12. Kannan D. and Bharucha-Reid A.T. — *Random integral equation formulation of a generalized Langevin equation*, Journal of Statistical Physics, 5 (1972), 209—233
13. Kannan D. — *Random Integrodifferential Equations, Probabilistic Analysis and Related Topics*, Vol. I, Ed. A.T. Bharucha-Reid, Academic Press, 1978, 87—167
14. Ladde G.S. and Lakshmikantham V. — *Random Differential Inequalities*, Academic Press, New York, 1980
15. Pachpatte, B.G. — *On Ito type Stochastic integrodifferential Equation*, Tamkang Jour. of Math. 10 (1979), 1—18
16. Tsokos C.P. and Padgett W.J. — *Random Integral Equations with Applications to Life Sciences and Engineering*, Academic Press, New York, 1974

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A NONSTANDARD THEORY OF HYPERSURFACES IN FINSLER SPACES

BY

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In memory of my teacher Mendel Haimovici (1906—1973), with the occasion of fifty years since the first paper on Finsler geometry was published in Romania

The theory of hypersurfaces in a Finsler space is a remarkable creation of Mendel Haimovici [7—10]. Fifty years ago, before Elie Cartan's monograph was published, Mendel Haimovici obtained the fundamental equations and Gauss-Codazzi equations for hypersurfaces of a Finsler space. Then he investigated important geometrical properties of totally geodesic and totally extremal hypersurfaces. For this reason Mendel Haimovici is considered as one of the founder of the theory of subspaces in a Finsler space.

In the same field, important results have been obtained by O. Varga [29], A. Rapcsák [26], S. Kikuchi [15], H. Rund [27, 28] and others.

M. Matsumoto has obtained, in his recent excellent paper [18] a synthesis of all results published in this field during about a half century. He pointed out gaps and considered the theory of hypersurfaces in Finsler spaces endowed with the important connections: Cartan, Berwald, Rund, Hashiguchi [19]. It is remarkable the rigurous proof of the famous theorem of Haimovici-Rapcsák-Kikuchi and his new results concerning the hyperplanes.

It is interesting to give here the following assertion of professor Makoto Matsumoto: „It seems, however, to the present author that there had to be unevitable obstructions to develope the theory of Finsler subspaces analogously to the Riemannian theory, and as a consequence, almost all the existing literature is not easy to understand and confused notations sometimes bewilder the readers. The first among those obstructions is perhaps surviving of quantities which are derived from Cartan's C-tensor and cause, for instance, the non-symmetry property of the second fundamental tensor. The second, a consequence of the first, is that the induced connection, defined by projection, does not generally coincide with the intrinsic connection ..... "

If we look carefully to this theory we see that the whole trouble comes from the choice of the non-linear connections $\overset{c}{N}$ and $\overset{c}{\underline{N}}$ in F^n and F^{n-1}, respectively, since $\overset{c}{N}$ has not the horizontality property with respect to $\overset{c}{\underline{N}}$ (see § 2). This situation implies the fact that the connection $IF\Gamma$, induced by F^n, is not the same with the intrinsic connection $C\Gamma\overset{c}{\underline{N}}$ of F^{n-1}. Moreover the intrinsic connection is not taken into account in this theory.

The main goal of the present paper is to give a theory based on the intrinsic connection of F^{n-1} and in this way it is similar to the theory of hypersurfeces in Riemannian spaces. More precisely, we start from the intrinsic connection $C\Gamma(\overset{c}{\underline{N}})$ and obtain the so called *normal* connection $F\Gamma(N)$ in F^n whose projection on F^{n-1} is just $C\Gamma(\overset{c}{\underline{N}})$. The nonlinear connection N of $F\Gamma(N)$ has the property of horizontality with respect to the non-linear connection $\underline{N}$ of $C\Gamma(\overset{c}{\underline{N}})$ on F^{n-1}. This geometrical property implies the following results: the fundamental equations of F^{n-1} are very simple, the second h- and v-fundamental tensors of F^{n-1} are symmetric and Gauss-Codazzi's equations are simple, too. Moreover in this case we can state a fundamental theorem (see theorem 6.3). The whole theory has the simplicity and elegance of the similar theory in Riemannian spaces.

For generality, we shall consider the theory of hypersurfaces in generalized Finsler spaces [23, 11, 31], $F^n=(M^n, g_{ij})$, where $g_{ij}(x, y)$ is a generalized Finsler metric, which is not necessarily obtained from a fundamental function.

For the reasons mentioned above we call this theory a „nonstandard" theory of hypersurfaces in Finsler spaces.

In the content of the paper we have: the moving frame on M^{n-1}; the (H)-property of the non-linear connections in F^n and F^{n-1}; the intrinsic connection in F^{n-1} and the normal connection $F\Gamma(N)$ in F^n; the equivalence of the induced and intrinsic connections on F^{n-1}; the relative covariant differential, Ricci formulas; fundamental equations of hypersurfaces in generalized Finsler spaces; Gauss-Codazzi equations; the fundamental theorem of hypersurfaces in F^n; hyperplanes; a particular case when F^n has a generalized regular metric g_{ij} and another one when we have hypersurfaces in a Finsler space (M^n, L).

We use notations and definitions from the papers [19, 23].

In a forthcoming joint paper with Aurel Bejancu we shall give a nonstandard theory of submanifolds M^m in F^n, $(1<m<n)$.

1. The moving frame on a hypersurface M^{n-1}. Let $F^n=(M^n, g_{ij}(x, y))$ be a generalized Finsler space (see: [23, 11]), where M^n is a real n-dimensional manifold and $g_{ij}(x, y)$ is a symmetric, non-degenerated and positive definite Finsler tensor field of the type (0, 2). We remark that $g_{ij}(x, y)$ is not necessarily obtained, as in the classical case, from a fundamental function. For this reason we call $g_{ij}(x, y)$ a generalized Finsler metric.

A hypersurface M^{n-1} in M^n may be represented parametrically by the equations

$$x^i=x^i(u^1, \dots, u^{n-1}); \quad (i, j, k, l, m; n, p, q=1, \dots, n), \tag{1.1}$$

such that the matrix $\|B^i_\alpha\|$ given by

$$B^i_\alpha(u)=\frac{\partial x^i}{\partial u^\alpha}, \quad (\alpha, \beta, \gamma, \delta, \varphi, \mu=1, \dots, n-1). \tag{1.2}$$

be of rank $n-1$. We assume that the supporting element y^i at each point (u^α) of M^{n-1} is tangent to M^{n-1}, i.e., we have

$$y^i=B^i_\alpha(u)v^\alpha. \tag{1.3}$$

Thus (v^α) is thought of as the supporting element of M^{n-1} at the point $(u^\alpha)\in M^{n-1}$. We denote by $y^i(u, v)$ the right hand side of (1.3).

Let TM^n and TM^{n-1} be the tangent bundles to M^n and M^{n-1}, respectively. Since M^{n-1} is a submanifold of M^n we have

$$\frac{\partial}{\partial u^\alpha}=B^i_\alpha\frac{\partial}{\partial x^i}+B^i_{0\alpha}\frac{\partial}{\partial y^i}, \quad \frac{\partial}{\partial v^\alpha}=B^i_\alpha\frac{\partial}{\partial y^i}, \quad B^i_{\alpha\beta}=\frac{\partial B^i_\alpha}{\partial u^\beta}, \quad B^i_{0\alpha}=v^\beta B^i_{\beta\alpha}, \tag{1.4}$$

where $\{\partial/\partial x^i, \partial/\partial y^i\}$ and $\{\partial/\partial u^\alpha, \partial/\partial v^\alpha\}$ are respectively the natural frames of $\mathcal{X}(TM^n)$ and $\mathcal{X}(TM^{n-1})$. In a similar way for the natural dual frame we have

$$dx^i=B^i_\alpha du^\alpha, \quad dy^i=B^i_{0\alpha}du^\alpha+B^i_\alpha dv^\alpha. \tag{1.4'}$$

The generalized Finsler metric $g_{ij}(x, y)$ induces on M^{n-1} the generalized Finsler metric $g_{\alpha\beta}(u, v)$ given by

$$g_{\alpha\beta}(u, v)=g_{ij}(x(u), y(u, v))B^i_\alpha(u)B^j_\beta(u). \tag{1.5}$$

The pair $F^{n-1}=(M^{n-1}, g_{\alpha\beta}(u, v))$ is called *the induced generalized Finsler space* from F^n to M^{n-1}.

We denote by $B^i(u, v)$ the unit Finsler vector field which is normal to M^{n-1} and is given by

$$g_{ij}(x(u), y(u, v))B^i_\alpha(u)B^j(u, v)=0, \quad g_{ij}(x(u), y(u, v))B^i(u, v)B^j(u, v)=1. \tag{1.6}$$

So we have obtained a moving frame on M^{n-1} in F^n:

$$\mathcal{R}=(x^i(u);\ B^i_\alpha(u), B^i(u, v)), \tag{1.7}$$

which is geometrically associated to the hypersurface M^{n-1}. Its dual frame $\mathcal{R}^*=(x^i(u);\ B^\alpha_i(u, v), B_i(u, v))$ satisfies the equalities:

$$B^i_\beta B^\alpha_i=\delta^\alpha_\beta, \quad B^i_\alpha B_i=0, \quad B^i B^\alpha_i=0, \quad B^i B_i=1, \quad B^i_\alpha B^\alpha_j+B^i B_j=\delta^i_j. \tag{1.8}$$

§2. The (H)-property. Let V be the vertical distribution on TM^n, locally spanned by $\{\partial/\partial y_i\}$. A distribution N on TM^n which is suplimentary to V is called a nonlinear connection on TM^n. A local basis in N is given by

$$\frac{\delta}{\delta x^i}=\frac{\partial}{\partial x^i}-N^j_i(x, y)\frac{\partial}{\partial y^j}. \tag{2.1}$$

The functions $N^j_i(x, y)$ are called the coefficients of the non-linear connection N. In a similar way, a suplimentary distribution $\underline{N}$ to the vertical distribution $\underline{V}$ on TM^{n-1} is a non-linear connection on TM^{n-1}. A local basis of $\underline{N}$ is given by

$$\frac{\delta}{\delta u^{\alpha}}=\frac{\partial}{\partial u^{\alpha}}-N^{\beta}_{\alpha}(u,v)\frac{\partial}{\partial v^{\beta}}, \tag{2.2}$$

where $N^{\beta}_{\alpha}(u,v)$ are the coefficients of $\underline{N}$.

If M^n is a paracompact manifold, then there exists a non-linear connection N on TM^n [22]. If $g_{ij}(x,y)$ is a regular generalized Finsler metric, then there exists a non-linear connection N, which depends only on the metric $g_{ij}(x,y)$ [23]. In this case, as we shall see, the induced generalized Finsler metric $g_{\alpha\beta}(u,v)$ is also regular.

For this reason we give:

Convention 1. We fixe a non-linear connection $\overset{c}{N}$ on TM^n and a non-linear connection $\overset{c}{\underline{N}}$ on TM^{n-1}.

We denote by $\overset{c}{N}{}^i_j$ the coefficients of N and by N^{α}_{β} the coefficients of $\overset{c}{\underline{N}}$.

Definition 2.1. *We say that a non-linear connection N on TM^n has the horizontality property (or briefly the (H)-property) with respect to the non-linear connection $\overset{c}{\underline{N}}$ if we have $\overset{c}{\underline{N}}\subset N$ at every point (u,v) of TM^{n-1}.*

By this definition the non-linear connection N on TM^n has the (H)-property with respect to $\overset{c}{\underline{N}}$ if $\overset{c}{\underline{N}}$ is a vector subbundle of N over TM^{n-1}. Of course, definition 2.1 has a geometrical meaning.

Theorem 2.1. *The non-linear connection N has the (H)-property with respect to the non-linear connection $\overset{c}{\underline{N}}$ if and only if, on TM^{n-1} we have*

$$B^j_{\alpha}N^i_j-N^{\beta}_{\alpha}B^i_{\beta}+B^i_{0\alpha}=0. \tag{2.3}$$

Proof. Suppose N has the (H)-property with respect to $\overset{c}{\underline{N}}$. Then we have $\delta/\delta u^{\alpha}=a^i_{\alpha}(u,v)\delta/\delta x^i$ and rank $\|a^i_{\alpha}(u,v)\|=n-1$. By (1.4), (2.1) and (2.2) we have

$$(B^i_{\alpha}-a^i_{\alpha})\frac{\partial}{\partial x^i}+(a^j_{\alpha}N^i_j-N^{\beta}_{\alpha}B^i_{\beta}+B^i_{0\alpha})\frac{\partial}{\partial y^i}=0,$$

which implies (2.3), since $\{\partial/\partial x^i, \partial/\partial y^i\}$ is a basis in $\mathcal{X}(TM^n)$.

Conversely, suppose (2.3) be satisfied. Then by using (1.4) we obtain

$$B^j_{\alpha}\frac{\delta}{\delta x^j}=B^j_{\alpha}\frac{\partial}{\partial x^j}-(N^{\beta}_{\alpha}B^j_{\beta}-B^j_{0\alpha})\frac{\partial}{\partial y^j}=\frac{\partial}{\partial u^{\alpha}}-N^{\beta}_{\alpha}\frac{\partial}{\partial v^{\beta}}=\frac{\delta}{\delta u^{\alpha}},$$

that is, N has the (H)-property with respect to $\overset{c}{\underline{N}}$. The proof is complete.

Corollary 2.1. *The non-linear connection N has the (H)-property with respect to $\overset{c}{\underline{N}}$ if and only if one of the following equalities holds*

$$\frac{\delta}{\delta u^{\alpha}}=B^i_{\alpha}\frac{\delta}{\delta x^i}; \qquad \delta y^i=B^i_{\alpha}\delta v^{\alpha}. \tag{2.4}$$

We have denoted by $\{\delta/\delta x^i, \partial/\partial y^i\}$ an adapted basis on the pair of distributions (N, V) and by $\{dx^i, \delta y^i\}$ its dual basis.

Corollary 2.2. *If N has the (H)-property with respect to $\overset{c}{\underline{N}}$ then we have*

$$B_i(B^j_{\alpha}N^i_j+B^i_{0\alpha})=0.$$

Consequently, not all non-linear connections on TM^n have the (H)-property with respect to $\overset{c}{\underline{N}}$.

Now, we define

$$\varphi^i_{\alpha}(u,v)=B^h_{\alpha}\overset{c}{N}{}^i_h-N^{\beta}_{\alpha}B^i_{\beta}+B^i_{0\alpha}. \tag{2.5}$$

Next, it is easily seen that $B^{\alpha}_j\varphi^i_{\alpha}$ is a Finsler tensor field on TM^{n-1}.

Theorem 2.2. *Suppose there exists a Finsler tensor field on TM^n $\varphi^i_j(x,y)$, such that its restriction on TM^{n-1} be just $B^{\alpha}_j\varphi^i_{\alpha}$. Then the non-linear connection N given by*

$$N^i_j(x,y)=\overset{c}{N}{}^i_j(x,y)-\varphi^i_j(x,y) \tag{2.6}$$

has the (H)-property with respect to the non-linear connection $\overset{c}{\underline{N}}$.

Proof. Since $\overset{c}{N}{}^i_j$ are the coefficients of a non-linear connection on TM^n and $\varphi^i_j(x,y)$ is a Finsler tensor field on TM^n, from (2.6) we obtain that N^i_j are the coefficients of a non-linear connection on TM^n, too. But on TM^{n-1}, by (2.6) we have

$$N^i_j(x(u),y(u,v))=\overset{c}{N}{}^i_j(x(u),y(u,v))-B^{\alpha}_j(u,v)\varphi^i_{\alpha}(u,v), \tag{2.6'}$$

which is a solution for (2.3).

Remark 1. If M^n is a paracompact manifold and TM^{n-1} is a closed submanifold in TM^n, there exists a differentiable prolongation φ^i_j of $B^{\alpha}_j\varphi^i_{\alpha}$.

Remark 2. We always know the distribution $N(N^i_j)$ given by (2.6′) on TM^{n-1}, by means of Convention 1.

Convention 2. Suppose that we have a fixed non-linear connection $N(N^i_j)$ on TM^n with (H)-property with respect to the non-linear connexion $\overset{c}{\underline{N}}(N^{\alpha}_{\beta})$ on TM^{n-1}.

§ 3. **Intrinsic connection on** F^{n-1}. From now on the non-linear connections N and $\overset{c}{\underline{N}}$ being fixed through the conventions 1 and 2, we can consider the most natural metrical Finsler connections in F^n and F^{n-1}.

Theorem 3.1. *There exists an unique Finsler connection $C\Gamma(\overset{c}{\underline{N}})=(F^{\alpha}_{\beta\gamma}, N^{\alpha}_{\beta}, C^{\alpha}_{\beta\gamma})$, in F^{n-1} such that the following conditions be satisfied:*

1°. *Its non-linear connection is $\overset{c}{\underline{N}}$,*

2°. *the h- and v-covariant derivatives of $g_{\alpha\beta}(u,v)$ with respect to $C\Gamma(\overset{c}{\underline{N}})$ vanish,* 3° $\quad T^{\alpha}_{\beta\gamma}=0,\ S^{\alpha}_{\beta\gamma}=0.$

The connection $C\Gamma(\overset{c}{\underline{N}})$ from theorem 3.1 is given by

$$(3.1)\qquad F^{\alpha}_{\beta\gamma}=\frac{1}{2}g^{\alpha\varphi}\left(\frac{\delta g_{\beta\varphi}}{\delta u^{\gamma}}+\frac{\delta g_{\gamma\varphi}}{\delta u^{\beta}}-\frac{\delta g_{\beta\gamma}}{\delta u^{\varphi}}\right),\quad C^{\alpha}_{\beta\gamma}=\frac{1}{2}g^{\alpha\varphi}\left(\frac{\partial g_{\beta\varphi}}{\partial v^{\gamma}}+\frac{\partial g_{\gamma\varphi}}{\partial v^{\beta}}-\frac{\partial g_{\beta\gamma}}{\partial v^{\varphi}}\right).$$

This result is proved by the present author in [23]. The Finsler connection $C\Gamma(\overset{c}{\underline{N}})$ is called *the canonical intrinsic connection* of F^{n-1}.

Similarly, we have

Theorem 3.2. *There exists an unique Finsler connection $F\Gamma(N)=(F^i_{jk}, N^i_j, C^i_{jk})$ in F^n, such that the following conditions be satisfied:*

1°. *Its non-linear connection is N, given by the convention 2,*

2° *the h- and v-covariant derivatives of $g_{ij}(x, y)$ with respect to $F\Gamma(N)$ vanish,*

3°. $T^i_{jk}=0,\qquad S^i_{jk}=0.$

The connection $F\Gamma(N)$ from theorem 3.2 is given by

$$(3.2)\qquad F^i_{jk}=\frac{1}{2}g^{ih}\left(\frac{\delta g_{jh}}{\delta x^k}+\frac{\delta g_{kh}}{\delta x^j}-\frac{\delta g_{jk}}{\delta x^h}\right),\quad C^i_{jk}=\frac{1}{2}g^{ih}\left[\frac{\partial g_{jh}}{\partial y^k}+\frac{\partial g_{kh}}{\partial y^j}-\frac{\partial g_{jk}}{\partial y^h}\right],$$

where $\delta/\delta x^i=\partial/\partial x^i-N^j_i\partial/\partial y^j$. The Finsler connection $F\Gamma(N)$ will be called the *normal Finsler connection of F^n*.

Now it is useful to consider the curvature tensor fields R^h_{ijk}, P^h_{ijk}, S^h_{ijk} of $F\Gamma(N)$:

$$R_i{}^h{}_{jk}=\frac{\delta F_i{}^h{}_j}{\delta x^k}-\frac{\delta F_i{}^h{}_k}{\delta x^j}+F_i{}^m{}_jF_m{}^h{}_k-F_i{}^m{}_kF_m{}^h{}_j+C_i{}^h{}_mR^m{}_{jk},$$

$$(3.3)\qquad P_i{}^h{}_{jk}=\frac{\partial F_i{}^h{}_j}{\partial y^k}-\frac{\delta C_i{}^h{}_k}{\delta x^j}+F_i{}^m{}_jC_m{}^h{}_k-C_i{}^m{}_kF_m{}^h{}_j+C_i{}^h{}_m\frac{\partial N_j{}^m}{\partial y^k},$$

$$S_i{}^h{}_{jk}=\frac{\partial C_i{}^h{}_j}{\partial y^k}-\frac{\partial C_i{}^h{}_k}{\partial y^j}+C_i{}^m{}_jC_m{}^h{}_k-C_i{}^m{}_kC_m{}^h{}_j.$$

By similar formulas are given the curvature tensor fields $R_\alpha{}^\varphi{}_{\beta\gamma}$, $P_\alpha{}^\varphi{}_{\beta\gamma}$ and $S_\alpha{}^\varphi{}_{\beta\gamma}$ of $C\Gamma(\overset{c}{\underline{N}})$. Next, we put:

$$(3.4)\qquad \begin{aligned}&R_{ijke}=g_{jh}R_i{}^h{}_{ke},\quad P_{ijke}=g_{jh}P_i{}^h{}_{ke},\quad S_{ijke}=g_{jh}S_i{}^h{}_{ke},\\ &R_{\alpha\beta\gamma\delta}=g_{\beta\varphi}R_\alpha{}^\varphi{}_{\gamma\delta},\quad P_{\alpha\beta\gamma\delta}=g_{\beta\varphi}P_\alpha{}^\varphi{}_{\gamma\delta},\quad S_{\alpha\beta\gamma\delta}=g_{\beta\varphi}S_\alpha{}^\varphi{}_{\gamma\delta}.\end{aligned}$$

Clearly, we have

$$(3.5)\qquad R_{iejk}=-R_{iekj},\quad S_{iejk}=-S_{iekj},\quad R_{\alpha\beta\gamma\delta}=-R_{\alpha\beta\delta\gamma},\quad S_{\alpha\beta\gamma\delta}=-S_{\alpha\beta\delta\gamma}.$$

Moreover we have:

Theorem 3.3. *The curvature tensors of the Finsler connections $F\Gamma(N)$ and $C\Gamma(\overset{c}{\underline{N}})$ satisfy*

$$(3.6)\qquad \begin{aligned}&R_{iejk}=-R_{eijk},\quad P_{iejk}=-P_{eijk},\quad S_{iejk}=-S_{eijk},\\ &R_{\alpha\beta\gamma\delta}=-R_{\beta\alpha\gamma\delta},\quad P_{\alpha\beta\gamma\delta}=-P_{\beta\alpha\gamma\delta},\quad S_{\alpha\beta\gamma\delta}=-S_{\beta\alpha\gamma\delta}.\end{aligned}$$

Proof. By using (3.2) and (3.3) we obtain

$$R_{ijke}=\frac{1}{2}\frac{\delta}{\delta x^k}\left(\frac{\delta g_{ie}}{\delta x^j}-\frac{\delta g_{je}}{\delta x^i}\right)-\frac{1}{2}\frac{\delta}{\delta x^e}\left(\frac{\delta g_{ik}}{\delta x_j}-\frac{\delta g_{jk}}{\delta x^i}\right)+$$

$$+F_i{}^h{}_eF_{jhk}-F_j{}^h{}_eF_{ihk}-\frac{1}{2}R^h{}_{ke}\left(\frac{\partial g_{ih}}{\partial y^j}-\frac{\partial g_{jh}}{\partial y^i}\right),$$

$$P_{ijke}=\frac{1}{2}\frac{\delta}{\delta x^k}\left(\frac{\partial g_{ie}}{\partial y^j}-\frac{\partial g_{je}}{\partial y^i}\right)-\frac{1}{2}\frac{\partial}{\partial y^e}\left(\frac{\delta g_{ik}}{\delta x^j}-\frac{\delta g_{jk}}{\delta x^i}\right)+$$

$$+C_i{}^h{}_eF_{jhk}-C_j{}^h{}_eF_{ihk}-\frac{1}{2}\frac{\partial N^h_k}{\partial y^e}\left(\frac{\partial g_{ih}}{\partial y^j}-\frac{\partial g_{jh}}{\partial y^i}\right),$$

$$S_{ijke}=\frac{1}{2}\frac{\partial}{\partial y^k}\left(\frac{\partial g_{ie}}{\partial y_j}-\frac{\partial g_{je}}{\partial y^i}\right)-\frac{1}{2}\frac{\partial}{\partial y^e}\left(\frac{\partial g_{ik}}{\partial y^j}-\frac{\partial g_{jk}}{\partial y^i}\right)+C_i{}^h{}_eC_{jhk}-C_j{}^h{}_eC_{ihk},$$

where $F_{ijk}=g_{hj}F_i{}^h{}_k$ and $C_{ijk}=g_{hj}C_i{}^h{}_k$. Thus the first line of the equalities in (3.6) is proved. In a similar way follows the second line.

The identities (3.6) will give a considerable simplification of the Gauss-Codazzi equations of M^{n-1} in F^n.

§ 4. **The induced connection.** We denote by D the operator of covariant differentiation with respect to the normal Finsler connection $F\Gamma(N)$. If $X^i(x, y)$ is a Finsler vector field in F^n, then we have

$$DX^i=X^i{}_{|k}dx^k+X^i|_k\delta y^k.$$

Similarly, we denote by $\bar{D}$ the operator of covariant differentiation with respect to a Finsler connection $F\bar{\Gamma}$ in F^{n-1}. Then for a Finsler vector field $X^\alpha(u, v)$ on F^{n-1} we have $\bar{D}X^\alpha=X^\alpha{}_{|\beta}du^\beta+X^\alpha|_\beta\delta v^\beta$.

Definition 4.1. *A Finsler connection $IF\Gamma=(E_\beta{}^\alpha{}_\gamma, \underline{N}^\alpha{}_\beta, \underline{C}_\beta{}^\alpha{}_\gamma)$ in F^{n-1} is called the induced Finsler connection on F^{n-1} by the normal Finsler connection $F\Gamma(N)$ if the following conditions are fulfilled:*

(i) *The non-linear connection N of $F\Gamma(N)$ has the (H)-property with respect to the non-linear connection $\underline{N}(\underline{N}^\alpha_\beta)$ of $IF\Gamma$.*

(ii) *For any Finsler vector field $X^i=B^i_\alpha X^\alpha$, tangent to M^{n-1}, we have $\bar{D}X^\alpha=B^\alpha_iDX^i$, where $\bar{D}$ is the operator of covariant differentiation with respect to $IF\Gamma$.*

Now we can state an important result given by:

Theorem 4.1. (i) *There exists an unique induced Finsler conection $IF\Gamma$ from the normal Finsler connection $F\Gamma(N)$.*

(ii) *The induced connection $IF\Gamma$ coincides with the canonical intrinsic connection $C\Gamma(\overset{c}{\underline{N}})$.*

Proof. (i) By the Convention 2 the coefficients $N^i_j(x, y)$ of N satisfy $B^i_\alpha N^i_j-B^i_\beta\underline{N}^\beta_\alpha+B^i_{0\alpha}=0$. Since N has the (H)-property with respect to $\underline{N}(\underline{N}^\alpha_\beta)$

we have $B^j_\alpha N^i_j - B^i_\beta N^\beta_\alpha + B^i_{0\alpha} = 0$. From these equalities we obtain

(4.1) $$\underline{N}^\beta_\alpha = N^\beta_\alpha = B^\beta_i (B^j_\alpha N^i_j + B^i_{0\alpha}).$$

By using $\overline{D}X^\alpha = B^\alpha_i DX^i$ we obtain

$$\left(\frac{\delta X^\alpha}{\delta u^\gamma} + F^\alpha_{\beta\gamma} X^\beta\right) du^\gamma + \left(\frac{\partial X^\alpha}{\partial v^\gamma} + \underline{C}^\alpha_{\beta\gamma} X^\beta\right)\delta v^\gamma = B^\alpha_i \left\{\left(\frac{\delta X^i}{\delta x^j} + F_h{}^i{}_j X^h\right) dx^j + \right.$$
$$\left. + \left(\frac{\partial X^i}{\partial y^j} + C_h{}^i{}_j X^h\right)\delta y^j\right\}.$$

On the other hand, by using

$$X^i = B^i_\alpha X^\alpha,\quad \frac{\delta}{\delta u^\alpha} = B^i_\alpha \frac{\delta}{\delta x^i},\quad \frac{\partial}{\partial v^\alpha} = B^i_\alpha \frac{\partial}{\partial y^i},\quad dx^i = B^i_\alpha du^\alpha,\quad \delta y^i = B^i_\alpha \delta v^\alpha,$$

the above relations imply

(4.2) $$F_\beta{}^\alpha{}_\gamma = B^\alpha_i \{B_\beta{}^i{}_\gamma + B^h_\beta B^j_\gamma F_h{}^i{}_j\},\quad \underline{C}_\beta{}^\alpha{}_\gamma = B^\alpha_i B^h_\beta B^j_\gamma C^i_{hj}.$$

Consequently, $IF\Gamma$ is uniquely determined by (4.1) and (4.2).

(ii) From (4.2) it follows

(4.2′) $$F_\beta{}^\alpha{}_\gamma = F_\gamma{}^\alpha{}_\beta,\quad C_\beta{}^\alpha{}_\gamma = \underline{C}_\gamma{}^\alpha{}_\beta.$$

Then in order to prove $IF\Gamma = C\Gamma(\overset{c}{\underline{N}})$ by (4.1) and Theorem 3.1 we see that it is sufficient to show that the h- and v-covariant derivatives of $g_{\alpha\beta}(u, v)$ with respect to $IF\Gamma$ vanish.

In order to do this we put

(4.3) $$H_{\alpha\beta} = B_i (B_\alpha{}^i{}_\beta + B^h_\alpha B^j_\beta F_h{}^i{}_j),\quad K_{\alpha\beta} = B_i B^h_\alpha B^j_\beta C_h{}^i{}_j$$

and we have

(4.4) $$H_{\alpha\beta} = H_{\beta\alpha},\quad K_{\alpha\beta} = K_{\beta\alpha}.$$

From (4.2) and (4.3) we get

(4.5) $$B_\alpha{}^i{}_\beta + B^h_\alpha B^j_\beta F_h{}^i{}_j - B^i_\varphi \underline{F}^\varphi_{\alpha\beta} = H_{\alpha\beta} B^i,\quad B^k_\alpha B^j_\beta C_k{}^i{}_j - B^i_\varphi \underline{C}_\alpha{}^\varphi{}_\beta = K_{\alpha\beta} B^i.$$

Now applying the h-covariant derivative with respect to $IF\Gamma$ to $g_{\alpha\beta}(u, v)$ we obtain

$$g_{\alpha\beta|\gamma} = \frac{\delta g_{\alpha\beta}}{\delta u^\gamma} - \underline{F}_\alpha{}^\varphi{}_\gamma g_{\varphi\beta} - F_\beta{}^\varphi{}_\gamma g_{\alpha\varphi} = \frac{\delta g_{\alpha\beta}}{\delta u^\gamma} - B^\varphi_i [B_\alpha{}^i{}_\gamma + F_h{}^i{}_j B^h_\alpha B^j_\gamma] g_{\varphi\beta} -$$
$$- B^\varphi_i [B_\beta{}^i{}_\gamma + F_h{}^i{}_j B^h_\beta B^j_\gamma] g_{\alpha\varphi}.$$

Since $g_{\alpha\beta} = g_{ij} B^i_\alpha B^j_\beta$ we have

$$\frac{\delta g_{\alpha\beta}}{\delta u^\gamma} = \frac{\delta g_{ij}}{\delta x^k} B^k_\gamma B^i_\alpha B^j_\beta + g_{ij} B^i_{\alpha\gamma} B^j_\beta + g_{ij} B^i_\alpha B^j_{\beta\gamma}.$$

This, together with $B^\varphi_i g_{\varphi\beta} = B^j_\beta g_{ji}$ are used in the previous relation and so, we obtain $g_{\alpha\beta|\gamma} = 0$. In a similar way it follows $g_{\alpha\beta}|_\gamma = 0$. The proof is complete.

Theorem 4.2. *The deflection tensors $D^i{}_j$ and $D^\alpha{}_\beta$ of the connections $F\Gamma(N)$ and $C\Gamma(\overset{c}{\underline{N}})$, respectively, are related by*

(4.6) $$B^j_\alpha D^i{}_j = B^i{}_\beta D^\beta{}_\alpha + H_{\alpha\beta} v^\beta B^i.$$

In fact, (4.5) and (2.3) imply (4.6). From (4.6) we see that if $D^i{}_j = 0$ then we have $D^\alpha{}_\beta = 0$ and $H_{\alpha\beta} v^\beta = 0$.

Theorem 4.3. *The torsions $P^i{}_{jk}$ and $P^\alpha{}_{\beta\gamma}$ of the connections $F\Gamma(N)$ and $C\Gamma(\overset{c}{\underline{N}})$ satisfy*

(4.7) $$P^i{}_{jk} B^j{}_\alpha B^k{}_\beta - P^\gamma{}_{\alpha\beta} B^i{}_\gamma = -H_{\alpha\beta} B^i.$$

Indeed, by differentiating (2.3) and then removing $B_\alpha{}^i{}_\beta$ from (4.5) and the obtained equation we get (4.7).

Then from (4.7) it follows that $P^i{}_{jk} = \partial N^i{}_j/\partial y^k - F_k{}^i{}_j = 0$ implies $P_\beta{}^\alpha{}_\gamma = \partial N^\alpha{}_\beta/\partial v^\gamma - F_\gamma{}^\alpha{}_\beta = 0$ and $H_{\alpha\beta} = 0$.

Also $\partial N^i{}_j/\partial y^k - \partial N^i{}_k/\partial y^j = 0$ implies $\partial N^\alpha_\beta/\partial v^\gamma - \partial N^\alpha{}_\gamma/\partial v^\beta = 0$.

Theorem 4.4. *Suppose the torsion $C_i{}^h{}_j$ of the normal connection $F\Gamma(N)$ vanishes. Then the torsion $C_\beta{}^\alpha{}_\gamma$ of $C\Gamma(\overset{c}{N})$ and $K_{\alpha\beta}$ vanish, too.*

The assertion of the theorem follows from the second equation in (4.2) and (4.3). Also, we have:

(4.8) $$R^i{}_{jk} B^j{}_\alpha B^k{}_\beta = B^i{}_\gamma R^\gamma{}_{\alpha\beta}.$$

§ 5. Relative covariant differential. By means of the normal Finsler connection $F\Gamma(N)$ and of the canonical intrinsic connection $C\Gamma(\overset{c}{\underline{N}})$ we shall obtain an operator of relative covariant differentiation (in the sense of van der Waerden-Bortolotti) in the algebra of Finsler tensor fields $\mathcal{T}_F$.

Definition 5.1. *A mapping $D: \mathcal{T}_F \to \mathcal{T}_F$ is called the relative covariant differential with respect to $(F\Gamma(N), C\Gamma(\overset{c}{\underline{N}}))$ if it satisfies the following conditions:*

1°. *DT is of the same type as T,*

2°. *D is an R-linear mapping,*

3°. *It commutes with the operation of contraction,*

4°. *It satisfies Leibnitz's rule,*

5°. *$D\varphi = d\varphi$ for any scalar field $\varphi(x(u), y(u, v))$,*

6°. *for every Finsler vector field X^i we have $DX^i = X^i{}_{|k} dx^k + X^i{}_{|k} \delta y^k$.*

7°. *for every Finsler vector field X^α we have $DX^\alpha = X^\alpha{}_{|\beta} du^\beta + X^\alpha|_\beta \delta v^\beta$.*

By using the well-known method, we prove the existence and unicity of D.

Thus for a relative Finsler tensor field $T^{i\alpha}_{j\beta}(u, v)$ one gets

(5.2) $$DT^{i\alpha}_{j\beta} = T^{i\alpha}_{j\beta\,|\gamma} du^\gamma + T^{i\alpha}_{j\beta}|_\gamma \delta v^\gamma,$$

where we have

$$(5.3)\qquad \begin{aligned} T^{i\alpha}_{j\beta|\gamma} &= \frac{\delta T^{i\alpha}_{j\beta}}{\delta u^{\gamma}} + T^{m\alpha}_{j\beta}F_{m}{}^{i}{}_{k}B^{k}_{\gamma} - T^{i\alpha}_{m\beta}F_{j}{}^{m}{}_{k}B^{k}_{\gamma} + T^{i\varphi}_{j\beta}F_{\varphi}{}^{\alpha}{}_{\gamma} - T^{i\alpha}_{j\varphi}F_{\beta}{}^{\varphi}{}_{\gamma}, \\ T^{i\alpha}_{j\beta}|_{\gamma} &= \frac{\partial T^{i\alpha}_{j\beta}}{\partial v^{\gamma}} + T^{m\alpha}_{j\beta}C_{m}{}^{i}{}_{k}B^{k}_{\gamma} - T^{i\alpha}_{m\beta}C_{j}{}^{m}{}_{k}B^{k}_{\gamma} + T^{i\varphi}_{j\beta}C_{\varphi}{}^{\alpha}{}_{\gamma} - T^{i\alpha}_{j\varphi}C_{\beta}{}^{\varphi}{}_{\gamma}. \end{aligned}$$

In particular, for a scalar field F we have

$$(5.4)\qquad F_{|\gamma} = F_{|k}B^{k}_{\gamma}, \quad F|_{\gamma} = F|_{k}B^{k}_{\gamma}.$$

Finally, for a Finsler tensor field $T^{i}_{j}(x(u), y(u, v))$ we have

$$(5.5)\qquad T^{i}_{j|\gamma} = T^{i}_{j|k}B^{k}_{\gamma}, \qquad T^{i}_{j}|_{\gamma} = T^{i}_{j}|_{k}B^{k}_{\gamma}.$$

Moreover, for the relative tensor field $B^{i}_{\alpha}(v)$ we obtain

$$(5.6)\qquad B^{i}_{\alpha|\gamma} = \frac{\partial^{k} x^{i}}{\partial u^{\alpha}\partial u^{\gamma}} + B^{j}_{\alpha}F_{j}{}^{i}{}_{k}B^{k}_{\gamma} - B^{i}_{\beta}F^{\beta}_{\alpha\gamma} = B^{i}_{\gamma|\alpha}, \quad B^{i}_{\alpha}|_{\gamma} = B^{j}_{\alpha}C_{j}{}^{i}{}_{k}B^{k}_{\gamma} - B^{i}_{\beta}C^{\beta}_{\alpha\gamma} = B^{i}_{\gamma}|_{\alpha}.$$

Therefore we obtain

Proposition 5.1. *The equations (4.5) are equivalent to*

$$(5.7)\qquad B^{i}_{\alpha|\beta} = H_{\alpha\beta}B^{i}, \quad B^{i}_{\alpha}|_{\beta} = K_{\alpha\beta}B^{i}.$$

The Finsler tensor field $H_{\alpha\beta}$ (resp. $K_{\alpha\beta}$) is called the second h-fundamental (resp. v-fundamental) tensor of the hypersurface M^{n-1}. As we shall see, these tensor fields have a great importance for the whole theory on the immersion of M^{n-1} in M^{n}.

Several applications are based on commutation formulas of the relative covariant differentiation.

Theorem 5.1. *The commutation formulas of the relative covariant differentiation are given by*

$$(5.8)\qquad \begin{aligned} T^{i\alpha}_{j\beta|\gamma|\delta} - T^{i\alpha}_{j\beta|\delta|\gamma} &= T^{m\alpha}_{j\beta}R_{m}{}^{i}{}_{ke}B^{k}_{\gamma}B^{e}_{\delta} - T^{i\alpha}_{m\beta}R_{j}{}^{m}{}_{ke}B^{k}_{\gamma}B^{e}_{\delta} + T^{i\varphi}_{j\beta}R_{\varphi}{}^{\alpha}{}_{\gamma\delta} - \\ &\quad T^{i\alpha}_{j\varphi}R_{\beta}{}^{\varphi}{}_{\gamma\delta} - R^{\varphi}{}_{\gamma\delta}T^{i\alpha}_{j\beta}|_{\varphi}, \\ T^{i\alpha}_{j\beta\,|\gamma}|_{\delta} - T^{i\alpha}_{j\beta}|_{\delta|\gamma} &= T^{m\alpha}_{j\beta}P_{m}{}^{i}{}_{ke}B^{k}_{\gamma}B^{e}_{\delta} - T^{i\alpha}_{m\beta}P_{j}{}^{m}{}_{ke}B^{k}_{\gamma}B^{e}_{\delta} + T^{i\varphi}_{j\beta}P_{\varphi}{}^{\alpha}{}_{\gamma\delta} - \\ &\quad - T^{i\alpha}_{j\varphi}P_{\beta}{}^{\varphi}{}_{\gamma\delta} - C_{\gamma}{}^{\varphi}{}_{\delta}T^{i\alpha}_{j\beta|\varphi} - P^{\varphi}{}_{\gamma\delta}T^{i\alpha}_{j\beta}|_{\varphi}, \\ T^{i\alpha}_{j\beta}|_{\gamma}|_{\delta} - T^{i\alpha}_{j\beta}|_{\delta}|_{\gamma} &= T^{m\alpha}_{j\beta}S_{m}{}^{i}{}_{ke}B^{k}_{\gamma}B^{e}_{\delta} - T^{i\alpha}_{m\beta}S_{j}{}^{m}{}_{ke}B^{k}_{\gamma}B^{e}_{\delta} + T^{i\varphi}_{j\beta}S_{\varphi}{}^{\alpha}{}_{\gamma\delta} - T^{i\alpha}_{j\varphi}S_{\beta}{}^{\varphi}{}_{\gamma\delta}. \end{aligned}$$

Of course, these formulas have a general character.

For a scalar field F (5.8) give

$$(5.9)\qquad \begin{aligned} &F_{|\gamma|\delta} - F_{|\delta|\gamma} = -R^{\varphi}{}_{\gamma\delta}F|_{\varphi}, \quad F_{|\gamma}|_{\delta} - F|_{\delta|\gamma} = -C_{\gamma}{}^{\varphi}{}_{\delta}F_{|\varphi} - P^{\varphi}{}_{\gamma\delta}F|_{\varphi}, \\ &F|_{\gamma}|_{\delta} - F|_{\delta}|_{\gamma} = 0. \end{aligned}$$

By the same formulas (5.8), for B^{i}_{α} and B^{i} we obtain

$$(5.10)\qquad \begin{aligned} B^{i}_{\alpha|\gamma|\delta} - B^{i}_{\alpha|\delta|\gamma} &= B^{m}_{\alpha}R_{m}{}^{i}{}_{jk}B^{j}_{\gamma}B^{k}_{\delta} - B^{i}_{\varphi}R_{\alpha}{}^{\varphi}{}_{\gamma\delta} - R^{\varphi}{}_{\gamma\delta}B^{i}_{\alpha}|_{\varphi}, \\ B^{i}_{\alpha|\gamma}|_{\delta} - B^{i}_{\alpha}|_{\delta|\gamma} &= B^{m}_{\alpha}P_{m}{}^{i}{}_{jk}B^{j}_{\gamma}B^{k}_{\delta} - B^{i}_{\varphi}P_{\alpha}{}^{\varphi}{}_{\gamma\delta} - C_{\gamma}{}^{\varphi}{}_{\delta}B^{i}_{\alpha|\varphi} - P^{\varphi}{}_{\gamma\delta}B^{i}_{\alpha}|_{\varphi}, \\ B^{i}_{\alpha}|_{\gamma}|_{\delta} - B^{i}_{\alpha}|_{\delta}|_{\gamma} &= B^{m}_{\alpha}S_{m}{}^{i}{}_{jk}B^{j}_{\gamma}B^{k}_{\delta} - B^{i}_{\varphi}S_{\alpha}{}^{\varphi}{}_{\gamma\delta}. \end{aligned}$$

and

$$(5.11)\qquad \begin{aligned} B^{i}{}_{|\gamma|\delta} - B^{i}{}_{|\delta|\gamma} &= B^{m}R_{m}{}^{i}{}_{jk}B^{j}_{\gamma}B^{k}_{\delta} - R^{\varphi}{}_{\gamma\delta}B^{i}|_{\varphi}, \\ B^{i}{}_{|\gamma}|_{\delta} - B^{i}|_{\delta|\gamma} &= B^{m}P_{m}{}^{i}{}_{jk}B^{j}_{\gamma}B^{k}_{\delta} - C_{\gamma}{}^{\varphi}{}_{\delta}B^{i}{}_{|\varphi} - P^{\varphi}{}_{\gamma\delta}B^{i}|_{\varphi}, \\ B^{i}|_{\gamma}|_{\delta} - B^{i}|_{\delta}|_{\gamma} &= B^{m}S_{m}{}^{i}{}_{jk}B^{j}_{\gamma}B^{k}_{\delta}. \end{aligned}$$

The equations (5.9)—(5.11) will be useful to obtain the Gauss-Codazzi equations for the immersion of M^{n-1} in F^{n}.

§ 6. The fundamental equations of a hypersurface M^{n-1} in F^{n}. For a geometrical study of the hypersurface M^{n-1} in F^{n} it is important to examine the variation of the moving frame $\mathcal{R} = (x^{i}(u)\,;\, B^{i}_{\alpha}(u), B^{i}(u,v))$ on M^{n-1}. This will lead to the fundamental equations of M^{n-1} in the generalized Finsler space F^{n}. This study is made by means of the normal Finsler connection $F\Gamma(N)$ and of the canonical intrinsic Finsler connection $\overset{c}{C\Gamma}(\underline{N})$.

Theorem 6.1. *The fundamental equations of the hypersurface M^{n-1} in the generalized Finsler spaces F^{n}, endowed with the Finsler connections $F\Gamma(N)$ and $\overset{c}{C\Gamma}(\underline{N})$ are given by*

$$(6.1)\qquad \begin{aligned} &\frac{\partial x^{i}}{\partial u^{\alpha}} = B^{i}_{\alpha}, \quad B^{i}_{\alpha|\beta} = H_{\alpha\beta}B^{i}, \quad B^{i}{}_{|\alpha} = -H^{\beta}_{\alpha}B^{i}_{\beta}, \\ &\frac{\partial x^{i}}{\partial v^{\alpha}} = 0, \quad B^{i}_{\alpha}|_{\beta} = K_{\alpha\beta}B^{i}, \quad B^{i}|_{\alpha} = -K^{\beta}_{\alpha}B^{i}_{\beta}, \end{aligned}$$

where we have

$$(6.2)\qquad H_{\alpha\beta} = H_{\beta\alpha}, \quad H^{\beta}_{\alpha} = g^{\beta\varphi}H_{\varphi\alpha}\,; \quad K_{\alpha\beta} = K_{\beta\alpha}, \quad K^{\beta}_{\alpha} = g^{\beta\varphi}K_{\varphi\alpha}.$$

Proof. The first two equations (6.1) follow from (1.2) and Proposition 5.1. The last equations are obtained by differentiating $g_{ij}B^{i}_{\alpha}B^{j} = 0$ and then taking into account of the Proposition 5.1.

Now we can state the following important result.

Theorem 6.2. *The Gauss-Codazzi equations of the hypersurface M^{n-1} in the generalized Finsler space F^{n} with respect to the normal connection $F\Gamma(N)$ and with the canonical intrinsic connection $\overset{c}{C\Gamma}(\underline{N})$ are given by*

$$(6.3)_{1}\qquad \begin{aligned} R_{ijke}B^{i}_{\alpha}B^{j}_{\beta}B^{k}_{\gamma}B^{e}_{\delta} - R_{\alpha\beta\gamma\delta} &= H_{\alpha\delta}H_{\beta\gamma} - H_{\alpha\gamma}H_{\beta\delta}, \\ P_{ijke}B^{i}_{\alpha}B^{j}_{\beta}B^{k}_{\gamma}B^{e}_{\delta} - P_{\alpha\beta\gamma\delta} &= K_{\alpha\delta}H_{\beta\gamma} - H_{\alpha\gamma}K_{\beta\delta}, \\ S_{ijke}B^{i}_{\alpha}B^{j}_{\beta}B^{k}_{\gamma}B^{e}_{\delta} - S_{\alpha\beta\gamma\delta} &= K_{\alpha\delta}K_{\beta\gamma} - K_{\alpha\gamma}K_{\beta\delta}, \end{aligned}$$

$$R_{ijkr}B^iB^j_\alpha B^k_\beta B^r_\gamma+K_{\alpha\varphi}R^\varphi{}_{\beta\gamma}=H_{\alpha\gamma|\beta}-H_{\alpha\beta|\gamma}\,,$$

$$(6.3)_2\qquad P_{ijkr}B^iB^j_\alpha B^k_\beta B^r_\gamma+H_{\alpha\varphi}C_\beta{}^\varphi{}_\gamma+K_{\alpha\varphi}P^\varphi{}_{\beta\gamma}=K_{\alpha\gamma|\beta}-H_{\alpha\beta}|_\gamma\,,$$

$$S_{ijkr}B^iB^j_\alpha B^k_\beta B^r_\gamma=K_{\alpha\gamma}|_\beta-K_{\alpha\beta}|_\gamma\,;$$

$$(6.3)_3\qquad \begin{array}{lll} R_{ijk}B^i_\alpha B^j_\beta B^k_\gamma=R_{\alpha\beta\gamma}\,, & P_{ijk}B^i_\alpha B^j_\beta B^k_\gamma=P_{\alpha\beta\gamma}\,, & C_{ijk}B^i_\alpha B^j_\beta B^k_\gamma=C_{\alpha\beta\gamma}\,, \\ R_{ijk}B^iB^j_\alpha B^k_\beta=0, & P_{ijk}B^iB^j_\alpha B^k_\beta=-H_{\alpha\beta}\,, & C_{ijk}B^i_\alpha B^jB^k_\beta=K_{\alpha\beta}\,; \end{array}$$

$$(6.3)_4\qquad D_{ij}B^i_\alpha B^j_\beta=D_{\alpha\beta}\,,\quad D_{ij}B^iB^j_\alpha=H_{0\alpha}.$$

Proof. Obviously we denoted

$$R_{ijk}=g_{ih}R^h{}_{jk},\quad P_{ijk}=g_{ih}P^h{}_{jk},\quad C_{ijk}=g_{jh}C_i{}^h{}_k,\quad D_{ij}=g_{ik}D^k{}_j$$

$$R_{\alpha\beta\gamma}=g_{\alpha\varphi}R^\varphi{}_{\beta\gamma}\,,\quad P_{\alpha\beta\gamma}=g_{\alpha\varphi}P^\varphi{}_{\beta\gamma}\,,\quad C_{\alpha\beta\gamma}=g_{\beta\varphi}C_\alpha{}^\varphi{}_\gamma\,,\quad D_{\alpha\beta}=g_{\alpha\gamma}D^\gamma{}_\beta\,,$$

and $H_{0\alpha}=H_{\beta\alpha}v^\beta$, $K_{0\alpha}=K_{\beta\alpha}v^\beta$. By applying the commutation formulas of the relative covariant differentiation (5.10) and (5.11) and by using the fundamental equations (6.1) we get 6 equations. Contracting these equations with B^α_i and with B_i, one obtains 12 equations. Thus, by theorem 3.3, we see that three of these equations are consequences of the others and other three are identicaly verified. There remains the equations $(6.3)_1$ and $(6.3)_2$. The torsions of the connections $F\Gamma(N)$ and $\overset{c}{C\Gamma}(\underline{N})$ are respectively $R^i{}_{jk}$, $C^i{}_{jk}$, $P^i{}_{jk}$ and $R^\alpha{}_{\beta\gamma}$, $C_\beta{}^\alpha{}_\gamma$, $P^\alpha{}_{\beta\gamma}$. From (5.9) we obtain $(6.3)_3$. Finally by differentiating h- and v-covariantly the relation $y^i=B^i_\alpha v^\alpha$, we obtain $(6.3)_4$ and $C_{0ij}B^i_\alpha B^j_\beta=C_{0\alpha\beta}$, $C_{0ij}B^iB^j_\alpha=K_{0\beta}$. The last equations are consequences of the last equalities $(6.3)_3$.

The next problem is to determine the solutions of the system of equations with partial derivatives (6.1). This system is a normal one with respect to the unknown functions x^i, y^i, B^i_α, B^i of (u^α, v^α). Between these functions we have a finite relation

$$(F_0)\qquad y^i=B^i{}_\alpha(u)v^\alpha.$$

Therefore, we consider the first integrability conditions for the system $\{(6.1), (F_0)\}$ and we denote them by (F_1). The conditions (F_1) are nothing but the Gauss-Codazzi's equations $(6.3)_1$—$(6.3)_4$. The second group of compatibility conditions is obtained by differentiating h- and v- covariantly the equations (F_1) and removing $B^i_{\alpha|\beta}$, $B^i_\alpha|_\beta$, $B^i{}_{|\alpha}$ and $B^i|_\alpha$ via (6.1). So we get a new system (F_2), and so on.

Then we obtain the sequence of systems of equations

$$(6.4)\qquad F_0, F_1, F_2, \ldots\ldots, F_p, \ldots\ldots$$

Now applying a well-known theorem (see Eisenhart [4], p. 14—18) we have the following fundamental result:

Theorem 6.3. *The necessary and sufficient conditions for the existence of an analytical hypersurface $M^{n-1}(x^i=x^i(u^\alpha))$ in the C^ω-generalized Finsler space $F^n=(M^n, g_{ij})$ endowed with a metrical Finsler connexion $F\Gamma(N)=(F_j{}^i{}_k, N^i{}_j, C_j{}^i{}_k)$ such that $F_j{}^i{}_k=F_k{}^i{}_j$, $C_j{}^i{}_k=C_k{}^i{}_j$ are the following:*

a. *Suppose there are given:*

1°. *A C^ω-non-linear connection $N^\alpha_\beta(u, v)$*

2°. *A C^ω-generalized Finsler metric $g_{\alpha\beta}(u, v)$,*

3°. *The C^ω-symmetric Finsler tensors $H_{\alpha\beta}(u, v)$ and $K_{\alpha\beta}(u, v)$, such that the metrical Finsler connection $\overset{c}{C\Gamma}(\underline{N})=(F_\beta{}^\alpha{}_\gamma, N^\alpha{}_\beta, C_\beta{}^\alpha{}_\gamma)$ with respect to $g_{\alpha\beta}(u, v)$ and $F_\beta{}^\alpha{}_\gamma=F_\gamma{}^\alpha{}_\beta$, $C_\beta{}^\alpha{}_\gamma=C_\gamma{}^\alpha{}_\beta$ can be constructed;*

b. *There exists a natural number p such that the system of equations*

$$(6.4')\qquad F_0, F_1, \ldots, F_p$$

has a solution $(x^i, y^i, B^i_\alpha, B^i)$ and this solution satisfies the equations of the system F_{p+1}. If the system (6.4') has q independent equations, then the solution $(x^i, y^i, B^i_\alpha, B^i)$ depends on $n(n+2)-q$ constants.

c. *A hypersurface $M^{n-1}, x^i=x^i(u^\alpha)$, the solution of (6.4') has the following properties: $B^i_\alpha(u)$ are tangent vectors to M^{n-1}; B^i are normal to M^{n-1}; $g_{\alpha\beta}(u, v)$ is the induced generalized Finsler metric on M^{n-1} by $g_{ij}(x, y)$; the non-linear connection $N^i{}_j$ has the (H)-property with respect to N^α_β; $H_{\alpha\beta}$ and $K_{\alpha\beta}$ are the second h- and v- fundamental tensors of M^{n-1}.*

Proof. If M^{n-1} is given by equations (1.1) and it is an analytical one, then, by theorem 6.3, we obtain that $(x^i, y^i, B^i_\alpha, B^i)$ satisfy all conditions of the theorem.

Conversely, we consider the normal system of equations with partial derivatives (6.1) where the h- and v-relative covariant derivatives are constructed from $F\Gamma(N)$ and $\overset{c}{C\Gamma}(\underline{N})$. By Eisenhart's theorem, ([4], p. 14—18), in our conditions, we have a solution of equations (6.1) given by

$$(x^i(u, v),\ y^i(u, v),\ B^i_\alpha(u, v),\ B^i(u, v)),$$

which is a solution of the system (6.4') and satisfies the equations F_{p+1}; This means $\partial x^i/\partial v^\alpha=0$, $\partial/\partial v^\beta(\partial x^i/\partial u^\alpha)=\partial B^i_\alpha/\partial v^\beta)=0$. Thus $x^i=x^i(u^\alpha)$ and $B^i_\alpha=B^i_\alpha(u)$. But (F_0) being satisfied, we can consider the functions

$$\Psi_{\alpha\beta}=g_{ij}(x(u), y(u, v))B^i_\alpha(u)B^j_\beta(u),\quad \Psi_\alpha=g_{ij}(x(u), y(u, v))B^i_\alpha(u)B^j(u, v),$$

$$\Psi=g_{ij}(x(u), y(u, v))B^i(u, v)B^j(u, v).$$

It follows that these functions satisfy the system of equations

$$\Psi_{\alpha\beta|\gamma}=H_{\alpha\gamma}\Psi_\beta+H_{\beta\gamma}\Psi_\alpha\,,\quad \Psi_{\alpha|\gamma}=H_{\alpha\gamma}\Psi-H^\varphi_\gamma\Psi_{\varphi\alpha}\,,\quad \Psi_{|\alpha}=-2H^\beta_\alpha\Psi_\beta\,,$$

$$\Psi_{\alpha\beta}|_\gamma=K_{\alpha\gamma}\Psi_\beta+K_{\beta\gamma}\Psi_\alpha\,,\quad \Psi_\alpha|_\gamma=K_{\alpha\gamma}\Psi-K^\varphi_\gamma\Psi_{\varphi\alpha}\,,\quad \Psi|_\alpha=-2K^\beta_\alpha\Psi_\beta\,.$$

This is a linear homogeneous system of equations with partial derivatives and has the solution $\Psi_{\alpha\beta}=g_{\alpha\beta}$, $\Psi_\alpha=0$, $\Psi=1$ and we consider it in what follows.

Now, from $(6.3)_4$, it follows $B^j_\alpha D^i_j-B^i_\beta D^\beta_\alpha=H_{0\alpha}B^i$. Taking into account that $(x^i(u), y^i(u, v), B^i_\alpha(u), B^i(u, v))$ satisfy (6.1), we obtain $B^j_\alpha N^i_j-B^i_\beta N^\beta_\alpha+B^i_{0\alpha}=0$. Finally, from (6.1) we get that $H_{\alpha\beta}$ and $K_{\alpha\beta}$ are, respectively the second h- and v-fundamental tensors of M^{n-1}.

§ **7. Hyperplanes.** Let M^{n-1} be a hypersurface in the generalized Finsler space $F^n=(M^n, g_{ij})$ and the induced generalized Finsler space $F^{n-1}=(M^{n-1}, g_{\alpha\beta})$. We consider in F^n the normal connection $F\Gamma(N)$ and in F^{n-1}

the canonical intrinsic Finsler connection $C\Gamma(\overset{c}{\underline{N}})$. We shall study here the case when both $H_{\alpha\beta}$ and $K_{\alpha\beta}$ vanish; to do this, we suppose that $g_{ij}(x, y)$ is zero-homogeneous with respect to y^i.

The length of a curve $(\tilde{C})$ $u^\alpha = u^\alpha(t)$, $t \in (a, b)$ in M^{n-1} is given by

$$s = \int_{t_0}^{t_1} \sqrt{g_{\alpha\beta}\left(u(t), \frac{du}{dt}\right) \frac{du^\alpha}{dt} \frac{du^\beta}{dt}}\, dt, \quad [t_0, t_1] \subset (a, b).$$

We take s as a parameter on $(\tilde{C})$ and give a curve (C) in F^{n-1} by the equations

$$u^\alpha = u^\alpha(s), \quad v^\alpha = v^\alpha(s), \quad s \in (s_0, s_1). \tag{7.1}$$

The curve (C) is given in the envelopping space F^n by equations

$$x^i = x^i(u^\alpha(s)), \quad y^i = B^i_\alpha(u^\gamma(s)) v^\alpha(s). \tag{7.2}$$

Now we consider on (C) the vector field

$$\frac{\delta v^\alpha}{ds} = \frac{dv^\alpha}{ds} + N^\alpha_\beta(u(s), v(s)) \frac{du^\beta}{ds}. \tag{7.3}$$

Thus we have in F^n the vector field

$$\frac{\delta y^i}{ds} = B^i_\alpha(u(s)) \frac{\delta v^\alpha}{ds}. \tag{7.4}$$

A *path* in F^{n-1} is a curve (C) satisfying the differential equation

$$\frac{d^2 u}{ds^2} + N^\alpha_\beta\left(u(s), \frac{du}{ds}\right) \frac{du^\beta}{ds} = 0.$$

Proposition 7.1. *Any path in F^{n-1} is a path in F^n.*

In fact, from (7.4) it follows

$$B^i_\alpha\left(\frac{d^2u^\alpha}{ds^2} + N^\alpha_\beta\left(u(s), \frac{du}{ds}\right)\frac{du^\beta}{ds}\right) = \frac{d^2x^i}{ds^2} + N^i_j(x(u(s)), y(u(s), v(s))) \frac{dx^j}{ds}$$

and the assertion is immediate.

This property of the paths was expected since N has the (H)-property with respect to $\overset{c}{\underline{N}}$.

The covariant differential DX^α on the curve (C) is given by

$$DX^\alpha = \frac{dX^\alpha}{ds} + \Gamma_\beta{}^\alpha{}_\gamma \frac{du^\beta}{ds}\frac{du^\gamma}{ds} + C_\beta{}^\alpha{}_\gamma \frac{du^\beta}{ds}\frac{du^\gamma}{ds}; \quad \Gamma_\beta{}^\alpha{}_\gamma = F_\beta{}^\alpha{}_\gamma + C_\beta{}^\alpha{}_\varphi N^\varphi_\gamma.$$

Then we obtain

$$\frac{\delta v^\alpha}{ds} = 0, \quad \frac{DX^\alpha}{ds} = \frac{dX^\alpha}{ds} + F_\beta{}^\alpha{}_\gamma \frac{du^\beta}{ds}\frac{du^\gamma}{ds}. \tag{7.5}$$

Definition 7.1. *A curve (C) in F^{n-1} is a h-path with respect to the Finsler connection $C\Gamma(\overset{c}{N})$ if it satisfies $\partial v^\alpha/ds = 0$, $D/ds(\partial u^\alpha/\partial s) = 0$. Similary, a h-path in F^n, with respect to the Finsler connection $F\Gamma(N)$ is given by $\delta y^i/ds = 0$, $D/ds(dx^i/ds) = 0$.*

Proposition 7.2. *If the curve (C) is a h-path in F^{n-1}, then on (C) we have*

$$H_{\alpha\beta} \frac{du^\alpha}{ds}\frac{du^\beta}{ds} = B_i \frac{D}{ds}\left(\frac{dx^i}{ds}\right). \tag{7.6}$$

Indeed, (7.6) is obtained from the following :

Proposition 7.3. *On any h-path (C) in F^{n-1} we have*

$$\frac{\delta y^i}{ds} = 0, \quad \frac{D}{ds}\left(\frac{dx^i}{ds}\right) = H_{\alpha\beta}\frac{du^\alpha}{ds}\frac{du^\beta}{ds} B^i. \tag{7.7}$$

Proof. From (7.4) it follows the first equality (7.7). Then on the h-path (C) we have

$$\frac{D}{ds}\left(\frac{dx^i}{ds}\right) = \frac{D}{ds}\left(B^i_\alpha \frac{du^\alpha}{ds}\right) = B^i_\alpha \frac{D}{ds}\left(\frac{du^\alpha}{ds}\right) +$$

$$+ \left(H_{\alpha\beta}\frac{du^\alpha}{ds} + K_{\alpha\beta}\frac{\delta v^\alpha}{ds}\right)\frac{du^\beta}{ds} B^i = H_{\alpha\beta}\frac{du^\alpha}{ds}\frac{du^\beta}{ds} B^i.$$

Remark. The left hand side of (7.6) is the normal curvature of an h-path (C).

Definition 7.2. *The hypersurface M^{n-1} is called a h-plane in F^n if every h-path in F^{n-1} is an h-path in F^n.*

Theorem 7.1. *The hypersurface M^{n-1} is an h-plane if and only if its second h-fundamental tensor $H_{\alpha\beta}$ vanishes.*

The proof is immediate by Proposition 7.3.

The fundamental equations and Gauss-Codazzi's equations for a h-plane are very simple. Thus in this case a fundamental theorem can be easely obtained.

Definition 7.3. *The hypersurface M^{n-1} is a hyperplane in F^n if $DB^i/ds = 0$, on a curve (C) in F^{n-1}.*

Theorem 7.2. *The hypersurface M^{n-1} is a hyperplane in F^n if and only if the second h- and v-fundamental tensors $H_{\alpha\beta}$ and $K_{\alpha\beta}$ vanish identicaly on M^{n-1}.*

Proof. From $DB^i/ds = B^i{}_{|\alpha} du^\alpha/ds + B^i|_\alpha \delta v^\alpha/ds = 0$ on an arbitrary curve (C) on F^{n-1} it follows $B^i{}_{|\alpha} = 0$ and $B^i|_\alpha = 0$ and conversely. But these equations are equivalent with $H_{\alpha\beta} = K_{\alpha\beta} = 0$.

From $(6.3)_1$—$(6.3)_4$ there are obtained the Gauss-Codazzi equations of hyperplanes by taking $H_{\alpha\beta} = K_{\alpha\beta} = 0$. Analogorsly a fundamental theorem of the type of theorem 6.3 can be stated in this particular case.

§ 8. Hypersurfaces in the generalized Finsler space with the regular metric. There is a remarkable class of generalized Finsler spaces $F^n = (M, g_{ij})$ in which the metrical Finsler connection $C\Gamma(\overset{c}{N})$ is given only by $g_{ij}(x, y)$.

These spaces are called with *regular metric*, [23, 11, 32].

Let us consider the function

$$\mathcal{L}(x, y) = g_{ij}(x, y)y^i y^j \tag{8.1}$$

which is called [23] *the absolute energy* of the structure $g_{ij}(x, y)$.

We denote

$$g^{\bullet}_{ij} = \frac{1}{2}\frac{\partial^2 \mathcal{L}}{\partial y^i \partial y^j}. \tag{8.2}$$

Then

$$g^{\bullet}_{ij} = g_{ij} + \frac{\partial g_{ki}}{\partial y^j} y^k. \tag{8.2'}$$

The generalized Finsler metric $g_{ij}(x, y)$ is called *regular* if it satisfies the following conditions:

$$g_{ij}y^j = \frac{1}{2}\frac{\partial \mathcal{L}}{\partial y^i}, \quad \det\|g^{\bullet}_{ij}\| \neq 0. \tag{8.3}$$

An interesting geometrical interpretation of these conditions is given by M. Hashiguchi [11].

We put $\|g^{\bullet ij}\| = \|g^{\bullet}_{ij}\|^{-1}$. The intrinsic absolute energy on F^{n-1} is defined by

$$\underline{\mathcal{L}}(u, v) = g_{\alpha\beta}(u, v)v^\alpha v^\beta \tag{8.4}$$

and we have from (8.1) and (1.3)

$$\underline{\mathcal{L}}(u, v) = \mathcal{L}(x(u), y(u, v)). \tag{8.5}$$

Theorem 8.1. *If the generalized Finsler metric g_{ij} is regular, then the induced metrical Finsler structure $g_{\alpha\beta}(u, v)$ is regular, too.*

Proof.

$$g_{\alpha\beta}v^\beta = g_{ij}B^i_\alpha B^j_\beta v^\beta = B^i_\alpha g_{ij}y^j = \frac{1}{2}B^i_\alpha \frac{\partial \mathcal{L}}{\partial y^i} = \frac{1}{2}\frac{\partial \underline{\mathcal{L}}}{\partial v^\alpha}.$$

Therefore the first condition (8.3) holds for $g_{\alpha\beta}$. Also, from $\partial\underline{\mathcal{L}}/\partial v^\alpha = B^i_\alpha \partial\mathcal{L}/\partial y^i$, we have $\partial^2\underline{\mathcal{L}}/\partial v^\alpha \partial v^\beta = B^i_\alpha B^j_\beta\, \partial^2\mathcal{L}/\partial y^i \partial y^j$. Consequently

$$g^{\bullet}_{\alpha\beta} = B^i_\alpha B^j_\beta g^{\bullet}_{ij}, \tag{8.6}$$

which shows that rank $\|g^{\bullet}_{\alpha\beta}\| = n - 1$.

Now we consider Christoffel's symbols

$$\gamma^i{}_{jk} = \frac{1}{2}g^{im}\left(\frac{\partial g_{jm}}{\partial x^k} + \frac{\partial g_{km}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^m}\right), \quad \gamma^i_{00} = \gamma^i_{jk}y^j y^k.$$

In [23] we have proved that the functions

$$\overset{c}{N}{}^i_j = \frac{1}{2}\frac{\partial}{\partial y^j}\{g^{\bullet im}g_{mk}\gamma^k_{00}\} \tag{8.7}$$

determine the coefficients of a non-linear connection $\overset{c}{N}$ in F^n. This connection depends only on the generalized regular Finsler metric g_{ij}. It is called *the canonical non-linear connection* of g_{ij}.

In the same paper [23] we proved:

Theorem 8.2. *In the generalized Finsler space F^n with the regular metric g_{ij} there exists an unique Finsler connection $C\Gamma(\overset{c}{N}) = (\overset{c}{F}{}^i_{jk}, \overset{c}{N}{}^i_j, \overset{c}{C}{}^i_{jk})$, with the properties:*

1. *The non-linear connection $\overset{c}{N}$ of $C\Gamma(\overset{c}{N})$ is given by* (8.7).
2. *$C\Gamma(\overset{c}{N})$ is a metrical one, with respect to g_{ij}.*
3. *$\overset{c}{T}{}^i_{jk} = 0$, $\overset{c}{S}{}^i_{jk} = 0$.*

This connection has the coefficients (8.7) and:

$$\overset{c}{F}{}^i_{jk} = \frac{1}{2}g^{im}\left(\frac{\overset{c}{\delta} g_{jm}}{\delta x^k} + \frac{\overset{c}{\delta} g_{km}}{\delta x^j} - \frac{\overset{c}{\delta} g_{jk}}{\delta x^m}\right), \quad \overset{c}{C}{}^i_{jk} = \frac{1}{2}g^{im}\left(\frac{\partial g_{jm}}{\partial y^k} + \frac{\partial g_{km}}{\partial y^j} - \frac{\partial g_{jk}}{\partial y^m}\right), \tag{8.8}$$

where $\overset{c}{\delta}/\delta x^k = \partial/\partial x^k - \overset{c}{N}{}^i_k \partial/\partial y^i$.

The connection $C\Gamma(\overset{c}{N})$ is called a *canonical* one for the generalized regular Finsler metric g_{ij}.

In a similar way, denoting by $\gamma^\alpha_{\beta\gamma}$ Christoffel's symbols of the generalized regular induced metric $g_{\alpha\beta}$ and by $\gamma^\alpha_{00} = \gamma^\alpha_{\beta\gamma}v^\beta v^\gamma$ and taking

$$\overset{c}{\underline{N}}{}^\alpha_\beta = \frac{1}{2}\frac{\partial}{\partial v^\beta}\{g^{\bullet\alpha\varphi}g_{\varphi\gamma}\gamma^\gamma_{00}\}. \tag{8.7'}$$

we have:

Theorem 8.3. *In the induced Finsler spaces $F^{n-1} = (M^{n-1}, g_{\alpha\beta})$, with the generalized regular induced metric, there exists an unique Finsler connection $C\Gamma(\overset{c}{\underline{N}}) = (\overset{c}{F}{}^\alpha_{\beta\gamma}, \overset{c}{N}{}^\alpha_\beta, \overset{c}{C}{}^\alpha_{\beta\gamma})$, metrical with respect to the metric $g_{\alpha\beta}$ and with the non-linear connection $\underline{N}$ given by (8.7') and $\overset{c}{T}{}^\alpha_{\beta\gamma} = \overset{c}{C}{}^\alpha_{\beta\gamma} = 0$. This connection $C\Gamma(\overset{c}{\underline{N}})$ is that one obtained in theorem 3.1, with $N^\alpha_\beta = \overset{c}{N}{}^\alpha_\beta$.*

Therefore, the Convention 1 is realised by taking the non-linear connections $\overset{c}{N}$ and $\overset{c}{\underline{N}}$ just the connections (8.7) and (8.7'), respectively.

Concerning the Convention 2 we remark that on TM^{n-1} the coefficients of the non-linear connection N^i_j, given by (2.6') depend only on g_{ij}, and $g_{\alpha\beta}$. Then, particularizing the theorem 2.2, we have:

Theorem 8.4. 1°. *The functions $N^i_j(x(u), y(u, v))$ given by* (2.6') *determine the coefficients of a non-linear connection, defined on TM^{n-1}, which depend only on the generalized, regular Finsler metrics $g_{\alpha\beta}$, g_{ij}.*

2°. *If there exists a prolongation $\varphi^i_j(x, y)$ to TM^n of the Finsler tensor field $B^\alpha_j \varphi^i_\alpha$, then*

$$N^i_j(x, y) = \overset{c}{N}{}^i_j(x, y) - \varphi^i_j(x, y) \tag{8.8}$$

is a non-linear connection on TM^n, which has the (H)-property with respect to the canonical non-linear connection $\overset{c}{\underline{N}}$.

3°. *The normal Finsler connection $F\Gamma(N)$ is uniquely determined by* (8.8) *and* (3.2).

Also, we have:

Theorem 8.5. *If the normal connection $F\Gamma(N)$ is given by the previous theorem, then the restriction of the transformation of Finsler connections $C\Gamma(\overset{C}{N}) \to F\Gamma(N)$ to F^{n-1} is given by:*

$$N^i_j = \overset{C}{N}{}^i_j - B^\alpha_j \varphi^i_\alpha ,$$

$$F^i_{jk} = \overset{C}{F}{}^i_{jk} + \frac{1}{2} g^{ip} \varphi^m_\alpha \left(B^\alpha_k \frac{\partial g_{jp}}{\partial y^m} + B^\alpha_j \frac{\partial g_{kp}}{\partial y^m} - B^\alpha_p \frac{\partial g_{jk}}{\partial y^m} \right), \quad C^i_{jk} = \overset{C}{C}{}^i_{jk} , \tag{8.9}$$

where $\varphi^i_\alpha = B^j_\alpha \overset{C}{N}{}^i_j - B^i_\beta \overset{C}{N}{}^\beta_\alpha + B^i_{0\alpha}$.

Proof. From (3.2) and (8.8) we get:

$$N^i_j = \overset{C}{N}{}^i_j - \varphi^i_j , \quad F^i_{jk} = \overset{C}{F}{}^i_{jk} + \frac{1}{2} g^{ip} \left(\varphi^m_k \frac{\partial g_{jp}}{\partial y^m} + \varphi^m_j \frac{\partial g_{kp}}{\partial y^m} - \varphi^m_p \frac{\partial g_{jk}}{\partial y^m} \right), \quad C^i_{jk} = \overset{C}{C}{}^i_{jk} ,$$

which, on F^{n-1}, gives (8.9).

Now the fundamental equations (6.1) and the Gauss-Codazzi equations (6.3) can be rewritten for this particular case.

The last theorem has as a consequence:

Theorem 8.6. *If the normal connection $F\Gamma(N)$ is given by the theorem 8.4, then it is the same with the canonical Finsler connection $C\Gamma(\overset{C}{N})$ in F^n if and only if the canonical non-linear connection N has the (H)-property with respect to the canonical non-linear connection $\overset{C}{\underline{N}}$.*

Indeed, $N^i_j = \overset{C}{N}{}^i_j$ iff $\varphi^i_j = 0$. Then, on F^{n-1}, we have $B^\alpha_j \varphi^i_\alpha = 0$, which implies $\varphi^i_\alpha = B^j_\alpha \overset{C}{N}{}^i_j - B^i_\beta \overset{C}{N}{}^\beta_\alpha + B^i_{0\alpha} = 0$. Now applying theorem 2.1 we have our assertion.

A Finsler space (M^n, L) is a particular generalized Finsler space $F^n = (M^n, g_{ij})$, with the regular Finsler metric $g_{ij}(x, y) = 1/2\ \partial^2 L^2/\partial y^i \partial y^j$, [23, 11, 31]. All considerations of this paper can be particularised for this case, taking $g^*_{ij}(x, y) = g_{ij}(x, y)$. The canonical Finsler connection $C\Gamma(\overset{C}{N})$ coincides with the Cartan connection $C\Gamma$ of (M^n, L) and the intrinsic canonical connection $C\Gamma(\overset{C}{\underline{N}})$ is the Cartan connection $C\underline{\Gamma}$ of the induced Finsler space $(M^{n-1}, \underline{L})$.

Evidently, the normal Finsler connection $F\Gamma(N)$ is completly different from the Cartan connection $C\Gamma$. For these reasons the classical Haimovici-Varga-Rund-Matsumoto theory is different from the present nonstandard theory of the hypersurfaces M^{n-1} in the Finsler spaces.

REFERENCES

1. Anh, K.Q. — *Geometry of Myller configurations in Finsler spaces.* Thesis, Iaşi, 1977, p. 142
2. Bejancu, A. — *Real hypersurfaces of a complex projective space.* Rendiconti di Matematica, 3—4, vol. 12, Sc. VI, 1979, p. 439—445

3. Čomić, I. — *Some special subspaces of a Finsler space.* Publ. de l'Inst. Math. N.S.t. 33 (17), 1983, p. 49—58
4. Eisenhart, L.P. — *Non-Riemannian Geometry.* Amer. Math. Soc., vol. VIII, 1927
5. Grifone, J. — *Sur les connexions induite et intrinseque d'une sous-variété d'une variété finslérienne.* C.R. Acad. Sci. Paris, 282, 1976, p. 599—602
6. Haimovici, A. — *Sur les espaces à connexion non linéaire à parallelisme absolu.* Rev. Roumaine Math. Pures Appl. 10, 1965, p. 1 121—1 128
7. Haimovici, M. — *Formules fondamentales dans la théorie des hypersurfaces d'un espace de Finsler.* C.R. Acad. Sci. Paris, 198, 1934, p. 426—427
8. Haimovici, M. — *Les formules fondamentales dans la théorie des hypersurfaces d'un espace général.* Ann. Sc. Univ. Jassy 20, 1935, p. 39—58
9. Haimovici, M. — *Sulle superficie totalmente geodetiche negli spazi di Finsler.* Rend. Accad. Naz. Lincei (6) 27, 1938, p. 633—641
10. Haimovivi, M. — *Variétés totalement extrémales et variétés totalement géodésiques dans les espaces de Finsler.* Ann Sci. Univ. Jassy I 25, 1939, p. 559—644
11. Hashiguchi, M. — *On generalized Finsler spaces.* An. şt. Univ. Iaşi. T. XXX, s. I, fasc. 1, 1984
12. Hashiguchi, M., Ichijyo, Y. — *On generalized Berwald spaces.* Rep. Fac. Sci. Kagoshima Univ. (Math. Phys. Chem.) Nr. 15, 1982, p. 19—32
13. Ichijyo, Y. — *Almost complex structure of tangent bundles and Finsler metrics.* J. of Math. of Kyoto Univ. vol. 6, Nr. 3, 1967, p. 419—452
14. Izumi, H., Sakaguchi, T. — *Identities in Finsler space.* Mem. of the Defence Academy. vol. 22, Nr. 1, 198 p.
15. Kikuchi, S. — *On the theory of subspaces in a Finsler space.* Tensor, N.S., 2, 1952, p. 67—79
16. Laugwitz, D. — *Geometrical methods in the differential geometry of Finsler spaces.* C.I.M.E. Geom. Calcolo Varia. 1—3, 1961, p. 49
17. Macovei, E. — *Properties of the indicatrices of the Finsler vector bundles* (to appear)
18. Matsumoto, M. — *The induced and intrinsic Finsler connections of a hypersurface and Finslerian projective geometry.* to appear in J. Math. Kyoto Univ. p. 42
19. Matsumoto, M. — *Foundation of Finsler geometry and special Finsler spaces.* Kyoto Univ. 1977 unpublished
20. Miron, R. — *On transformation groups of Finsler connections.* Tensor N.S. 35, 1981, p. 235—240
21. Miron, R. — *Introduction to the theory of Finsler spaces.* Proc. of the Nat. Seminar on Finsler spaces. Univ. Braşov, 1980, p. 130—183
22. Miron, R. — *Vector bundles Finsler geometry.* Proc. of the Nat. Seminar on Finsler spaces. Univ. Braşov, 1982, p.
23. Miron, R. — *Metrical Finsler structures and metrical Finsler connections.* J. Math. Kyoto Univ. vol. 23, Nr. 2, 1983, p. 219—224
24. Miron, R., Hashiguchi, M. — *Metrical Finsler connections.* Rep. Fac. Sci. Kagoshima Univ. (Math. Phys. Chem.) 12, 1979, p. 21—35
25. Miron, R., Papuc, D. — *Un théorème local d'existence pour les variétés plongées dans un espace a connexion affine.* C.R. Acad. Sc. Paris. t. 260, 1965, p. 2 695—2 697
26. Rapcsák, A. — *Eine neue Charakterisierung Finslerscher Räume skalarer und konstanter Krümmung und projektiv-ebene Räume.* Acta Math. Acad. Sci. Hungar. 8, 1957, p. 1—18
27. Rund, H. — *The differential geometry of Finsler spaces.* Springer-Verlag, Berlin—Heidelberg—Götingen, 1959
28. Rund, H. — *Subspaces of manifolds with direction-dependent connections.* Tensor N.S., 35, 1981, p. 139—154
29. Varga, O. — *Über den inneren und induzierten Zusammenhang für Hyperflachen in Finslerschen Räumen.* Publ. Math. Debrecen, 8, 1961, p. 208—217
30. Wagner, V.V. — *Geometria del calcolo delle variationi.* C.I.M.E., 1965
31. Watanabe, S., Ikeda, F. — *On metrical Finsler connection of a metrical Finsler structure.* Tensor N.S., 39, 1982, p. 37—41
32. Watanabe, S., Ikeda, S., Ikeda, F. — *On a metrical Finsler connection of a generalized Finsler metric.* Tensor N.S. 40, 1983, p. 97—102

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