Ax(u,\omega),\omega)du+$$
$$+\int_0^s g(u,x(u,\omega),\dot{x}(u,\omega),Bx(u,\omega),\omega)d\beta(u)]ds$$

a.s. if $\omega\in\Omega_N$. It follows that $x(t,\omega)$ is a solution of (2.1)—(2.2).

Finally, using Lemma 2 and Fatou's lemma we get (4.4).

In order to prove the uniqueness, let $x(t,\omega)$ and $y(t,\omega)$ denote two a.s. continuous solutions of (2.1)—(2.2) such that $x(0,\omega)=y(0,\omega)$, $\dot{x}(0,\omega)=\dot{y}(0,\omega)$, a.s. Define for $N>0$ and $t\in[0,T_0]$

$$\Phi(t,\omega)=1 \text{ if } \sup_{0\leq t\leq T_0}[|x(t,\omega)|+|\dot{x}(t,\omega)|]\leq N,\ \sup_{0\leq t\leq T_0}[|y(t,\infty)|+|\dot{y}(t,\omega)|]\leq N,$$

and $\Phi(t,\omega)=0$ otherwise.

Since $\Phi(t,\omega)=\Phi(t,\omega)\Phi(s,\omega)$, $s\leq t$, it follows that for each $t\in[0,T_0]$

$$E[\Phi(t,\omega)|\dot{x}(t,\omega)-\dot{y}(t,\omega)|^2]\leq C_{1N}\int_0^t E[\Phi(s,\omega)|\dot{x}(s,\omega)-\dot{y}(s,\omega)|^2]dG(s),\ 0\leq t\leq T_0$$

where

$$C_{1N}=2(T_0+1)M_N^2\left[\{T_0\Delta G(T_0)+1\}+M_N^2\Delta G(T_0)\left\{\frac{T_0\Delta G(T_0)}{2}+1\right\}\right].$$

Proceeding as in Theorem 2, we get

$$E[\Phi(t,\omega)|\dot{x}(t,\omega)-\dot{y}(t,\omega)|^2]=0,\ 0\leq t\leq T_0,$$

i.e.

$$(4.9)\qquad P[\dot{x}(t,\omega)\neq\dot{y}(t,\omega)]\leq P[\sup_{0\leq t\leq T_0}\{|x(t,\omega)|+|\dot{x}(t,\omega)|\}>N]+$$
$$+P[\sup_{0\leq t\leq T_0}\{|y(t,\omega)|+|\dot{y}(t,\omega)|\}>N].$$

Since $x(t,\omega)$, $y(t,\omega)$ and $\dot{x}(t,\omega)$, $\dot{y}(t,\omega)$ are continuous, the right side of (4.9) converges to 0 as $N\to\infty$. Hence $P[\dot{x}(t,\omega)\neq\dot{y}(t,\omega)]=0$. Further, by the continuity of $\dot{x}(t,\omega)$ and $\dot{y}(t,\omega)$, we have $P[\dot{x}(t,\omega)=\dot{y}(t,\omega),\ 0\leq t\leq T_0]=1$. Now it follows that

$$x(t,\omega)=x(0,\omega)+\int_0^t\dot{x}(s,\omega)ds=y(0,\omega)+\int_0^t\dot{y}(s,\omega)ds=y(t,\omega)$$

a.s. for $0\leq t\leq T_0$. This completes the proof of the theorem.

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A NONSTANDARD THEORY OF HYPERSURFACES IN FINSLER SPACES

BY

RADU MIRON

In memory of my teacher Mendel Haimovici (1906—1973), with the occasion of fifty years since the first paper on Finsler geometry was published in Romania

The theory of hypersurfaces in a Finsler space is a remarkable creation of Mendel Haimovici [7—10]. Fifty years ago, before Elie Cartan's monograph was published, Mendel Haimovici obtained the fundamental equations and Gauss-Codazzi equations for hypersurfaces of a Finsler space. Then he investigated important geometrical properties of totally geodesic and totally extremal hypersurfaces. For this reason Mendel Haimovici is considered as one of the founder of the theory of subspaces in a Finsler space.

In the same field, important results have been obtained by O. Varga [29], A. Rapcsák [26], S. Kikuchi [15], H. Rund [27, 28] and others.

M. Matsumoto has obtained, in his recent excellent paper [18] a synthesis of all results published in this field during about a half century. He pointed out gaps and considered the theory of hypersurfaces in Finsler spaces endowed with the important connections: Cartan, Berwald, Rund, Hashiguchi [19]. It is remarkable the rigurous proof of the famous theorem of Haimovici-Rapcsák-Kikuchi and his new results concerning the hyperplanes.

It is interesting to give here the following assertion of professor Makoto Matsumoto: „It seems, however, to the present author that there had to be unevitable obstructions to develope the theory of Finsler subspaces analogously to the Riemannian theory, and as a consequence, almost all the existing literature is not easy to understand and confused notations sometimes bewilder the readers. The first among those obstructions is perhaps surviving of quantities which are derived from Cartan's C-tensor and cause, for instance, the non-symmetry property of the second fundamental tensor. The second, a consequence of the first, is that the induced connection, defined by projection, does not generally coincide with the intrinsic connection ..... "