a) *The set of all infinitesimal* HP-*transformations of* M, *denoted by* $\overline{hp}(M)$ *is a Lie algebra of dimension* $\leqslant n^2/2+2n=\dim P$.

b) *The subset of* $\overline{hp}(M)$ *consisting of complete vector fields, denoted by* $hp(M)$ *is subalgebra of* $\overline{hp}(M)$.

c) *The group of* HP-*transformations of* M, *denoted by* $HP(M)$ *is a Lie transformation group with Lie algebra* $hp(M)$.

d) *If the* HP-*structure is complete, every infinitesimal* HP-*transformation is complete, i.e.* $hp(M)=\overline{hp}(M)$.

Theorem 8.5. *If the Lie algebra* $hp(M)$ *of infinitesimal* HP-*transformations of* M *is of dimension* $n^2/2+2n$, *then the normal* HP-*connection of* P *has vanishing curvature.*

Theorem 8.6. *A* HP-*structure* P *on a manifold* M *is flat iff the normal* HP-*connection has vanishing curvature.*

Corollary 8.7. *A* HP-*structure* P *on a manifold of dimension* $\geqslant 4$ *is flat iff* HP-*curvature tensor vanishes.*

Theorem 8.8. *Let* P *be a complete flat* HP-*structure on a simply connected manifold* M, $\dim M=n$. *Then there is* HP-*diffeomorphism of* M *onto* $G(n+2,2)$.

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THE d_{J_r}-COHOMOLOGY ON THE TANGENT BUNDLE OF ORDER k

BY

MANUEL DE LEON

1. Introduction. Let V be a paracompact manifold and J an 1-vector form on V with vanishing Nijenhuis tensor. Willmore in [7] showned that if J has maximal rank, then the d_J-cohomology, defined by the derivation d_J associated to J, is isomorphic to the de Rham cohomology of V. If J has no maximal rank, then the Poincaré lemma is not true, that is, a d_J-closed form is not d_J-exact unless if certain conditions are verified. Here, we consider the canonical 1-vector forms $J_1, \ldots, J_k$ on T^kM, the tangent bundle of order k of a manifold M, and study the cohomology defined by d_{J_r}, $1\leqslant r\leqslant k$. We obtain the following results:

(1) The d_{J_r}-cohomology is isomorphic to the de Rham cohomology of any fiber of the submersion $\pi^k_{k-r}: T^kM\to T^{k-r}M$.

(2) A semibasic p-form of type $k-r+1$ (that is, a p-form in $\operatorname{Im}J_r^*$) d_{J_r}-closed is d_{J_r}-exact.

So, we generalize the results obtained by Ayassou for the case $k=1$ [2], and those in [3] for the case $k=2$. We remit to [4] for notations and results of § 2. The formalism of Frölicher-Nijenhuis is also widely used.

2. Differential calculus on T^kM. Let M be a paracompact manifold of dimension n. The tangent bundle of order k of M is the $(k+1)n$-dimensional manifold of k-jets at $0\in R$ of differentiable mappings $f: R\to M$. If $\pi^k_{r-1}: T^kM\to T^{r-1}M$, $1\leqslant r\leqslant k$, denotes the canonical mapping defined by $j_0^kf\to j_0^{r-1}f$, where T^0M is identified with M and $\pi^k_0=\pi^k$, then T^kM has a bundle structure over $T^{r-1}M$ with projection map π^k_{r-1}.

Let (U, x^i) be a coordinate neighbourhood of M and denote by $(y^{(0)i}, y^{(1)i}, \ldots, y^{(k)i})$ the induced system of coordinates in $(\pi^k)^{-1}(U)$.

Let J_r, $1\leqslant r\leqslant k$, be the canonical 1-vector forms on T^kM; they are locally expressed with respect to the induced coordinates in T^kM by

$$J_r=\begin{pmatrix} 0 & 0 \\ I_{rn} & 0 \end{pmatrix}.$$

where I_{rn} denotes the unit matrix of order rn. Then, rank $J_r=rn$.

Proposition 2.1.

$$(J_r)^s=\begin{cases} 0, & \text{if } s(k-r+1)\geqslant k+1 \\ J_{k+1-s(k-r+1)}, & \text{if } s(k-r+1)<k+1 \end{cases} \tag{1}$$

(2) $[J_r, J_s]=0$, *for every* $1\leq r, s\leq s$; *therefore*, J_k *defines an integrable almost tangent structure of order* k *on* T^kM.

For each r, $1\leq r\leq k$, the 1-vector form J_r defines and endomorphism J_r^* of the exterior algebra $\Lambda(T^kM)$, which will be called the r-th vertical operator.

Proposition 2.2. *We have*

$$(J_r^*)^s=\begin{cases}0, & if\ s(k-r+1)\geq k+1\\ J^*_{k+1-s(k-r+1)}, & if\ s(k-r+1)<k+1.\end{cases}$$

Moreover, J_r^* *is locally expressed with respect to the induced coordinates by*

$$J_r^*(dy^{(s)})=\begin{cases}0, & if\ r+s<k+1\\ dy^{(r+s-k-1)}, & if\ r+s\geq k+1.\end{cases}$$

Now, let us consider the derivation i_{J_r} of type i_* associated to J_r; i_{J_r} is the derivation of degree 0 of $\Lambda(T^kM)$ defined by

$$(i_{J_r}\omega)(X_1, ..., X_p)=\sum_{i=1}^{p}\omega(X_1, ..., J_rX_i, ..., X_p)$$

for any differential p-form ω on T^kM. We shall call i_{J_r} the r-th vertical derivation. In fact, i_{J_r} is the unique derivation of degree 0 of $\Lambda(T^kM)$ which verifies $i_{J_r}f=0$, $i_{J_r}(df)=J_r^*(df)$, for every function f on T^kM.

With respect to the induced coordinates, i_{J_r} is locally expressed by

$$i_{J_r}(dy^{(s)})=\begin{cases}0, & \text{if } r+s<k+1\\ dy^{(r+s-k-1)}; & \text{if } r+s\geq k+1.\end{cases}$$

On the other hand, let d_{J_r} be the derivation of $\Lambda(T^kM)$ of type $d*$ associated to J_r, d_{J_r} is the derivation of degree 1 defined by $d_{J_r}=[i_{J_r}, d]$, and it will be called the r th vertical differentiation. In fact, d_{J_r} is uniquely characterized by the following relations

$$\begin{cases}d_{J_r}f=J_r^*(df), & \text{for every function } f \text{ on } T^kM\\ d_{J_r}d=-dd_{J_r}.\end{cases}$$

Consequently, d_{J_r} is locally expressed by

$$d_{J_r}f=\sum_{\lambda=k-r+1}^{k}\frac{\partial f}{\partial y^{(\lambda)}}dy^{(r+\lambda-k-1)},$$

with respect to the induced coordinates.

As a direct consequence of the Frölicher-Nijenhuis differential concomitances, we have $d^2_{J_r}=d_{N_{J_r}}$, where N_{J_r} is the Nijenhuis tensor of J_r. Then, from (2) in Proposition 2.1, we obtain

Theorem 2.3. *The r-th vertical differentiation verifies* $(d_{J_r})^2=0$.

Moreover, a direct computation leads to the following.

Proposition 2.4.

(1) $$J_r^*d_{J_s}=0, \quad if\ r+s<k+2$$

(2) $$d_{J_r}J_r^*=J_r^*d.$$

Definition 2.5. *A differential form* ω *on* T^kM *is said semibasic of type* r *if* $\omega\in \mathrm{Im} J_{k-r+1}$.

The collection SB_r of semibasic forms of type r on T^kM is a graduated subalgebra of $\Lambda(T^kM)$, that is,

$$SB_r=\bigoplus_p S^pB_r,\quad S^pB_r\wedge S^qB_r\subset S^{p+q}B_r,$$

where S^pB_r denotes the set of semibasic forms of degree p.

In fact, we have a decreasing sequence of subalgebras $SB_1\supset SB_2\supset ...\supset \supset SB_{k-1}\supset SB_k$.

The following results can be easily proved

Proposition 2.6. *Let* ω *be a* p*-form on* T^kM. *Then* ω *is semibasic of type* r *if and only if* $i_V\omega=0$, *for every tangent vector* $V\in V^{\pi^k_{k-r}}(T^kM)$.

Corollary 2.7. *The derivation* i_{J_r} *vanishes on* SB_r.

Proposition 2.8.

(1) $$d_{J_{k-r+1}}(SB_r)\subset SB_r$$

(2) $$d_{J_{k-r+1}}f\in SB_r, \text{ for every differentiable function } f \text{ on } T^kM.$$

Finally, let us remark that the algebra SB_r is locally generated by the differentiable functions on T^kM and $dy^{(0)}$, $dy^{(1)}$, ..., $dy^{(k-r)}$.

3. The d_{J_r}-cohomology. Proposition 3.1. *For every* $z\in T^kM$, *there exists a neighbourhood* U *of* z *and an endomorphism* K *of* $\Lambda(U)$ *such that*:

(1) $$K(\Lambda^p(U))\subset\Lambda^{p-1}(U), \text{ for } p\geq 1.$$

(2) *For every* p*-form* ω *on* U, *there is a* p*-form* ω_0 *on* U *such that*

$$\omega=Kd\omega+dK\omega+\omega_0$$

and, moreover, $i_{J_r}\omega_0=0$ *and* $J_r^*\omega_0=0$.

(3) $$If\ J_r^*\omega=0, \text{ then } J_r^*K\omega=0.$$

Proof. We use the following classic lemma [6]:

Lemma. *Let* j_l, $l=0, 1$, *be the canonical injections of* R^{n-m} *in* $R^{n-m}\times R$ *defined by* $j_l(z)=(z, l)$. *Then, there exists a linear map*

$$k:\Lambda(R^{n+m}\times R)\to\Lambda(R^{n+m})$$

with the following properties:

(1) $$k(\Lambda R^{n-m}\times R))\subset\Lambda^{p-1}(R^{n+m}), \text{ for } p\geq 1$$

(2) $$dk+kd=j_1^*-j_0^*.$$

Now, we shall prove the Proposition. Let $\tilde{U}$ be a neighbourhood of $\pi^k(z)$ in M : then $U=(\pi^k)^{-1}(\tilde{U})$ is a neighbourhood of z in T^kM. We define

$$H : U\times R\to R$$

by $$H(y^{(0)}, y^{(1)}, \dots, y^{(k)} ; t)=(y^{(0)}, \dots, y^{(k-r)}, ty^{(k-r+1)}, \dots, ty^{(k)}).$$

By using the lemma, we obtain $(dk+kd)H^*\omega=j_1^*H^*\omega-j_0^*H^*\omega$. But $j_1^*H^*\omega=(H\circ j_1)^*\omega=\omega$, because $H\circ j_1=\mathrm{id}$. So, $\omega=dkH^*\omega+kdH^*\omega+(H\circ j_0)^*\omega$.

We define $K\omega=kH^*\omega$ and $\omega_0=(H\circ j_0)^*\omega$; thus, $K\omega\in\Lambda^{p-1}(U)$, $\omega_0\in\Lambda^p(U)$ and $\omega=dK\omega+Kd\omega+\omega_0$.

Moreover, since

$$(H\circ j_0)(y^{(0)}, \dots, y^{(k)})=(y^{(0)}, \dots, y^{(k-r)}, 0, \dots, 0),$$

then $(H\circ j_0)^*\omega$ only involves $dy^{(0)}, \dots, dy^{(k-r)}$, and therefore $J_r^*\omega_0=0$ and $i_{J_r}\omega_0=0$.

Finally, if $J_r^*\omega=0$, then some $dy^{(\lambda)}$, $0\leqslant\lambda\leqslant k-r$, will be always present in every term of ω, and the same is true for $K\omega$. This implies $J_r^*K\omega=0$.

Proposition 3.2. *Let ω be a p-form on U. If $J_r^*\omega$ is d_{J_r}-closed, then there exists a $(p-1)$-form τ on U such that $J_r^*\omega=d_{J_r}(J_r^*\tau)$.*

Proof. Let $\tau=K\omega$. Then $\tau\in\Lambda^{p-1}(U)$. On the other hand,

$d_{J_r}(J_r^*\tau)=d_{J_r}J_r^*K\omega=J_r^*dK\omega$ (by Proposition 2.4) $=J_r^*(\omega-Kd\omega-\omega_0)=J_r^*\omega-J_r^*Kd\omega-J_r^*\omega_0=J_r^*\omega$, because, taking into account Proposition 3.1, $J_r^*\omega_0=0$ and, at the same time, $J_r^*d\omega=0$ implies $J_r^*Kd\omega=0$.

Let $\mathcal{F}$ denote the differentiable functions on T^kM and let $\mathcal{F}_{k-r+1}=\{f\in\mathcal{F}/f$ is constant along the fibers of the submersion $\pi^k_{k-r} : T^kM\to T^{k-1}M\}$. Then we have the following theorem.

Theorem 3.3. *For every $z\in T^kM$, there exists a neighbourhood U of z such that the following sequence is exact*

$$0\to\mathcal{F}_{k-r+1}(U)\to\mathcal{F}(U)\xrightarrow{d_{J_r}} S^1B_{k-r+1}(U)\xrightarrow{d_{J_r}}$$

$$\dots\xrightarrow{d_{J_r}} S^{rn-1}B_{k-r+1}U)(\xrightarrow{d_{J_r}} S^{rn}B_{k-r+1}(U)\xrightarrow{d_{J_r}} 0.$$

Proof. Let U be as that in Proposition 3.1. Firstly, SB_{k-r+1} is stable by d_{J_r}, and, moreover, $S^{rn}B_{k-r+1}\xrightarrow{d_{J_r}}0$ since SB_{k-r+1} is locally generated by $dy^{(0)}, \dots, dy^{(r-1)}$.

Now, we shall prove the exactness of the above sequence.

On one hand, $d^2_{J_r}=0$. On the other, if for an arbitrary $f\in\mathcal{F}(U)$ we have $d_{J_r}f=0$, that is,

$$\sum_{J=k-r+1}^{k}\frac{\partial f}{\partial y^{(\lambda)}}\,dy^{(r+\lambda-k-1)}=0,$$

then, assumed U connected, we deduce that f is constant with respect to $y^{(k-r+1)}, \dots, y^{(k)}$, *i.e.*, $f\in\mathcal{F}_{k-r+1}(U)$, which shows the exactness at $\mathcal{F}(U)$.

Finally, if $\omega\in S^pB_{k-r+1}(U)$, $p\geqslant1$, then, by definition of semibasic form, $\omega=J_r^*\alpha$, with $\alpha\in\Lambda^p(U)$. Then, if $d_{J_r}\omega=0$, from Proposition 3.2, there exists $\tau\in\Lambda^{p-1}(U)$ such that $J_r^*\alpha=d_{J_r}J_r^*\tau$, so $\omega=d_{J_r}J_r^*\tau$ and $J_r^*\tau$ is semibasic of type $k-r+1$, which shows the exactness at $S^pB_{k-r+1}(U)$, $p\geqslant1$.

Theorem 3.4. *The d_{J_r}-cohomology is isomorphic to the de Rham cohomology of any fiber of the submersion $\pi^k_{k-r} : T^kM\to T^{k-r}M$.*

Proof. Let S be an arbitrary fiber of $\pi^k_{k-r} : T^kM\to T^{k-r}M$; then, S is a submanifold of T^kM.

The mapping

$$\Lambda^p(S)\xrightarrow{J_r^*} S^pB_{k-r+1}(S)$$

being an isomorphism, the result follows taking into account the identity

$$J_r^*d=d_{J_r}J_r^*.$$

Proposition 3.5. *Let ω be a semibasic p-form of type $k-r+1$, with $p\geqslant1$, and d_{J_r}-closed. Then, there exists $\varphi\in\Lambda^{p-1}(T^kM)$ such that $\omega=d_{J_r}\varphi$ and φ being semibasic of type $k-r+1$.*

Proof. Let $\{\tilde{U}_i\}$ be an open covering of M and let $\{\tilde{f}_i\}$ be a partition of unity subordinate to $\{\tilde{U}_i\}$. Then $\{U_i=(\pi^k)^{-1}(\tilde{U}_i)\}$ is an open covering of T^kM and $\{f_i=\tilde{f}_i\circ\pi^k\}$ is a partition of unity subordinate to $\{U_i\}$. Moreover, for every U_i we have the exact sequence

$$0\to\mathcal{F}_{k-r+1}(U_i)\to\mathcal{F}(U_i)\xrightarrow{d_{J_r}} S^1B_{k-r+1}(U_i)\xrightarrow{d_{J_r}}$$

$$\dots\xrightarrow{d_{J_r}} S^{rn-1}B_{k-r+1}(U_i)\xrightarrow{d_{J_r}} S^{rn}B_{k-r+1}(U_i)\xrightarrow{d_{J_r}} 0$$

of Theorem 3.3.

Now, let be $\omega_i=\omega|U_i$; then ω_i is semibasic of type $k-r+1$ and d_{J_r} closed. Consequently, there exists $\varphi_i\in S^{p-1}B_{k-r+1}(U_i)$ such that $\omega_i=d_{J_r}\varphi_i$. If we put $\varphi=\sum_i f_i\varphi_i$, then φ is semibasic of type $k-r+1$ and, moreover, $d_{J_r}\varphi=\omega$. In fact,

$$d_{J_r}\varphi=d_{J_r}(\sum_i f_i\varphi_i)=\sum_i(d_{J_r}f_i)\Lambda\varphi_i+\sum_i f_i(d_{J_r}\varphi_i)=\sum f_i\omega_i=\omega$$

since $d_{J_r}f_i=0$.

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INTEGRABILITY OF ALMOST QUATERNAL STRUCTURES

BY

VASILE OPROIU

1. Introduction. In this paper we continue the study of the integrability problem of the almost quaternal structures initiated in [6]. Recall that an almost quaternal structure is defined on a $4m$-dimensional manifold M by a vector subbundle V, with fibre dimension 3, of the bundle of $(1,1)$-type tensors on M, having canonical bases. That means that for every point $p \in M$ there exists a neighbourhood U of p and 3 tensor fields F_1, F_2, F_3, sections of the restriction of V to U, satisfying the relations:

$$F_a^2 = -I, \qquad F_aF_b = -F_bF_a = F_c; \qquad a = 1, 2, 3, \tag{1}$$

where (a, b, c) is a cyclic permutation of $(1, 2, 3)$. The tensor fields (F_a), (F'_a) defining canonical bases of on U, U', respectively, are related on $U \cap U'$ by:

$$F'_a = \sum_{b=1}^{3} s_{ab}F_b, \tag{2}$$

where $[s_{ab}] \in SO(3)$.

The structural group is $G = GL(m, \mathbf{H})$. Sp (1) and the Lie algebra $\mathfrak{g}$ of G is of finite type.

Denote by N_a the Nijenhuis tensor field of F_a and let: $S = N_1 + N_2 + N_3$. If α_a is the 1-form defined by:

$$\alpha_a(X) = [1/(4m-2)]\, \mathrm{trace}(Y \to F_aS(X, Y)),$$

then the Chern-Bernard invariant (the torsion invariant) of the given almost quaternal structure is trivial if and only if S is expressed as:

$$S(X, Y) = \sum_{a=1}^{3} \{\alpha_a(Y)F_aX - \alpha_a(X)F_aY\}. \tag{3}$$

This result was obtained in [6] by looking for a torsion free adapted connection to the almost quaternal structure. Remark that the 1-forms α_a; $a = 1, 2, 3$, satisfy the relation:

$$\sum_{a=1}^{3} \alpha_a(F_aX) = 0. \tag{4}$$

In this paper we show that if the torsion invariant of the almost quaternal structure is trivial then the family of the torsion-free adapted connections is obtained by *Q-projective transformations only*. Then, using the curva-