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HOLOMORPHICALLY PROJECTIVE STRUCTURES
I. THE CONNECTIONS

BY

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§0. Introduction. The projective structures were considered by E. Cartan by his method of moving frames [1], [2] and later it was rigorous investigated using principal fibre bundles by Tanaka, Kobayashi and Nagano [14], [6], [5].

In the present paper we shall study holomorphically projective structures (in the real form) by the method of Kobayashi-Nagano; we shall give also a G-theoretic approach to the theory of holomorphically projective related connections and holomorphically projective transformations.

Throughout this paper we always assume the differentiability of class C^∞. We also use the theory of bundles of frames of higher order and canonical forms according to [4], [13], [5]. We use the abbreviation HP for holomorphically projective, e.g. HP-structure, HP-related, HP-connection, HP-transformation.

§1. HP.-spaces and HP-groups. Let F be an almost complex operator of the real space $\mathbf{R}^n$. Let

$$\tilde{F}=\begin{bmatrix} 0 & 0 & -1 \\ 0 & F & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

be a new almost complex structure attached to $\mathbf{R}^{n+2}$. In the space $(\mathbf{R}^{n+2}, \tilde{F})$, we consider the set of all 2-dimensional subspaces invariant with respect to $\tilde{F}$. This set is denoted by $G(n+2,2)$. If a vector $p=(x, x^i, y)$, $i=1, \ldots, n$, belongs to a 2-dimensional subspace π, then a vector $\tilde{F}p=(-y, F^i_j x^j, x)\in\pi$. The set U consisting of all 2-dimensional subspaces, all representatives of which $(p, \tilde{F}_p)$ satisfy the condition $\det\begin{vmatrix} x & -y \\ y & x \end{vmatrix}\neq 0$ is open. On the set U we can define an inhomogeneous coordinate system

$$\pi\to z^j=\frac{xx^j-yF^j_i x^i}{x^2+y^2},\quad i, j=1,\ldots, n. \tag{1.1}$$

It is obvious that the manifold $G(n+2,2)$ is diffeomorphic with the real form of $\mathbf{C}P^{n/2}$.

The elements A of the group $Gl(n+2, \mathbf{R})$ satisfying $AF=FA$ form a closed subgroup $\widetilde{\mathrm{HP}}(n+2)$. An element of $\widetilde{\mathrm{HP}}(n+2)$ sends points of $G(n+2,2)$ into points of $G(n+2,2)$ and is a matrix of the form

$$(1.2)\qquad \begin{bmatrix} a & u & b \\ \xi & B & F\xi \\ -b & -uF & a \end{bmatrix}$$

where u is a row n-vector, ξ is a column n-vector, $B \in Gl(n, \mathbf{R})$ and satisfies the condition $BF=FB$. Notice that the elements of a closed subgroup $\tilde{H}$ consisting of all the matrices of the form

$$(1.3)\qquad \begin{bmatrix} \alpha & 0 & -\beta \\ 0 & \alpha I-\beta F & 0 \\ \beta & 0 & \alpha \end{bmatrix}$$

transform representatives of any point of $G(n+2,2)$ into some representatives of the same point. Notice also that the group $\tilde{H}$ is a normal subgroup of $\widetilde{\mathrm{HP}}(n+2)$. Then the group $\mathrm{HP}(n+2)=\widetilde{\mathrm{HP}}(n+2)/\tilde{H}$ becomes a Lie transitive transformation group of $G(n+2,2)$ (it is in fact the real form of $PGl\left(\frac{n}{2}, \mathbf{C}\right)$.

Hence if $\det\begin{vmatrix} a, & b \\ -b, & a \end{vmatrix} \neq 0$, then we set

$$(1.4)\qquad \begin{gathered} a^i=\frac{a}{a^2+b^2}\xi^i+\frac{b}{a^2+b^2}F^i_z\xi^z,\ a^i_j=\frac{a}{a^2+b^2}b^i_j+\frac{b}{a^2+b^2}F^z_j b^i_z, \\ a_j=\frac{a}{a^2+b^2}u_j+\frac{b}{a^2+b^2}F^z_j u_z. \end{gathered}$$

for the local coordinate system on $\mathrm{HP}(n+2)$, where u_j are coordinates of u, ξ^j coordinates of ξ, and b^i_j entires of the matrix B. If (z^j), $j=1,\dots,n$, are inhomogeneous coordinates on $G(n+2,2)$ and (a^i, a^i_j, a_j), $i, j=1, \dots n$, coordinates of an element of $\mathrm{HP}(n+2)$, then the induced holomorphically projective transformation is given, in terms of the inhomogeneous coordinate system (z^j), $j=1,\dots, n$ by the following transformation

$$(1.5)\qquad z^j \to \frac{a^j+a^ja_iz^i+a^j_iz^ia_sz^s+a^j_iz^i+a_iF^i_sz^sa^j_rF^r_pz^p+a_iF^i_sz^sF^j_ta^t}{1+(a_jz^i)^2+(a_iF^i_sz^c)^s+2a_iz^i},$$

The isotropy subgroup of HP $(n+2)$ at a point $(0,\dots, 0)$ of $G(n+2,2)$ is $H=\tilde{\tilde{H}}/\tilde{H}$ where $\tilde{\tilde{H}}$ is a closed subgroup of $\widetilde{\mathrm{HP}}(n+2)$ consisting of all the matrices of the form

$$(1.6)\qquad \begin{bmatrix} a & u & b \\ 0 & B & 0 \\ -b & -uF & a \end{bmatrix}.$$

It is evident, that in the local coordinate system (a^i, a^i_j, a_j) of $\mathrm{HP}(n+2)$, the subgroup H is defined by $a^i=0$, $i=1,\dots, n$. The transformation induced by an element $a \in H$ is therefore given by the equation of the form:

$$(1.7)\qquad z^j \to \frac{a^j_iz^i+a^j_iz^ia_sz^s+a_iF^i_sz^sa^j_rF^r_pz^p}{1+(a_iz^i)^2+(a_iF^i_sz^s)^2+2a_iz^i}.$$

It is easy to see that $\mathrm{HP}(n+2)=S\widetilde{\mathrm{HP}}(n+2)/S\tilde{H}$ where $S\widetilde{\mathrm{HP}}(n+2)=\{A \in \widetilde{\mathrm{HP}}(n+2), \det A=1\}$ and $S\tilde{H}=\{A \in \tilde{H}, \det A=1\}$.

Now, we intend to consider the Lie algebras of above mentioned groups. Let $hp(n+2)=s\widetilde{hp}(n+2)/s\tilde{h}$ be the Lie algebra of the group $\mathrm{HP}(n+2)$. Let us define

$$(1.8)\qquad \begin{gathered} g_{-1}=\begin{bmatrix} 0 & 0 & 0 \\ \xi & 0 & F\xi \\ 0 & 0 & 0 \end{bmatrix}+s\tilde{h},\quad g_0=\begin{bmatrix} a & 0 & b \\ 0 & B & 0 \\ -b & 0 & a \end{bmatrix}+s\tilde{h},\ \operatorname{tr} B+2a=0, \\ g_1=\begin{bmatrix} 0 & u & 0 \\ 0 & 0 & 0 \\ 0 & -uF & 0 \end{bmatrix}+s\tilde{h}. \end{gathered}$$

Then $hp(n+2)=g_{-1}+g_0+g_1$ and $[g_p, g_q] \subset g_{p+q}$ for $p, q \geq -1$ and $[X, g_{-1}] \neq 0$ for each nonzero $X \in g_p$, $p \geq 0$.

Let us consider a group $L_0\tilde{H}=\{A \in \tilde{\tilde{H}}, \det A=1\}/S\tilde{H}$; then $G(n+2,2)=S\widetilde{\mathrm{HP}}(n+2)/S\tilde{H}/L_0\tilde{H}/S\tilde{H}=S\widetilde{\mathrm{HP}}(n+2)/L_0\tilde{H}$ is a connected homogeneous space on which the Lie group $\mathrm{HP}(n+2)=S\widetilde{\mathrm{HP}}(n+2)/S\tilde{H}$ acts transitively. Since $L_0\tilde{H}/S\tilde{H}$ is the isotropy subgroup at the origin $(0,\dots, 0)$ of $G(n+2,2)=\mathrm{HP}(n+2)/H$ there is a natural representation of $H=L_0\tilde{H}/S\tilde{H}$, called the linear isotropy representation of H on the tangent space of $G(n+2,2)$ at the origin. Let $L_1\tilde{H}$ denote the kernel of the isotropy representation, then g_1 is the Lie algebra of $L_1\tilde{H}$, g_0+g_1 is the Lie algebra of $L_0\tilde{H}$, so that g_0 is the Lie algebra of $L_0\tilde{H}/L_1\tilde{H}$.

Now, we shall engage in the equations of M a u r e r - C a r t a n of $\mathrm{HP}(n+2)$.

Proposition 1.1. *Let $(\omega^i, \omega^i_j, \omega_j)$ be the M a u r e r - C a r t a n forms on $\mathrm{HP}(n+2)$ which coincide with da^i, da^i_j, da_j at the identity. Then the equations of Maurer-Cartan of $\mathrm{HP}(n+2)$ are given by:*

$$(1.9)\qquad \begin{gathered} d\omega^i=-\omega^i_k \wedge \omega^k,\ d\omega^i_j=-\omega^i_t \wedge \omega^t_j-\omega^i \wedge \omega_j+\delta^i_j\omega_t \wedge \omega^t-\omega_sF^s_t \wedge \omega^tF^i_j+ \\ +F^i_t\omega^t \wedge \omega_sF^s_j\quad d\omega_j=-\omega^t \wedge \omega^t_j. \end{gathered}$$

Proof: Let us set

$$(1.10)\qquad \omega=\begin{bmatrix} \bar{\omega}^0_0 & \bar{\omega}^0_j & \bar{\omega}^0_x \\ \bar{\omega}^i_0 & \bar{\omega}^i_j & \bar{\omega}^i_x \\ \bar{\omega}^x_0 & \bar{\omega}^x_j & \bar{\omega}^x_x \end{bmatrix}=\begin{bmatrix} a & u & b \\ \xi & B & F\xi \\ -b & -uF & a \end{bmatrix}^{-1}\cdot d\begin{bmatrix} a & u & b \\ \xi & B & F\xi \\ -b & -uF & a \end{bmatrix}$$

then we have $\bar{F}\bar{\omega}=\bar{\omega}\bar{F}$. If we set $\begin{bmatrix} a & u & b \\ \xi & B & F\xi \\ -b & -uF & a \end{bmatrix}=I$, then we get $\bar{\omega}=$
$=d\begin{bmatrix} a & u & b \\ \xi & B & F\xi \\ -b, & -uF & a \end{bmatrix}$. On the other hand we get from (1.4)

$$da^i=d\xi^i,\quad da^i_j=ab^i_j-\delta^i_j da-F^i_j db,\quad da_j=du_j \tag{1.11}$$

at the identity. Moreover $\omega^i=da^i$, $\omega^i_j=da^i_j$, $\omega_j=da_j$ at the identity, hence we have (1.9). q.e.d.

The dual of above Proposition may be formulated as follows.
Let

$$m=\begin{bmatrix} 0 & 0 & 0 \\ \xi & 0 & F\xi \\ 0 & 0 & 0 \end{bmatrix}+\tilde{h},\quad g=\begin{bmatrix} 0 & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & 0 \end{bmatrix}+\tilde{h},\quad m^*=\begin{bmatrix} 0 & u & 0 \\ 0 & 0 & 0 \\ 0 & -uF & 0 \end{bmatrix}+\tilde{h}.$$

Then the Lie algebra of $HP(n+2)$ is the direct sum $hp(n+2)=m+g+m^*$ with the following bracket operation: if $u, v\in m$, u^*, $v^*\in m^*$ and $U, V\in g$ where

$$u=\begin{bmatrix} 0 & 0 & 0 \\ u & 0 & Fu \\ 0 & 0 & 0 \end{bmatrix}+\tilde{h},\quad u^*=\begin{bmatrix} 0 & u & 0 \\ 0 & 0 & 0 \\ 0 & -uF & 0 \end{bmatrix}+\tilde{h},$$

$$U=\begin{bmatrix} 0 & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & 0 \end{bmatrix}+\tilde{h},\text{ then } [u, v]=\tilde{h},\ [u^*, v^*]=\tilde{h},$$

$$[U, u]=\begin{bmatrix} 0 & 0 & 0 \\ Uu & 0 & FUu \\ 0 & 0 & 0 \end{bmatrix}+\tilde{h},\quad [u^*, U]=\begin{bmatrix} 0 & u^*U & 0 \\ 0 & 0 & 0 \\ 0 & -u^*UF & 0 \end{bmatrix}+\tilde{h},$$

$$[U, V]=\begin{bmatrix} 0 & 0 & 0 \\ 0 & UV-VU & 0 \\ 0 & 0 & 0 \end{bmatrix}+\tilde{h}, \tag{1.13}$$

$$[u, u^*]=\begin{bmatrix} 0 & 0 & 0 \\ 0 & uu^*+u^*uI-Fuu^*F-Fu^*Fu & 0 \\ 0 & 0 & 0 \end{bmatrix}+\tilde{h}.$$

The left invariant vector fields on $HP(n+2)$, correspond with $\partial/\partial a^i$, $\partial/\partial a^i_j$, $\partial/\partial a_j$ at the identity and form a natural basis for m, g and m^*.

§2. Cartan connections. In this section we shall study the Cartan connections in our context; so for the definitions and the basic facts we refer the reader to [5], [6].

Let H be the isotropy group of $HP(n+2)$ so that $HP(n+2)/H=$ $=G(n+2,2)$. Let M be a manifold and P be a principal fibre bundle over M with structure group H. We fix the natural basis for the Lie algebra $hp(n+2)$ as described in §1. A Cartan connection ω in P is then given, with respect to this basis, by a set of 1-forms ω^i, ω^i_j, ω_j $i, j=1,\dots, n$ on P. The structure equations of the Cartan connections are given by

$$\begin{aligned} &d\omega^i=-\omega^i_k\wedge\omega^k+\Omega^i,\quad d\omega^i_j=-\omega^i_t\wedge\omega^t_j-\omega^i\wedge\omega_j+\delta^i_j\omega_t\wedge\omega^t-\omega_s F^s_t\wedge\omega^t F^i_j+ \\ &\qquad +F^i_t\omega^t\wedge\omega_s F^s_j+\Omega^i_j\quad d\omega_j=-\omega_t\wedge\omega^t_j+\Omega_j. \end{aligned} \tag{2.1}$$

As usual, we shall take these equations as a definition of the 2-forms Ω^i, Ω^i_j, Ω_j, $i, j=1,\dots, n$. We call (Ω^i) the torsion form of the Cartan connection ω and (Ω^i_j, Ω_j) the curvature form of ω. From the theorems proved in [5], [6] follows that the torsion and curvature forms of the Cartan connection can be written as follows.:

$$\Omega^i=1/2K^i_{kl}\omega^k\wedge\omega^l,\quad \Omega^i_j=1/2K^i_{jkl}\omega^k\wedge\omega^l,\quad \Omega_j=1/2K_{jkl}\omega^k\wedge\omega^l. \tag{2.2}$$

In order that the forms ω^i, ω^i_j, ω_j define a Cartan connection in P, one has to pose the following conditions on ω^i, and ω^i_j:

(2.3) $\omega^i(A^*)=0$ and $\omega^i_j(A^*)=A^i_j$ for every $A=(A^i_j, A_j)\in g+m^*$, where A^* is the fundamental vector field corresponding to A.

(2.4) $R^*_b(\omega^i, \omega^i_j)=ad(b^{-1})(\omega^i, \omega^i_j)$ for every $b\in H$, where $ad(b^{-1})\colon m+g\to m+g$ is the mapping $hp(n+2)/m\to hp(n+2)/m$ induced by $ad(b^{-1})\colon hp(n+2)\to$ $\to hp(n+2)$.

(2.5) If X is a tangent vector to P such that $\omega^i(X)=0$, then X is vertical.

Proposition 2.1. *Let P be a principal fibre bundle over M with structure group H. Given ω^i and ω^i_j satisfying (2.3), (2.4), (2.5) and*

$$d\omega^i=-\omega^i_k\wedge\omega_k, \tag{2.6}$$

then there exists a unique Cartan connection $\omega=(\omega^i, \omega^i_j, \omega_j)$ with the following properties:

$$K^i_{ijk}=0,\quad K^i_{jil}=0. \tag{2.7}$$

Proof: We shall study first the relationship between two Cartan connections ω and $\bar{\omega}$ with the given (ω^i, ω^i_j). By the definition of the Cartan connection we can write

$$\bar{\omega}_j-\omega_j=A_{jk}\omega^k, \tag{2.8}$$

where the coefficients A_{jk} are functions on P. We have

(2.9) $\bar{\Omega}^i_j-\Omega^i_j=(1/2)(\bar{K}^i_{jkl}-K^i_{jkl})\omega^k\wedge\omega^l=(A_{tk}\delta^i_j-A_{sk}F^s_t F^i_j-A_{jt}\delta^i_k-F^i_k A_{st}F^s_j)\omega^k\wedge\omega^t$

that is

$$\begin{aligned} \bar{K}^i_{jkt}-K^i_{jkt}&=A_{tk}\delta^i_j-A_{kt}\delta^i_j-A_{sk}F^s_t F^i_j+A_{st}F^s_k F^i_j-A_{jt}\delta^i_k+ \\ &\quad +A_{jk}\delta^i_t+F^i_k A_{st}F^s_j-F^i_t A_{sk}F^s_j. \end{aligned} \tag{2.10}$$

Hence

$$(2.11)\qquad \begin{aligned} K^i_{ikt}-\bar K^i_{ikt}&=(n+2)(A_{tk}-A_{kt}),\\ \bar K^i_{jit}-K^i_{jit}&=A_{tj}-(n+1)A_{jt}-F^i_t(A_{st}+A_{ts})F^s_j. \end{aligned}$$

Now we prove the uniqueness of a Cartan connection as stated above. Let ω and $\bar\omega$ be Cartan connection with properties (2.7). Then the left hand sides of (2.11) vanish. Hence $A_{tk}=A_{kt}$, and

$$(2.12)\qquad nA_{jt}+2F^i_tA_{si}F^s_j=0.$$

Let us rewrite the formula (2.12) in the form $W^{si}_{jt}A_{si}=0$, where $W^{si}_{jt}=(n/2)\delta^i_t\delta^s_j+F^i_tF^s_j$.
Since $W^{si}_{jt}U^{jt}_{ba}=U^{si}_{jt}W^{jt}_{ba}=\delta^i_a\delta^s_b$ where $U^{jt}_{ba}=[2n/(n^2-4)]\delta^t_a\delta^j_b-[4/(n^2-4)]F^t_aF^j_b$ then U^{jt}_{ba} is the inverse matrix to the matrix W^{si}_{jt}. Thus we have

$$(2.13)\qquad A_{st}=0$$

if $n=\dim M\geqslant 4$.

Assuming that there is at least one Cartan connection $\bar\omega=(\omega^i,\omega^i_j,\bar\omega_j)$ with the given (ω^i,ω^i_j) satisfying (2.6) we shall show the existence of a Cartan connection $\omega=(\omega^i,\omega^i_j,\omega_j)$ satisfying (2.7). In equations (2.11) we set

$$(2.14)\qquad K^i_{ikt}=0 \text{ and } K^i_{jit}=0.$$

Then

$$(2.15)\qquad W^{sp}_{jt}A_{sp}=-\tfrac{1}{2}\bar K^i_{jit}+1/[2(n+2)]\bar K^i_{ijt}-1/[2(n+2)]\bar K^i_{isp}F^s_jF^p_t.$$

From this equation we have

$$A_{wu}=[1/[(n^2-4)]\,\bar K^i_{iwu}-[n/(n^2-4)]\bar K^i_{wiu}-[1/(n^2-4)]F^t_u\bar K^i_{ijt}F^j_w+$$
$$+[2/(n^2-4)]F^t_u\bar K^i_{jit}F^j_w.$$

If we set

$$(2.17)\qquad \omega_j=\bar\omega_j-A_{jk}\omega^k,$$

then $\omega=(\omega^i,\omega^i_j,\omega_j)$ is a Cartan connection with the required properties.

To complete the proof of our proposition, we have only to prove that there is at least one Cartan connection ω with the given (ω^i,ω^i_j). But the proof is the same as the case of projective connections (c.f. [6]). q.e.d.

Proposition 2.2. *Let P a principal fibre bundle over M with structure group H. If $\omega=(\omega^i,\omega^i_j,\omega_j)$ is a Cartan connection with the properties* (2.6), (2.7), *then its curvature forms has the following properties:*

$$(2.18)\qquad \Omega^i_j\wedge\omega^j=0,\ \textit{that is}\ K^i_{jkl}+K^i_{klj}+K^i_{ljk}=0,$$

$$(2.19)\qquad \Omega_j\wedge\omega^j=0,\ \textit{that is}\ K_{jkl}+K_{klj}+K_{ljk}=0,$$

$$(2.20)\qquad \text{If } \Omega^i_j=0 \textit{ and } \dim M\geqslant 4, \text{ then } \Omega_j=0.$$

Proof. The formulae (2.18), (2.19) are obvious. By applying exterior differentiation to the structure equation (2.1.2) and setting $\Omega^i_j=0$ we obtain

$$\Omega_s\wedge F^s_t\omega^tF^i_j+F^i_t\omega^t\wedge\Omega_sF^s_j=0,$$

that is $V^{sr}_{tj}\omega^t\wedge\Omega_s=0$, where $V^{sr}_{tj}=F^s_tF^r_j-\delta^s_j\delta^r_t$. A matrix $Z^{jb}_{ra}=[1/(n+1)]F^j_rF^b_a-\delta^j_a\delta^b_r$, is an inverse matrix to the matrix V^{sr}_{tj}, then, $Z^{jb}_{ra}V^{sr}_{tj}=V^{jb}_{ra}Z^{sr}_{tj}=\delta^s_a\delta^b_t$.
Thus we have

$$\omega^t\wedge\Omega_s=0.$$

Hence we have $\Omega_s=0$ provided that $\dim M\geqslant 4$. (q.e.d.)

§ 3. HP-structures and HP-connections. Proposition 3.1. *A subset H^2_n of L^2_n consisting of elements (s^i_j, s^i_{jk}) with $F^t_is^i_j=F^i_js^t_i$ and $s^i_{jk}=-s^i_ks_j-s^i_ks_j+s_tF^t_js^i_rF^r_k+s_tF^t_ks^i_rF^r_j$ forms a subgroup of L^2_n of dimension $(n^2/2)+n$.*

If one explicitly calculate the jet of (1.7) then obtain

Proposition 3.2. *For each element $a\in H$ let f be the transformation* (1.7) *of $G(n+2,2)$ induced by a. Then $a\to j^2_0f$ gives an isomorphism of H onto H^2_n. Moreover if $a\in H$ has coordinates (a^i, a^i_j, a_j) where $a^i=0$, with respect to the coordinate system in* HP$(n+2)$ *given by* (1.4), *then the corresponding element of H^2_n has coordinates*

$$(3.1)\qquad (a^i_j,\ -a^i_ja_s-a^i_sa_j+a_tF^t_ja^i_rF^r_s+a_tF^t_sa^i_pF^p_j).$$

Let $\theta=(\theta^i,\theta^i_j)$ be the canonical form on P^2M and let $\bar\theta=(\bar\theta^i,\bar\theta^i_j)$ be the restriction of θ to P.

Definition 3.2. *A subbundle $P\subset P^2M$ is said to be a* HP*-structure on M if a structural group of P is H^2_n. A* HP*-connection associated with a* HP*-structure is a Cartan connection ω on P such $\omega^i=\bar\theta^i$.*

From the Proposition 2.1 and the theory of cannonical forms we have

Theorem 3.3. *For each* HP*-structure P on a manifold M, there is a unique* HP*-connection $\omega=(\omega^i,\omega^i_j,\omega_j)$ such that*

$$(3.2)\qquad \omega^i=\bar\theta^i \textit{ and } \omega^i_j=\bar\theta^i_j, \textit{ so that } d\omega^i=-\omega^i_k\wedge\omega^k$$

$$(3.3)\qquad \Omega^i_i=0,\qquad (3.4)\qquad K^i_{jit}=0.$$

On the manifold $G(n+2,2)$ one has a natural HP-structure. Let us imbed $\mathbf{R}^n$ into $G(n+2,2)$ by sending $(x^1,\dots,x^n)$ into the point with inhomogeneous coordinates $(x^1,\dots,x^n)$. Let us consider the subbundle $P^2G(n+2,2)$ consisting of elements of the form j^2_0f, where f is a HP-transformation of $R^n\to M$.

If we write $G(n+2,2)=\mathrm{HP}(n+2)/H$, then we have the following commutative diagram:

$$(3.5)\qquad \begin{tikzcd} \mathrm{HP}(n+2) \arrow[rr, "\eta"] \arrow[dr] & & P \arrow[dl] \\ & G(n+2,2) & \end{tikzcd}$$

where η is defined by

$$(3.6)\qquad \eta(f)=j^2_0f,\ f\in\mathrm{HP}(n+2)$$

is a bundle isomorphism compatible with isomorphism $H\to H^2_n$. The normal HP-connection $(\omega^i,\omega^i_j,\omega_j)$ associated with P corresponds to the canonical left invariant form of the group HP$(n+2)$ and its structure equations coincide with the equations of Maurer-Cartan for the group HP$(n+2)$ so that $(\Omega^i, \Omega^i_j, \Omega_j)=0$.

§ 4. Natural frames and coefficients of HP**-connections.** Let P be a HP-structure on a manifold M and $(U, (x^1,\dots,x^n))$ a local chart in M. Let $\sigma : U \to P$ be a cross-section and let $U\times H\approx P \mid U$ the isomorphism induced by σ. Let σ be given by $(x^i)\to(x^i, \sigma^i_j, \sigma^i_{jk})$. Let (a^i_j, a^i_{jk}) be the coordinate system in H^2_n, where $a^i_t F^t_j = F^i_t a^t_j$ and $a^i_{jk} = -a^i_j a_k - a^i_k a_j + a_t F^t_j a^i_r F^r_k + a_t F^t_k a^i_r F^r_j$. Then the natural coordinate system (u^i, u^i_j, u^i_{jk}) in $P \mid U$ can be written as:

$$u^i = x^i,\quad u^i_j=\sigma^i_k a^k_j,\quad u^i_{jk}=\sigma^i_t a^t_{jk}+\sigma^i_{tm}a^t_j a^m_k. \tag{4.1}$$

Let us set

$$\psi^i=\sigma^*\bar\theta^i,\quad \psi^i_j=\sigma^*\bar\theta^i_j. \tag{4.2}$$

Then we obtain the following formulae

$$\theta^i=\bar a^{-1}{}^i_t\psi^t,\quad \theta^i_j=\bar a^{-1}{}^i_t\psi^t_s a^s_j+\bar a^{-1}{}^i_t da^t_j+a_j\theta^i+\delta^i_j a_t\theta^t-a_tF^t_p\theta^pF^i_j-a_tF^t_jF^i_p\theta^p. \tag{4.3}$$

Let $(\omega^i, \omega^i_j, \omega_j)$ be the normal HP-connection in P and set

$$\psi^i=\sigma^*\omega^i=\Pi^i_k dx^k,\quad \psi^i_j=\sigma^*\omega^i_j=\Pi^i_{kj}dx^k,\quad \psi_j=\sigma^*\omega_j=\Pi_{jk}dx^k. \tag{4.4}$$

Then after some straightforward calculations, we obtain the following formulae:

$$\begin{aligned}\omega^i&=a^{-1}{}^i_j\psi^j,\quad \omega^i_j=\bar a^{-1}{}^i_t\psi^t_s a^s_j+\bar a^{-1}{}^i_t da^t_j+a_j\omega^i+\delta^i_j a_t\omega^t-a_tF^t_s\omega^sF^i_j-a_tF^t_jF^i_s\omega^s,\\ \omega_j&=da_j-a_k\omega^k_j+a_k\omega^k a_j+a^k_j\psi_k-(1/2)a_tF^t_k\omega^kF^s_j a_s.\end{aligned} \tag{4.5}$$

We call Π^i_k, Π^i_{jk}, Π_{jk} the coefficients of the normal HP-connection with respect to the local cross-section σ.

Proposition 4.1. *Let P be a* HP*-structure on M and $(\omega^i, \omega^i_j, \omega_j)$ the normal* HP*-connection in P. Let U be a coordinate neighbourhood in M with local coordinate system $(x^1,\dots,x^n)$. Then there is a unique crooss-section $\sigma : U\to P^2M$ such that*

$$\sigma^*\omega^i=dx^i \textit{ and } \sigma^*\omega^i_i=0. \tag{4.6}$$

If we set

$$\sigma^*\omega^i_j=\Pi^i_{kj}dx^k \textit{ and } \sigma^*\omega_j=\Pi_{kj}dx^k \tag{4.7}$$

then

$$\Pi^i_{jk}=\Pi^i_{kj} \textit{ and } \Pi_{jk}=\Pi_{kj}. \tag{4.8}$$

Proof. Let $\bar\sigma : U\to P$ be a cross-section such that $\bar\sigma^*(\omega^i)=dx^i$. In terms of the local coordinate system (u^i, u^i_j, u^i_{jk}) in P^2M, the cross-section is given by

$$u^i=x^i,\quad u^i_j=\delta^i_j,\quad u^i_{jk}=-\Gamma^i_{jk}, \tag{4.9}$$

where Γ^i_{jk} are certain function of $(x^1,\dots, x^n)$. We take σ as the cross-section given by

$$u^i=x^i,\quad u^i_j=\delta^i_j,\quad u^i_{jk}=-\Pi^i_{jk}. \tag{4.10}$$

where

$$\Pi^i_{jk}=\Gamma^i_{jk}-\frac{1}{n+2}(\Gamma^t_{tj}\delta^i_k+\delta^i_j\Gamma^t_{tk}-\Gamma^t_{tp}F^p_kF^i_j-\Gamma^t_{tu}F^u_jF^i_k). \tag{4.11}$$

Then we have

$$\sigma^*\omega^i_j=\Pi^i_{kj}dx^k. \tag{4.12}$$

Clearly, σ is a cross-section with desired properties.

The uniqueness is obvious using (4.6).

The rest is an immediate consequence of the facts that $\Omega^i=0$ and $\Omega^i_j=0$. q.e.d.

We call σ in Proposition 4.1 the natural cross-section or the natural frame of P associated with $(x^1,\dots,x^n)$.

§ 5. Complex connections and HP**-connections.** Let us consider the group G^2_n consisting of elements (a^i_j, a^i_{jk}) with $a^i_jF^j_l = F^i_ja^j_l$ and $a^i_{tk}F^t_j = F^i_ta^t_{jk}$. The group $Gl_{\mathbf{R}}(n/2, \mathbf{C})$ consisting of elements $A\in Gl(n, \mathbf{R})$ satisfying $AF=FA$ can be considered as the subgroup of G^2_n consisting of elements $(a^i_j, 0)$. Thus

$$Gl_{\mathbf{R}}(n/2, \mathbf{C})\subset H^2_n\subset G^2_n\subset L^2_n. \tag{5.1}$$

If P^1_FM is the reduced principal frame bundle with the structure group $Gl_{\mathbf{R}}(n/2, \mathbf{C})$, and this structure is, by assumption integrable then there exists an atlas on M, such that the transition functions of P^1_FM

$$\varphi^i_j : U\cap V\to Gl(n, \mathbf{R}) \tag{5.2}$$

satisfy

$$\varphi^i_jF^j_l=F^i_j\varphi^j_l. \tag{5.3}$$

Then for new transition functions

$$\varphi^i_{jk} : U\cap V\to L^2_n \tag{5.4}$$

which satisfy

$$\varphi^i_{jk}F^j_l=F^i_j\varphi^j_{lk}, \tag{5.5}$$

we can construct a principal fibre bundle $P^2_FM\cup P^2M$ with structure group G^2_n.

We consider the associated bundles $P^2_FM/Gl_{\mathbf{R}}(n/2, \mathbf{C})$ and P^2_FM/H with fibres $G^2_n/Gl_{\mathbf{R}}(n/2, \mathbf{C})$ and G^2_n/H^2_n respectively.

Proposition. 5.1. a) *The cross-sections $M\to P^2_FM/Gl_{\mathbf{R}}(n/2, \mathbf{C})$ are in one-to-one correspondence with the torsionless connections in P^1_FM.* b) *The cross-sections $M\to P^2_FM/H^2_n$ are in one-to-one correspondence with the* HP*-structures on M.*

Proof. Let (u^i, u^i_j, u^i_{jk}) be the local coordinate system in P^2_FM induced from a local coordinate system $(x^1,\dots,x^n)$ in M. We introduce a local coordinate system (z^i, z^i_{jk}) in $P^2_FM/Gl_{\mathbf{R}}(n/2, \mathbf{C})$ in such a way that the natural projection $P^2_FM\to P^2_FM/Gl_{\mathbf{R}}(n/2, \mathbf{C})$ is given by the equations left invariant form of the group $HP(n+2)$ and its structure equations coincide with the equations of Maurer-Cartan for the group $HP(n+2)$ so that $(\Omega^i, \Omega^i_j, \Omega_j)=0$.

§ 4. Natural frames and coefficients of HP-connections. Let P be a HP-structure on a manifold M and $(U, (x^1,\dots,x^n))$ a local chart in M. Let $\sigma: U\to P$ be a cross-section and let $U\times H\approx P\mid U$ the isomorphism induced by σ. Let σ be given by $(x^i)\to(x^i, \sigma^i_j, \sigma^i_{jk})$. Let (a^i_j, a^i_{jk}) be the coordinate system in H^2_n, where $a^i_t F^t_j = F^i_t a^t_j$ and $a^i_{jk} = -a^i_j a_k - a^i_k a_j + a_t F^t_j a^i_r F^r_k + a_t F^t_k a^i_r F^r_j$. Then the natural coordinate system (u^i, u^i_j, u^i_{jk}) in $P\mid U$ can be written as:

$$u^i = x^i,\quad u^i_j = \sigma^i_k a^k_j,\quad u^i_{jk} = \sigma^i_t a^t_{jk} + \sigma^i_{tm} a^t_j a^m_k. \tag{4.1}$$

Let us set

$$\psi^i = \sigma^*\bar{\theta}^i,\quad \psi^i_j = \sigma^*\bar{\theta}^i_j. \tag{4.2}$$

Then we obtain the following formulae

$$\theta^i = \bar{a}^{-1i}_t\psi^t,\quad \theta^i_j = \bar{a}^{-1i}_t\psi^t_s a^s_j + \bar{a}^{-1i}_t da^t_j + a_j\theta^i + \delta^i_j a_t\theta^t - a_t F^t_p\theta^p F^i_j - a_t F^i_j F^t_p\theta^p. \tag{4.3}$$

Let $(\omega^i, \omega^i_j, \omega_j)$ be the normal HP-connection in P and set

$$\psi^i = \sigma^*\omega^i = \Pi^i_k dx^k,\quad \psi^i_j = \sigma^*\omega^i_j = \Pi^i_{kj} dx^k,\quad \psi_j = \sigma^*\omega_j = \Pi_{jk} dx^k. \tag{4.4}$$

Then after some straightforward calculations, we obtain the following formulae:

$$\begin{gathered}\omega^i = a^{-1i}_j\psi^j,\quad \omega^i_j = \bar{a}^{-1i}_t\psi^t_s a^s_j + \bar{a}^{-1i}_t da^t_j + a_j\omega^i + \delta^i_j a_t\omega^t - a_t F^t_s\omega^s F^i_j - a_t F^i_j F^t_s\omega^s,\\ \omega_j = da_j - a_k\omega^k_j + a_k\omega^k a_j + a^k_j\psi_k - (1/2)a_t F^t_k\omega^k F^s_j a_s.\end{gathered} \tag{4.5}$$

We call Π^i_k, Π^i_{jk}, Π_{jk} the coefficients of the normal HP-connection with respect to the local cross-section σ.

Proposition 4.1. *Let P be a HP-structure on M and $(\omega^i, \omega^i_j, \omega_j)$ the normal HP-connection in P. Let U be a coordinate neighbourhood in M with local coordinate system $(x^1,\dots,x^n)$. Then there is a unique crooss-section $\sigma: U\to P^2M$ such that*

$$\sigma^*\omega^i = dx^i \text{ and } \sigma^*\omega^i_i = 0. \tag{4.6}$$

If we set

$$\sigma^*\omega^i_j = \Pi^i_{kj}dx^k \text{ and } \sigma^*\omega_j = \Pi_{kj}dx^k \tag{4.7}$$

then

$$\Pi^i_{jk} = \Pi^i_{kj} \text{ and } \Pi_{jk} = \Pi_{kj}. \tag{4.8}$$

Proof. Let $\bar{\sigma}: U\to P$ be a cross-section such that $\bar{\sigma}^*(\omega^i) = dx^i$. In terms of the local coordinate system (u^i, u^i_j, u^i_{jk}) in P^2M, the cross-section is given by

$$u^i = x^i,\quad u^i_j = \delta^i_j,\quad u^i_{jk} = -\Gamma^i_{jk}, \tag{4.9}$$

where Γ^i_{jk} are certain function of $(x^1,\dots, x^n)$. We take σ as the cross-section given by

$$u^i = x^i,\quad u^i_j = \delta^i_j,\quad u^i_{jk} = -\Pi^i_{jk}, \tag{4.10}$$

where

$$\Pi^i_{jk} = \Gamma^i_{jk} - \frac{1}{n+2}(\Gamma^t_{tj}\delta^i_k + \delta^i_j\Gamma^t_{tk} - \Gamma^t_{tp}F^p_k F^i_j - \Gamma^t_{tu}F^u_j F^i_k). \tag{4.11}$$

Then we have

$$\sigma^*\omega^i_j = \Pi^i_{kj}dx^k. \tag{4.12}$$

Clearly, σ is a cross-section with desired properties.

The uniqueness is obvious using (4.6).

The rest is an immediate consequence of the facts that $\Omega^i=0$ and $\Omega^i_j=0$. q.e.d.

We call σ in Proposition 4.1 the natural cross-section or the natural frame of P associated with $(x^1,\dots,x^n)$.

§ 5. Complex connections and HP-connections. Let us consider the group G^2_n consisting of elements (a^i_j, a^i_{jk}) with $a^i_j F^t_j = F^i_t a^t_j$ and $a^i_{tk}F^t_j = F^i_t a^t_{jk}$. The group $Gl_{\mathbf{R}}(n/2, \mathbf{C})$ consisting of elements $A\in Gl(n, \mathbf{R})$ satisfying $AF=FA$ can be considered as the subgroup of G^2_n consisting of elements $(a^i_j, 0)$. Thus

$$Gl_{\mathbf{R}}(n/2, \mathbf{C})\subset H^2_n\subset G^2_n\subset L^2_n. \tag{5.1}$$

If $P^1_F M$ is the reduced principal frame bundle with the structure group $Gl_{\mathbf{R}}(n/2, \mathbf{C})$, and this structure is, by assumption integrable then there exists an atlas on M, such that the transition functions of $P^1_F M$

$$\varphi^i_j: U\cap V\to Gl(n, \mathbf{R}) \tag{5.2}$$

satisfy

$$\varphi^i_j F^j_t = F^i_j\varphi^j_t. \tag{5.3}$$

Then for new transition functions

$$\varphi^i_{jk}: U\cap V\to L^2_n \tag{5.4}$$

which satisfy

$$\varphi^i_{jk}F^k_t = F^i_j\varphi^j_{tk}, \tag{5.5}$$

we can construct a principal fibre bundle $P^2_F M\cup P^2M$ with structure group G^2_n.

We consider the associated bundles $P^2_F M/Gl_{\mathbf{R}}(n/2, \mathbf{C})$ and $P^2_F M/H$ with fibres $G^2_n/Gl_{\mathbf{R}}(n/2, \mathbf{C})$ and G^2_n/H^2_n respectively.

Proposition. 5.1. a) *The cross-sections $M\to P^2_F M/Gl_{\mathbf{R}}(n/2, \mathbf{C})$ are in one-to-one correspondence with the torsionless connections in $P^1_F M$.* b) *The cross-sections $M\to P^2_F M/H^2_n$ are in one-to-one correspondence with the HP-structures on M.*

Proof. Let (u^i, u^i_j, u^i_{jk}) be the local coordinate system in $P^2_F M$ induced from a local coordinate system $(x^1,\dots,x^n)$ in M. We introduce a local coordinate system (z^i, z^i_{jk}) in $P^2_F M/Gl_{\mathbf{R}}(n/2, \mathbf{C})$ in such a way that the natural projection $P^2_F M\to P^2_F M/Gl_{\mathbf{R}}(n/2, \mathbf{C})$ is given by the equations

(5.6) $$z^i = u^i, \quad z^i_{jk} = u^i_{pq} u^{-1p}_{\ j} u^{-1q}_{\ k}.$$

Then a cross-section $\Gamma : M \to P^2_F M / G1_{\mathbf{R}}(n/2, \mathbf{C})$ is given locally, by a set of functions

(5.7) $$z^i_{jk} = -\Gamma^i_{jk}(x^1, \ldots, x^n) \text{ with } \Gamma^i_{jk} = \Gamma^i_{kj}.$$

If we consider the action of the group G^2_n on the fibre $G^2_n / G1_{\mathbf{R}}(n/2, \mathbf{C})$ then we see that the functions Γ^i_{jk} behave under the change of coordinate system as coefficients of a connection. Besides, the properties of the group G^2_n imply that the tensor F^i_j is parallel with respect to Γ^i_{jk}.

Since the reductions of structure group to H^2_n and the cross-sections $M \to P^2_F M / H^2_n$ are in one-to-one correspondence, *b)* is evident. q.e.d.

Every complex connection without torsion $\Gamma : M \to P^2_F M / G1_{\mathbf{R}}(n/2, \mathbf{C})$, composed with the natural projection $P^2_F M / G1_{\mathbf{R}}(n/2, \mathbf{C}) \to P^2_F M / H^2_n$ gives a HP-structure $M \to P^2_F M / H^2_n$. A HP-connection without torsion is said to belong to a HP-structure P if Γ induces P in the manner described above. We say that two complex connections without torsion are HP-related if they belong to the same HP-structure. This definition is equivalent to the one given by O t s u k i and T a s h i r o in [11]. Indeed an element $(a^i_j, -a^i_j a_k - a^i_k a_j + a_t F^t_j a^i_r F^r_k + a_t F^t_k a^i_r F^r_j)$ of H^2_n induces the transformation of $P^2_F M$ given by

(5.8) $$(u^i, u^i_j, u^i_{jk}) \to (u^i, u^i_p a^p_j, u^i_{qr} a^q_j a^r_k - u^i_p a^p a_k - u^i_p a^p_k a_j + u^i_p a_t F^t_j a^p_r F^r_k + u^i_p a_t F^t_k a^p_r F^r_j).$$

It induces the transformation of $P^2_F M$ given by

(5.9) $$(z^i, z^i_{jk}) \to (z^i, -\delta^i_j a_k - \delta^i_k a_j + a_t F^t_j F^i_k + a_t F^t_k F^i_j).$$

It means that the connection coefficients are in the following relation

(5.10) $$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_k a_j - a_t F^t_j F^i_k - a_t F^t_k F^i_j + \delta^i_j a_k.$$

Let $\Gamma : M \to P^2_F M / G1_{\mathbf{R}}(n/2, \mathbf{C})$ be a complex connection without torsion. It corresponds, in natural manner, to a reduction of the structure group to $G1_{\mathbf{R}}(n/2, \mathbf{C})$. In other words, it induces an isomorphism of $P^1_F M$ into $P^2_F M$. Thus, a complex connection Γ without torsion belongs to a HP-structure P iff the corresponding subbundle of $P^2_F M$ with structure group $G1_{\mathbf{R}}(n/2, \mathbf{C})$ is contained in P.

Proposition 5.2. *Let Γ be a complex connection without torsion of M and $\gamma : P^1_F M \to P^2_F M$. Let (θ^i, θ^i_j) be the cannonical form of $P^2_F M$. Then $\gamma^* \theta^i$ is the canonical form of $P^1_F M$ and $\gamma^* \theta^i_j$ is the connection form of the complex connection Γ.*

Proof. This is immediate from the expression of (θ^i, θ^i_j) in terms of local coordinate system and Proposition (7.3) in [8]. q.e.d.

Let P be a HP-structure on M. We shall the explain HP-curvature of P. Let $(\omega^i, \omega^i_j, \omega_j)$ be the normal projective connection associated with P. Let M^* be the kernel of the natural homomorphism $H^2_n \to G1_{\mathbf{R}}(n/2, \mathbf{C})$, so that $P^1_F M = P / M^*$. Let m^* be the Lie algebra of M^*. Then we have

Proposition 5.3.

(5.11) $$i_{A^*} \Omega^i_j = 0, \quad L_{A^*} \Omega^i_j = 0,$$

for every $A \in m^*$, where A^* denotes the fundamental vector field corresponding to $A \in m^*$.

From the above Proposition 5.3 we see that the 2-form Ω^i_j can be projected down to the bundle $P^1_F M = P / M^*$. It follows that Ω^i_j define a tensor field of type (1,3) just as the curvature form defines the curvature tensor field. This field is called the HP-curvature tensor it depends only on the HP-structure P.

§ 6. Geodesics and completeness. Let P be a HP-structure on a manifold M and $(\omega^i, \omega^i_j, \omega_j)$ the nromal HP-connection associated with. With each element $\xi^i = (\xi^1, \ldots, \xi^n) \in \mathbf{R}^n$ we can associate a unique vector field B on P with the following properties :

(6.1) $$\omega^i(B) = \xi^i, \quad \omega^i_j(B) = 0, \quad \omega_j(B) = 0.$$

We call B the standard orizontal vector field corresponding to ξ.

Definition 6.1. *A curve x_t in M is called a geodesic of the given HP-structure if $x_t = \pi((\exp tB) u_0)$ for some standard horizontal vector field B and for some point $u_0 \in P$, where $\pi : P \to M$ is the projection. We call t a HP-parameter of the geodesic x_t.*

Definition 6.2. *The HP-structure P is called complete if every standard horizontal vector field B is complete, that is, generates a 1-parameter group of global transformations.*

Definition 6.3. *A curve in M is called a holomorphically projective flat curve of given complex structure if*

(6.2) $$\frac{d^2 x^k}{dt^2} + \Gamma^k_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} = \alpha(t) \frac{dx^k}{dt} + \beta(t) F^k_r \frac{dx^r}{dt}$$

for some parameter t, where α, β are some functions of the parameter t.

Proposition 6.4. *Let P be a HP-structure on M and Γ a complex connection without torsion belonging to P. If we disregard parametrizations, then the geodesic of P are the same as holomorphically projective flat curves of Γ.*

Proof. a) If two complex connections without torsion belong to the same HP-structure, then they have the same HP-flat curves. Let Γ^i_{jk} and $\bar{\Gamma}^i_{jk}$ be the coefficients of these connections, then

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j \varphi_k + \delta^i_k \varphi_j - \varphi_t F^t_j F^i_k - \varphi_t F^t_k F^i_j.$$

If $x^i(t)$, $i = 1, \ldots, n$, be the equation of HP-flat curve of the connection Γ, then $x^i(t)$ is also HP-flat curve of $\bar{\Gamma}$, as one can easily check by straightforward calculations.

b) We shall show that every geodesic of P is HP-flat curve of the connection Γ. Let B a standard horizontal vector field on P and $u_0 \in P$. Set $u_t = (\exp tB) u_0$. Then $x_t = \pi(u_t)$ is a geodesic of the HP-structure P. Let $\bar{\Gamma}$ be a complex connection belonging to P such that the corresponding isomorphism $\bar{\gamma} : P^1_F M \to P^2_F M$ has the following property :

(6.3) $$u_t \in \bar{\gamma}(P^1_F M) \text{ for } 0 \leq t \leq \delta \text{ for some } \delta \in \mathbf{R}.$$

The existence of such a $\bar{\Gamma}$ is obvious if we consider $\bar{\Gamma}$ as a cross-section

(6.4) $$M \to P / G1_{\mathbf{R}}(n/2, \mathbf{C}) \subset P^2_F M / G1_{\mathbf{R}}(n/2, \mathbf{C})$$

it is just a question of extending a cross-section defined on x_t, $t\in[0,\delta]$ to a global cross-section. By a/ it suffices to prove that x_t, $t\in[0,\delta]$ is HP-flat curve of $\bar{\Gamma}$. Let v_t be the curve in $P^1_F M$ defined by $v_t=\bar{\gamma}^{-1}(u_t)$, $t\in[0,\delta]$. Denote by $\dot{u}_t$ and $\dot{v}_t$ the tangent vectors to these curves u_t and v_t. Then

$$(6.5)\qquad (\bar{\gamma}^*\theta^i)(\dot{v}_t)=\omega^i(\dot{u}_t),\quad (\bar{\gamma}^*\theta^i_j)(\dot{v}_t)=\omega^i_j(\dot{u}_t)=0.$$

Of course, $\dot{u}_t=B(u_t)$. Let U be a coordinate neighbourhood in M with local coordinate system $(x^1,\dots,x^n)$. Let (a^i_j, a_j) be the coordinate system on H^2_n, then (x^i, a^i_j, a_j) be the coordinate system in $P|U$. Let (B^i, B^i_j, B_j) be the components of the standard horizontal-vector field B associated with $\xi\in\mathbf{R}^n$ with respect to the natural basis $\partial/\partial x^i$, $\partial/\partial a^i_j$, $\partial/\partial a_j$. From (4.5) we have

$$B_i=a^i_k\xi^k,$$

$$(6.6)\qquad B^i_j=-a_j\frac{dx^i}{dt}-a^i_ja_k\xi^k+a_pF^p_k\xi^ka^i_iF^i_j+a_pF^p_ja^i_iF^i_k\xi^k-\frac{dx^r}{dt}\bar{\Gamma}^i_{rs}a^s_j,$$

$$B_j=-a_jb_t\xi^t-a^k_j\Pi_{ik}\frac{dx^i}{dt}+\frac{1}{2}a_pF^p_k\xi^kF^t_ja_t.$$

Set $u_t=(\exp tB)u_0=(x^i(t), a^i_j(t), a_j(t))$, then we get

$$(6.7)\qquad \frac{dx^i}{dt}=B^i,\quad \frac{da^i_j}{dt}=B^i_j,\quad \frac{da_j}{dt}=B_j.$$

Then from (6.5) and (6.7) we have

$$\frac{d^2x^i}{dt^2}+\Gamma^i_{kl}\frac{dx^k}{dt}\frac{dx^l}{dt}=\alpha(t)\frac{dx^i}{dt}+\beta(t)F^i_p\frac{dx^p}{dt}.$$

By a/ we proved our assertion b/.

To prove the converse it is sufficient to use the uniqueness of the solution of the differential equation. q.e.d.

§ 7. Complex connections. The Proposition 5.2 implies that there exists a cross-section $\sigma: U\to P$ such that

$$(7.1)\qquad \psi^i=\sigma^*\bar{\theta}^i=dx^i,\quad \psi^i_j=\sigma^*\bar{\theta}^i_j=\Gamma^i_{kj}dx^k,$$

where Γ^i_{kj} are components of a complex connection without torsion.

Proposition. 7.1. *Let $(\omega^i, \omega^i_j, \omega_j)$ be the normal* HP*-connection on P and σ the above mentioned cross-section. If we set for such σ the*

$$(7.2)\qquad \psi_j=\sigma^*\omega_j=\Pi_{kj}dx^k,$$

then

$$(7.3)\qquad \Pi_{ab}=\frac{1}{n-2}\left(R_{ba}+\frac{1}{n+2}(\delta^j_b\delta^t_a-F^j_bF^t_a)(R_{jt}+R_{tj})\right),$$

where R_{ij} denote the components of the Ricci tensor.

Proof. From 4.5 we have

$$\omega^i=\bar{a}^{-1}{}^i_kdx^k,$$

$$\omega^i_j=\bar{a}^{-1}{}^i_l\Gamma^l_{ks}a^s_jdx^k+\bar{a}^{-1}{}^i_tda^t_j+a_j\omega^i+\delta^i_ja_t\omega^t-a_tF^t_s\omega^sF^i_j-a_tF^t_jF^i_s\omega^s$$

$$(7.4)\qquad \bar{\omega}_j=da_j-a_k\omega^k_j+a_ja_l\omega^l-\frac{1}{2}a_lF^l_k\omega^kF^s_ja_s+$$
$$+a^k_j\left\{-\frac{1}{n+2}\left(R_{ij}+\frac{1}{n+2}(\delta^l_j\delta^s_i-F^l_jF^s_i)(R_{ls}+R_{sl})\right)\right\}dx^i.$$

Then

$$(7.5)\qquad \psi^i=\sigma^*\omega^i=dx^i,\quad \psi^i_j=\sigma^*\omega^i_j=\Gamma^i_{kj}dx^k,\quad \psi_j=\sigma^*\omega_j=\Pi_{kj}dx^k.$$

From the structure equation (2.1.2) we have

$$(7.6)\qquad \sigma^*\Omega^i_j=\frac{1}{2}\{R^i_{jkt}+\Pi_{tj}\delta^i_k-\Pi_{kj}\delta^i_t-(\Pi_{kt}-\Pi_{tk})\delta^i_j+$$
$$+(\Pi_{ks}F^s_t-\Pi_{ts}F^s_k)F^i_j-F^i_k\Pi_{ts}F^s_j+F^i_t\Pi_{ks}F^s_j\}\,dx^k\wedge dx^t.$$

If we set $\Omega^i_j=\frac{1}{2}K^i_{jkl}\omega^k\wedge\omega^l$, and

$$(7.7)\qquad P^i_{jkt}=R^i_{jkt}+\Pi_{tj}\delta^i_k-\Pi_{kj}\delta^i_t-(\Pi_{kt}-\Pi_{tk})\delta^i_j+$$
$$+(\Pi_{ks}F^s_t-\Pi_{ts}F^s_k)F^i_j-F^i_k\Pi_{ts}F^s_j+F^i_t\Pi_{ks}F^s_j$$

then

$$(7.8)\qquad \sigma^*K^i_{jkl}=P^i_{jkl}.$$

It is easy to see that $P^i_{ikl}=0$ and $P^i_{jil}=0$. Hence $K^i_{ikl}=0$ and $K^i_{jil}=0$. This proves that $(\omega^i, \omega^i_j, \bar{\omega}_j)$ is the normal HP-connection. q.e.d.

The P^i_{jkl} are the components of HP-curvature tensor (c.f. [16], [17]).

§ 8. HP-transformations and flat HP-structures. Let P and P' be HP-structures on manifolds M and M' of the same dimension n.

Definition 8.1. *A diffeomorphism $f: M\to M'$ is called holomorphically projective (with respect to P and P') iff, prolonged to a mapping $P^2M\to P^2M'$, maps P onto P'. In particular, a transformation f of M is called holomorphically projective if it maps P onto itself.*

Definition 8.2. *A* HP*-structure P on a manifold M is called flat, if for each point of M there exists a neighbourhood U and a* HP*-diffeomorphism onto an open subset of $G(n+2,2)$ (cf. [17]).*

Every vector field X on M generates a 1-parameter local group of local transformations. This local group prolonged to P^2M induces a vector field on P^2M, which will be denoted by $\tilde{X}$.

Definition 8.3. *We call X an infinitesimal* HP*-transformation if $\tilde{X}$ is tangent to P at each point of P.*

The following theorems have standard proofs.

Proposition 8.4. *Let $\omega=(\omega^i, \omega^i_j, \omega_j)$ be the normal* HP*-connection associated with P. For a vector field X on M the following conditions are equivalent:*

a) *X is an infinitesimal* HP*-transformation of M,*

b) *$\tilde{X}$ is tangent to P at every point of P.*

c) *$L_{\tilde{X}}\omega=0$,*

d) *$L_{\tilde{X}}B_\xi=0$ for every $\xi\in R^n$, where B_ξ is the standard horizontal vector field corresponding to ξ.*

Theorem 8.4. *Let P be a* HP*-structure on a manifold M of dimension n. Then*

a) *The set of all infinitesimal* HP-*transformations of* M, *denoted by* $\overline{hp}(M)$ *is a Lie algebra of dimension* $\leqslant n^2/2+2n=\dim P$.

b) *The subset of* $\overline{hp}(M)$ *consisting of complete vector fields, denoted by* $hp(M)$ *is subalgebra of* $\overline{hp}(M)$.

c) *The group of* HP-*transformations of* M, *denoted by* $HP(M)$ *is a Lie transformation group with Lie algebra* $hp(M)$.

d) *If the* HP-*structure is complete, every infinitesimal* HP-*transformation is complete, i.e.* $hp(M)=\overline{hp}(M)$.

Theorem 8.5. *If the Lie algebra* $hp(M)$ *of infinitesimal* HP-*transformations of* M *is of dimension* $n^2/2+2n$, *then the normal* HP-*connection of* P *has vanishing curvature.*

Theorem 8.6. *A* HP-*structure* P *on a manifold* M *is flat iff the normal* HP-*connection has vanishing curvature.*

Corollary 8.7. *A* HP-*structure* P *on a manifold of dimension* $\geqslant 4$ *is flat iff* HP-*curvature tensor vanishes.*

Theorem 8.8. *Let* P *be a complete flat* HP-*structure on a simply connected manifold* M, $\dim M=n$. *Then there is* HP-*diffeomorphism of* M *onto* $G(n+2,2)$.

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THE d_{J_r}-COHOMOLOGY ON THE TANGENT BUNDLE OF ORDER k

BY

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1. Introduction. Let V be a paracompact manifold and J an 1-vector form on V with vanishing Nijenhuis tensor. W i l l m o r e in [7] showned that if J has maximal rank, then the d_J-cohomology, defined by the derivation d_J associated to J, is isomorphic to the de Rham cohomology of V. If J has no maximal rank, then the Poincaré lemma is not true, that is, a d_J-closed form is not d_J-exact unless if certain conditions are verified. Here, we consider the canonical 1-vector forms $J_1, \ldots, J_k$ on T^kM, the tangent bundle of order k of a manifold M, and study the cohomology defined by d_{J_r}, $1\leqslant r\leqslant k$. We obtain the following results:

(1) The d_{J_r}-cohomology is isomorphic to the de Rham cohomology of any fiber of the submersion $\pi^k_{k-r}: T^kM\to T^{k-r}M$.

(2) A semibasic p-form of type $k-r+1$ (that is, a p-form in $\operatorname{Im} J_r^*$) d_{J_r}-closed is d_{J_r}-exact.

So, we generalize the results obtained by A y a s s o u for the case $k=1$ [2], and those in [3] for the case $k=2$. We remit to [4] for notations and results of § 2. The formalism of Frölicher-Nijenhuis is also widely used.

2. Differential calculus on T^kM. Let M be a paracompact manifold of dimension n. The tangent bundle of order k of M is the $(k+1)n$-dimensional manifold of k-jets at $0\in R$ of differentiable mappings $f: R\to M$. If $\pi^k_{r-1}: T^kM\to T^{r-1}M$, $1\leqslant r\leqslant k$, denotes the canonical mapping defined by $j^k_0f\to j^{r-1}_0f$, where T^0M is identified with M and $\pi^k_0=\pi^k$, then T^kM has a bundle structure over $T^{r-1}M$ with projection map π^k_{r-1}.

Let (U, x^i) be a coordinate neighbourhood of M and denote by $(y^{(0)i}, y^{(1)i}, \ldots, y^{(k)i})$ the induced system of coordinates in $(\pi^k)^{-1}(U)$.

Let J_r, $1\leqslant r\leqslant k$, be the canonical 1-vector forms on T^kM; they are locally expressed with respect to the induced coordinates in T^kM by

$$J_r=\begin{pmatrix} 0 & 0 \\ I_{rn} & 0 \end{pmatrix}.$$

where I_{rn} denotes the unit matrix of order rn. Then, rank $J_r=rn$.

Proposition 2.1.

$$(J_r)^s=\begin{cases} 0, & if\ s(k-r+1)\geqslant k+1 \\ J_{k+1-s(k-r+1)}, & if\ s(k-r+1)<k+1 \end{cases} \tag{1}$$