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INTEGRABILITY OF ALMOST QUATERNAL STRUCTURES

BY

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1. Introduction. In this paper we continue the study of the integrability problem of the almost quaternal structures initiated in [6]. Recall that an almost quaternal structure is defined on a $4m$-dimensional manifold M by a vector subbundle V, with fibre dimension 3, of the bundle of (1,1)-type tensors on M, having canonical bases. That means that for every point $p \in M$ there exists a neighbourhood U of p and 3 tensor fields F_1, F_2, F_3, sections of the restriction of V to U, satisfying the relations:

$$F_a^2 = -I, \qquad F_a F_b = -F_b F_a = F_c; \qquad a = 1, 2, 3, \tag{1}$$

where (a, b, c) is a cyclic permutation of $(1, 2, 3)$. The tensor fields (F_a), (F'_a) defining canonical bases of on U, U', respectively, are related on $U \cap U'$ by:

$$F'_a = \sum_{b=1}^{3} s_{ab} F_b, \tag{2}$$

where $[s_{ab}] \in SO(3)$.

The structural group is $G = GL(m, \mathbf{H})$. Sp (1) and the Lie algebra $\mathfrak{g}$ of G is of finite type.

Denote by N_a the Nijenhuis tensor field of F_a and let: $S = N_1 + N_2 + N_3$. If α_a is the 1-form defined by:

$$\alpha_a(X) = [1/(4m-2)] \operatorname{trace}(Y \to F_a S(X, Y)),$$

then the Chern-Bernard invariant (the torsion invariant) of the given almost quaternal structure is trivial if and only if S is expressed as:

$$S(X, Y) = \sum_{a=1}^{3} \{\alpha_a(Y) F_a X - \alpha_a(X) F_a Y\}. \tag{3}$$

This result was obtained in [6] by looking for a torsion free adapted connection to the almost quaternal structure. Remark that the 1-forms α_a; $a = 1, 2, 3$, satisfy the relation:

$$\sum_{a=1}^{3} \alpha_a(F_a X) = 0. \tag{4}$$

In this paper we show that if the torsion invariant of the almost quaternal structure is trivial then the family of the torsion-free adapted connections is obtained by *Q-projective transformations only*. Then, using the curvature invariant obtained as a curvature invariant of the Q-projective transformations, it is shown that the structure is integrable if and only if this curvature invariant is trivial. In [4] there were considered the manifolds carrying integrable almost quaternal structures, i.e. the quaternion local charts are related by coordinate transformations, differentiable in the generalized quaternionic sense. It was shown that such coordinate transformations are necessarily projective. Thus, the result of [4] is framed now in the general integrability problem for G-structures.

2. The torsion free adapted connections. The following tensor fields N_{ab} ; $a \neq b = 1, 2, 3$ are stronly related to the integrability problem for almost quaternal structures :

(5) $$N_{ab}(X, Y) = [F_aX, F_bY] + [F_bX, F_aY] - F_a[F_bX, Y] - F_b[F_aX, Y] - -F_a[X, F_bY] - F_b[X, F_aY] ; \quad a \neq b = 1, 2, 3.$$

The tensor fields S, N_a, N_{ab} are related by the following formulae which can be proved by straightforward computations :

(6) $$6N_{ab}(X, Y) = F_aS(X, F_bY) + F_bS(X, F_aY) + F_aS(F_bX, Y) + F_bS(F_aX, Y) - -S(F_aX, F_bY) - S(F_bX, F_aY),$$

(7) $$2S(X, Y) = F_aN_{ab}(X, F_bY) + F_bN_{ab}(X, F_aY) + F_aN_{ab}(F_bX, Y) + +F_bN_{ab}(F_aX, Y) - N_{ab}(F_aX, F_bY) - N_{ab}(F_bX, F_aY),$$

(8) $$6N_a(X, Y) = S(X, Y) + F_aS(F_aX, Y) + F_a(X, F_aY) - S(F_aX, F_aY),$$

(9) $$6N_a(X, Y) = F_aN_{ab}(X, F_bY) + F_bN_{ab}(X, F_aY) + F_aN_{ab}(F_bX, Y) + +F_bN_{ab}(F_aX, Y) - N_{ab}(F_aX, F_bY) - N_{ab}(F_bX, F_aY) - F_cN_{ab}(X, Y) + +N_{ab}(X, F_cY) + N_{ab}(F_cX, Y) + F_aN_{ab}(F_cX, F_aY) + F_aN_{ab}(F_aX, F_cY) + +F_cN_{ab}(F_aX, F_aY).$$

Throughout this paper (a, b, c) will be thought of as a cyclic permutation of $(1, 2, 3)$ and by the symbol $\sum_{(a,b,c)}$ we mean the summation after the three cyclic permutations $(1, 2, 3)$, $(2, 3, 1)$, $(3, 1, 2)$ of $(1, 2, 3)$. From the formulae (6), (7) we deduce that the vanishing of S is equivalent to the vanishing of one of N_{ab} ; $a \neq b = 1, 2, 3$ and from (8), (9) that the vanishing of S hence of N_{ab} implies the vanishing of *all* N_a, $a = 1, 2, 3$. Thus the vanishing of S assures the integrability of all three local almost complex structures defined by F_a. However, in the local chart in which the components of F_a are constant the components of F_b may be not constant. Due to its symmetry, S is more convenient to study the integrability problems.

Proposition 1. *We have the following connection defined on U :*

(10) $$\dot{\nabla}_X Y = (1/12) \sum_{(a,b,c)} \{F_a[F_bX, F_cY] - F_a[F_cX, F_bY]\} + (1/6) \sum_{a=1}^{3} \{F_a[F_aX, Y] - -F_a[X, F_aY]\} + (1/2)[X, Y]$$

whose torsion tensor field is trivial and for which :

(11) $$(\dot{\nabla}_X F_a)Y = (1/12)\{F_aS(X, Y) - S(X, F_aY)\}.$$

Proof. One verifies easily that $\dot{\nabla}$ has the properties of a linear connection. Then, using the definition $\dot{T}(X, Y) = \dot{\nabla}_X Y - \dot{\nabla}_Y X - [X, Y]$ of the torsion tensor field, one obtains $\dot{T} = 0$. Finally, the formula (11) is proved using the definition :

$$(\dot{\nabla}_X F_a)Y = \dot{\nabla}_X(F_aY) - F_a\dot{\nabla}_X Y.$$

Proposition 2. *The relation between the connections $\dot{\nabla}$, $\dot{\nabla}'$, defined on U, U' by the canonical bases (F_a), (F'_a), respectively, is :*

(12) $$\dot{\nabla}'_X Y = \dot{\nabla}_X Y - (1/6) \sum_{a=1}^{3} \{\lambda_a(F_aX)Y + \lambda_a(F_aY)X\} - (1/12) \sum_{(a,b,c)} \{[4\lambda_a(X) + +\lambda_b(F_cX) - \lambda_c(F_bX)]F_aY + [4\lambda_a(Y) + \lambda_b(F_cY) - \lambda_c(F_bY)]F_aX\},$$

where the 1-forms λ_a are defined on $U \cap U'$ by :

(13) $$\lambda_a = s_{1b}ds_{1c} + s_{2b}ds_{2c} + s_{3b}ds_{3c} = -(s_{1c}ds_{1b} + s_{2c}ds_{2b} + s_{3c}ds_{3c}).$$

The proof is obtained by using the formulae (2) and the properties of the bracket.

Proposition 3. *If S is given by* (3) *then :*

(14) $$12(\dot{\nabla}_X F_a)Y = -2\alpha_b(X)F_cY + 2\alpha_c(X)F_bY - \alpha_a(Y)X - \alpha_a(F_aY)F_aX + +\{\alpha_b(Y) - \alpha_c(F_aY)\}F_cX - \{\alpha_c(Y) + \alpha_b(F_aY)\}F_bX.$$

Thus, even if the torsion of the almost quaternal structure is trivial, the connection $\dot{\nabla}$ is not adapted. However, we can now obtain an adapted connection.

Proposition 4. *The connection $\ddot{\nabla}$ defined on U by :*

(15) $$\ddot{\nabla}_X Y = \dot{\nabla}_X Y + (1/12) \sum_{a=1}^{3} \{\alpha_a(X)F_aY + \alpha_a(Y)F_aX\},$$

is torsion free and :

(16) $$(\ddot{\nabla}_X F_a)Y = -(1/3)\alpha_b(X)F_cY + (1/3)\alpha_c(X)F_bY.$$

Proof. If $\ddot{\nabla} = \dot{\nabla} + P$ with P a tensor field of type (1, 2), then :

$$(\ddot{\nabla}_X F_a)Y = (\dot{\nabla}_X F_a)Y + P(X, F_aY) - F_aP(X, Y).$$

If P is given by (15) it follows (16).

Proposition 5. *The connections $\ddot{\nabla}$, $\ddot{\nabla}'$ defined on U, U' respectively are related on $U \cap U'$ by :*

(17) $$\ddot{\nabla}'_X Y = \ddot{\nabla}_X Y - (1/6) \sum_{a=1}^{3} \{\lambda_a(F_aX)Y + \lambda_a(F_aY)X\} - (1/6) \sum_{(a,b,c)} \{(\lambda_a(X) + +\lambda_b(F_cX) - \lambda_c(F_bX))F_aY + (\lambda_a(Y) + \lambda_b(F_cY) - \lambda_c(F_bY))F_aX\}.$$

Proof. Using the definition of S we obtain the transformation formula:

(18) $$S'(X, Y)=S(X, Y)+\sum_{(a,b,c)}\{-2\lambda_a(X)+\lambda_b(F_cX)-\lambda_c(F_bX)\}F_aY- -\sum_{(a,b,c)}\{-2\lambda_a(Y)+\lambda_b(F_cY)-\lambda_c(F_bY)\}F_aX.$$

Next, using the definition of the 1-forms α_a one gets:

(19) $$\sum_{b=1}^{3} s_{ba}\alpha'_b(X)=\alpha_a(X)+2\lambda_a(X)-\lambda_b(F_cX)+\lambda_c(F_bX).$$

Since: $F_a=\sum_{b=1}^{3} s_{ba}F'_b$, we have:

(20) $$\sum_{a=1}^{3}\alpha'_a(X)F'_a=\sum_{a=1}^{3}\alpha_a(X)F_a+\sum_{(a,b,c)}\{2\lambda_a(X)-\lambda_b(F_cX)+\lambda_c(F_bX)\}F_a$$

and then get easily (17).

Finally we should obtain a globally defined adapted connection related to the locally defined connections $\ddot\nabla$, $\ddot\nabla'$, ... To this end, let us compare the 1-forms $-(1/3)\alpha_a$ corresponding to the connection $\ddot\nabla$ in (16) and the 1-forms η_a corresponding to the globally defined adapted connection ∇, i.e.:

(21) $$\nabla_XF_a=-\eta_b(X)F_c+\eta_c(X)F_b\ ;\ a=1, 2, 3.$$

The transformation formulae for η_a are:

(22) $$\sum_{b=1}^{3} s_{ba}\eta'_b=\eta_a+\lambda_a\ ;\ a=1, 2, 3,$$

and thus we get:

(23) $$\sum_{b=1}^{3} s_{ba}(\eta'_b-(1/3)\alpha'_b)(X)=(\eta_a-(1/3)\alpha_a)(X)+(1/3)\{\lambda_a(X)+\lambda_b(F_cX)-\lambda_c(F_bX)\}.$$

Next, just like in the formula (20):

(24) $$\sum_{a=1}^{3}\{\eta'_a(F'_aX)-1/3\alpha'_a(F'_aX)\}=\sum_{a=1}^{3}\{\eta_a(F_aX)-(1/3)\alpha_a(F_aX)\}+\sum_{a=1}^{3}\lambda_a(F_aX).$$

Recall that α_a satisfy the relation (4), such that, denoting:

(25) $$\sum_{a=1}^{3}\eta_a(F_aX)=-6\pi(X),$$

one gets:

(26) $$\pi'(X)=\pi(X)-(1/6)\sum_{a=1}^{3}\lambda_a(F_aX),$$

and then:

$$\sum_{b=1}^{3} s_{ba}\pi'(F'_bX)=\pi(F_aX)+(1/6)\{\lambda_a(X)+\lambda_b(F_cX)-\lambda_c(F_bX)\}.$$

Thus the 1-forms $\tilde\eta_a$ defined by:

(27) $$\tilde\eta_a(X)=(1/3)\alpha_a(X)+2\pi(F_aX)$$

have the same transformation rule as the 1-forms η_a.

Theorem 6. *The torsion-free adapted connection $\tilde\nabla$ with the corresponding 1-forms $\tilde\eta_a$ given by (27) is:*

(28) $$\tilde\nabla_XY=\ddot\nabla_XY-\pi(X)Y-\pi(Y)X+\sum_{a=1}^{3}\{\pi(F_aX)F_aY+\pi(F_aY)F_aX\}.$$

Proof. We can do a strightforward verification that $\tilde\nabla$ is an adapted connection with the corresponding 1-forms $\tilde\eta_a$. Another proof can be obtained if we recall that an adapted connection is obtained from $\ddot\nabla$ by the formula:

$$\nabla_X=\ddot\nabla_X+\sum_{a=1}^{3}\{\eta_a(X)-(1/3)\alpha_a(X)\}F_a+(1/4)(I\otimes I-\sum_{a=1}^{3}F_a\otimes F_a)Q_X,$$

where Q is an arbitrary tensor field of type (1, 2), [6]. Then obtain $\tilde\nabla$ taking:

$$Q(X, Y)=-\pi(X)Y-4\pi(Y)X.$$

Theorem 7. *If ∇ is a torsion-free adapted connection for the almost quaternal structure, then every other torsion-free adapted connection $\tilde\nabla$ is obtained by the formula:*

(29) $$\tilde\nabla_XY=\nabla_XY+\omega(X)Y+\omega(Y)X-\sum_{a=1}^{3}\{\omega(F_aX)F_aY+\omega(F_aY)F_aX\},$$

where ω is a 1-form globally defined on M. The 1-form ω is related to the 1-forms η_a, $\tilde\eta_a$ corresponding to ∇, $\tilde\nabla$ by the formulae:

(30) $$\omega(X)=(1/6)\sum_{a=1}^{3}\{\tilde\eta_a(F_aX)-\eta_a(F_aX)\}\ ;\ \tilde\eta_a(X)=\eta_a(X)-2\omega(F_aX).$$

Proof. Let: $\tilde\nabla=\nabla+P$, where P is a tensor field of type (1, 2) such that $P(X, Y)=P(Y, X)$. Then P must satisfy the relations:

(31) $$P(X, F_aY)-F_aP(X, Y)=-(\tilde\eta_b-\eta_b)(X)F_cY+(\tilde\eta_c-\eta_c)(X)F_bY.$$

For convenience denote $\tilde\eta_a(X)-\eta_a(X)=\beta_a(F_aX)$. Then the following relations are obtained from (31):

$$P(X, Y)+F_aP(X, F_aY)=\beta_b(F_bX)F_bY+\beta_c(F_cX)F_cY,$$
$$P(X, Y)+F_bP(F_bX, Y)=\beta_a(F_aY)F_aX+\beta_c(F_cY)F_cX,$$
$$-F_aP(X, F_aY)-F_cP(F_bX, F_aY)=-\beta_a(Y)X+\beta_c(F_bY)F_bX,$$
$$-F_bP(F_bX, Y)+F_cP(F_bX, F_aY)=-\beta_b(X)Y+\beta_c(F_aX)F_aY.$$

Summing up, we get:

(32) $$2P(X, Y)=-\beta_b(X)Y-\beta_a(Y)X+\beta_c(F_aX)F_aY+\beta_a(F_aY)F_aX+ +\beta_b(F_bX)F_bY+\beta_c(F_bY)F_bX+\beta_c(F_cX)F_cY+\beta_c(F_cY)F_cX.$$

Due to the symmetry of P it follows:

$$\mathrm{trace}(Y\to P(X, Y))=\mathrm{trace}(Y\to P(Y, X)).$$

hence:

$$-2\beta_a(X)-4m\beta_b(X)-2\beta_c(X)=-4m\beta_a(X)-2\beta_b(X)-2\beta_c(X).$$

Since (a, b, c) is an arbitrary cyclic permutation of $(1, 2, 3)$, it follows: $\beta_a=\beta_b=\beta_c=-2\omega$. Then, from (32) we get (29). The expression (30) of ω is obtained easily from the formulae $\tilde{\eta}_a(X)=\eta_a(X)-2\omega(F_aX)$.

Remark. Since the 1-form ω is uniquely determined by the 1-forms $\tilde{\eta}_a-\eta_a$ it follows that a torsion-free adapted connection is well determined by a linear connection on V having its local connection forms η_a.

Remark. The connection transformation (29) was obtained also in [2] as the transformation preserving the Q-flat paths.

3. Integrability of almost quaternal structures. Let us find now the curvature invariant of the almost quaternal structure given by a tensor field, related to the curvature tensor field, invariant under the transformations (29). It is the second and last obstruction to the integrability of the almost quaternal structures. If $\tilde{\nabla}=\nabla+P$ the corresponding curvature tensor fields $\tilde{R}$, R are related by:

$$\tilde{R}(X,Y)Z=R(X,Y)Z+(\nabla_XP)(Y,Z)-(\nabla_YP)(X,Z)+P(X,P(Y,Z))-$$
$$-P(Y,P(X,Z)).$$

If P is obtained from (29), then:

$$(33)\quad \tilde{R}(X,Y)Z=R(X,Y)Z+\{L(X,Y)-L(Y,X)\}Z+L(X,Z)Y-L(Y,Z)X-$$
$$-\sum_{a=1}^{3}\{L(X,F_aY)-L(Y,F_aX)\}F_aZ-\sum_{a=1}^{3}\{L(X,F_aZ)F_aY-L(Y,F_aZ)F_aX\},$$

where:

$$(34)\qquad L(X,Y)=(\nabla_X\omega)(Y)-\omega(X)\omega(Y)+\sum_{a=1}^{3}\omega(F_aX)\omega(F_aY).$$

The relation between the corresponding Ricci tensor fields $\tilde{r}$, r, is:

$$(35)\qquad \tilde{r}(X,Y)=r(X,Y)-(4m+4)L(X,Y)+\{L(X,Y)+L(Y,X)\}-$$
$$-\sum_{a=1}^{3}\{L(F_aX,F_aY)+L(F_aY,F_aX)\}.$$

From this relation we obtain by some usual computations:

$$L(X,Y)-L(Y,X)=-[1/(4(m+1))]\{\tilde{r}(X,Y)-\tilde{r}(Y,X)-r(X,Y)+r(Y,X)\}$$
$$L(X,Y)+L(Y,X)=[1/(8m(m+2))]\{\tilde{r}(F_aX,F_aY)+\tilde{r}(F_aY,F_aX)-$$
$$-r(F_aX,F_aY)-r(F_aY,F_aX)\}-[(2m+3)/(8m(m+2))]\{\tilde{r}(X,Y)+\tilde{r}(Y,X)-$$
$$-r(X,Y)-r(Y,X)\}.$$

Thus:

$$(36)\quad L(X,Y)=-[1/(4(m+1))]\{\tilde{r}(X,Y)-r(X,Y)\}-[(m+3)/(16m(m+1)$$

$$(m+2)]\{\tilde{r}(X,Y)+\tilde{r}(Y,X)-r(X,Y)-r(Y,X)\}+[1/(16m(m+2)$$
$$\{\sum_{a=1}^{3}\{\tilde{r}(F_aX,F_aY)+\tilde{r}(F_aY,F_aX)-r(F_aX,F_aY)-r(F_aY,F_aX)\}.$$

Replacing this expression of $L(X,Y)$ in (33) we get: $\tilde{W}(X,Y)Z=W(X,Y)Z$, where:

$$(37)\quad W(X,Y)Z=R(X,Y)Z+\{Q(X,Y)-Q(Y,X)\}Z+Q(X,Z)Y-Q(Y,Z)X-$$
$$-\sum_{a=1}^{3}\{Q(X,F_aY)-Q(Y,F_aX)\}F_aZ-\sum_{a=1}^{3}\{Q(X,F_aZ)F_aY-Q(Y,F_aZ)F_aX\},$$

$$(38)\quad Q(X,Y)=[1/(4(m+1))]r(X,Y)+[(m+3)/(16m(m+1)(m+2))]\{r(X,Y)+$$
$$+r(Y,X)\}-[1/(16m(m+2))]\sum_{a=1}^{3}\{r(F_aX,F_aY)+r(F_aY,F_aX)\}.$$

The tensor field W is invariant under all the changes of the torsion-free adapted connections. Thus it can be called the *curvature invariant* of the given almost quaternal structure. We see that the vanishing of the curvature invariant assures the integrability of the almost quaternal structure.

Theorem 8. *If the Chern-Bernard and the curvature invariants of almost quaternal structure are trivial then there exists, locally, a flat adapted connection.*

Proof. If S is given by (3) there exists a torsion free adapted connection ∇. Then $W=0$ implies that the curvature tensor field is:

$$(39)\qquad R(X,Y)Z=-\{Q(X,Y)-Q(Y,X)\}Z-Q(X,Z)Y+Q(Y,Z)X+$$
$$+\sum_{a=1}^{3}\{Q(X,F_aY)-Q(Y,F_aX)\}F_aZ+\sum_{a=1}^{3}\{Q(X,F_aZ)F_aY-Q(Y,F_aZ)F_aX\}.$$

Then the Ricci tensor field is:

$$(40)\quad r(X,Y)=(4m+4)Q(X,Y)-\{Q(X,Y)+Q(Y,X)\}+\sum_{a=1}^{3}\{Q(F_aX,F_aY)+$$
$$+Q(F_aY,F_aX)\}.$$

To solve our problem we have to do a transformation (29) such that $\tilde{R}=0$. Comparing (33) with (37), we see that if ω satisfies the partial differential system:

$$(41)\qquad (\nabla_X\omega)(Y)=\omega(X)\omega(Y)-\sum_{a=1}^{3}\omega(F_aX)\omega(F_aY)+Q(X,Y),$$

then certainly $\tilde{R}=W=0$. Thus, let us study the complete integrability of the partial differential system (41). The complete integrability condition is:

$$(\nabla_X\nabla_Y\omega)(Z)-(\nabla_Y\nabla_X\omega)(Z)-(\nabla_{[X,Y]}\omega)(Z)=-\omega(R(X,Y)Z).$$

Using (21) and (41) it follows, after a quite long computation:

$$(\nabla_X\nabla_Y\omega)(Z)=2\omega(X)\omega(Y)\omega(Z)-2\omega(X)\sum_{a=1}^{3}\omega(F_aY)\omega(F_aZ)-$$

$$-2\omega(Y)\sum_{a=1}^{3}\omega(F_aX)\omega(F_aZ)-2\omega(Z)\sum_{a=1}^{3}\omega(F_aX)\omega(F_aY)+(\nabla_XQ)(Y,Z)+$$

$$+Q(X,Y)\omega(Z)+Q(X,Z)\omega(Y)-\sum_{a=1}^{3}\{Q(X,F_aY)\omega(F_aZ)+Q(X,F_aZ)\omega(F_aY)\}+$$

$$+\omega(\nabla_XY)\omega(Z)-\sum_{a=1}^{3}\omega(F_a\nabla_XY)\omega(F_aZ)+Q(\nabla_XY,Z).$$

Then the the complete integrability condition becomes:

$$(\nabla_XQ)(Y,Z)-(\nabla_YQ)(X,Z)+\{Q(X,Y)-Q(Y,X)\}\omega(Z)+Q(X,Z)\omega(Y)-$$

$$-Q(Y,Z)\omega(X)-\sum_{a=1}^{3}\{Q(X,F_aY)-Q(Y,F_aX)\}\omega(F_aZ)-\sum_{a=1}^{3}\{Q(X,F_aZ)\omega(F_aY)-$$

$$-Q(Y,F_aZ)\omega(F_aX)\}=-\omega(R(X,Y)Z),$$

which, due to the expression (39) of R can be written as:

$$B(X,Y,Z)=(\nabla_XQ)(Y,Z)-(Q_YQ)(X,Z)=0. \tag{42}$$

We shall deduce $B=0$ from the second Bianchi identity satisfied by R:

$$(\nabla_ZR)(X,Y)+(\nabla_XR)(Y,Z)+(\nabla_YR)(Z,X)=0. \tag{43}$$

Using (21), we have:

$$(\nabla_ZR)(X,Y)\tilde{Z}=-\{(\nabla_ZQ)(X,Y)-(\nabla_ZQ)(Y,X)\}\tilde{Z}-(\nabla_ZQ)(X,\tilde{Z})Y+$$

$$+(\nabla_ZQ)(Y,\tilde{Z})X+\sum_{a=1}^{3}\{(\nabla_ZQ)(X,F_aY)-(\nabla_ZQ)(Y,F_aX)\}F_a\tilde{Z}+$$

$$+\sum_{a=1}^{3}\{(\nabla_ZQ)(X,F_a\tilde{Z})F_aY-(\nabla_ZQ)(Y,F_a\tilde{Z})F_aX\}.$$

Next, taking the trace in (43) we obtain:

$$(4m+4)\{B(X,Y,Z)+B(Y,Z,X)+B(Z,X,Y)\}=0. \tag{44}$$

The following relation is obtained from (43) by taking the trace of the mapping $\tilde{Z}\to(\nabla_{\tilde{Z}}R)(X,Y)Z$:

$$(4m+1)B(X,Y,Z)+\sum_{a=1}^{3}\{B(X,F_aY,F_aZ)-B(Y,F_aZ,F_aX)+$$
$$+B(F_aX,Y,F_aZ)-B(F_aZ,X,F_aY)\}=0 \tag{45}$$

Replacing Y, Z by F_aY, F_aZ respectively, summing up for $a=1,2,3$, and doing cyclic permutations of X, Y, Z we obtain:

$$\sum_{a=1}^{3}\{B(X,F_aY,F_aZ)+B(Y,F_aZ,F_aX)+B(Z,F_aX,F_aY)\}=0.$$

Then we obtain easily from (45):

$$(4m+1)\{B(X,Y,Z)+B(X,Z,Y)\}+3\sum_{a=1}^{3}\{B(X,F_aY,F_aZ)+$$
$$B(X,F_aZ,F_aY)\}=0. \tag{46}$$

In (46) replace Y, Z by F_aY, F_aZ respectively and sum for $a=1,2,3$:

$$(4m+7)\sum_{a=1}^{3}\{B(X,F_aY,F_aZ)+B(X,F_aZ,F_aY)\}+9\{B(X,Y,Z)+$$
$$+B(X,Z,Y)\}=0. \tag{47}$$

From (46), (47) it follows: $B(X,Y,Z)+B(X,Z,Y)=0$. Then, using (44), we obtain $B(X,Y,Z)=0$. Thus (41) is completely integrable and, using a local solution ω we obtain the local connection $\tilde{\nabla}$ with $\tilde{R}=0$.

Theorem 9. *The almost quaternal structure is integrable if and only if the Chern-Bernard and curvature invariants are trivial.*

Proof. If S is given by (3) and $W=0$, let $\tilde{\nabla}$ be a, locally defined, flat connection. There exists a local chart on M such that the connection coeficients in this local chart are trivial. Thus, in this local chart the usual derivative does coincide with the covariant derivative with respect to $\tilde{\nabla}$. Next, using (2) we replace the local canonical basis (F_a) of V by the local canonical basis (F'_a) such that the corresponding 1-forms $\tilde{\eta}'_{1a}$ are trivial. To this end let us remark that the curvature R of the adapted connection ∇ with its corresponding 1-forms η_{1a} satisfies the relation:

$$R(X,Y)F_aZ-F_aR(X,Y)Z=-(d\eta_{1b}+\eta_{1c}\wedge\eta_a)(X,Y)F_cZ+(d\eta_{1c}+\eta_{1a}\eta_{1b})(X,Y)F_b$$

obtained from (21) by using the definition of R. If ∇ is replaced by $\tilde{\nabla}$ with $\tilde{R}=0$ then:

$$d\tilde{\eta}_a+\tilde{\eta}_b\wedge\tilde{\eta}_c=0,\quad a=1,2,3. \tag{48}$$

Now, if $\tilde{\eta}'_{1a}=0$, using the transformation formula (22) for η_{1a}, the matrix $[s_{ab}]$ must satisfy the differential system:

$$s_{ab}ds_{ac}+s_{bb}ds_{bc}+s_{cb}ds_{cc}=-\tilde{\eta}_{1a};\quad a=1,2,3.$$

To show that this system is completely integrable, remark that it is equivalent to:

$$ds_{aa}=s_{ab}\tilde{\eta}_{1c}-s_{ac}\tilde{\eta}_b,\quad ds_{ab}=s_{ac}\tilde{\eta}_{1a}-s_{aa}\tilde{\eta}_{1c},\quad ds_{ac}=s_{aa}\tilde{\eta}_{1b}-s_{ab}\tilde{\eta}_{1a}.$$

Recall that (a,b,c) is allways a cyclic permutation of (1, 2, 3). Then the complete integrability conditions:

$$0=d^2s_{aa}=s_{ab}d\ _c-s_{ac}d\tilde{\eta}_b+ds_{ab}\wedge\tilde{\eta}_{1c}-ds_{ac}\wedge\tilde{\eta}_{1b}=s_{ab}(d\tilde{\eta}_{1c}+\tilde{\eta}_{1a}\wedge\tilde{\eta}_{1b})-$$
$$-s_{ac}(d\tilde{\eta}_{1b}+\eta_{1c}\wedge\tilde{\eta}_{1a}),$$

$$0=d^2s_{ab}=s_{ac}d\ _a-s_{aa}d\tilde{\eta}_{1c}+ds_{ac}\wedge\tilde{\eta}_{1a}-ds_{aa}\wedge\tilde{\eta}_{1c}=s_{ac}(d\tilde{\eta}_{1a}+\tilde{\eta}_{1b}\wedge\tilde{\eta}_{1c})-$$
$$-s_{aa}(d\tilde{\eta}_{1c}+\tilde{\eta}_{1a}\wedge\tilde{\eta}_{1b}).$$

$$0=\mathrm{d}^2 s_{ac}=s_{aa}\mathrm{d}\tilde{\gamma}_{lb}-s_{ab}\mathrm{d}\tilde{\gamma}_{la}+\mathrm{d}s_{aa}\wedge\tilde{\eta}_{lb}-\mathrm{d}s_{ab}\wedge\tilde{\gamma}_{la}=s_{aa}(\mathrm{d}\tilde{\gamma}_{lb}+\tilde{\gamma}_{lc}\wedge\tilde{\gamma}_{la})-$$
$$-s_{ab}(\mathrm{d}\tilde{\gamma}_{la}+\tilde{\gamma}_{lb}\wedge\tilde{\eta}_{c})$$

are satisfied according to (48).

Thus, for the connection $\tilde{\nabla}$ and the local canonical basis F'_a we have :

$$\tilde{\nabla}_i(F'_a)^k_j=\partial_i(F'_a)^k_j=0,\ a=1,2,3;\ i,j,k=1,\dots,4m,$$

hence the local expressions of F'_a are constant. By a linear coordinate transformation we can give to the matrices associated with F'_a the usual canonical expressions. Hence the almost quaternal structure is integrable. Conversely if the structure is integrable there exist local charts such that the local expressions of F_a in these local charts are constant. For such a chart consider the local connection having trivial connections coefficients. This connection is obviously adapted with the corresponding 1-forms $\gamma_{la}=0$. Then $T=0$, $R=0$, thus the local expressions of S and W are zero. Due to their definition, they are globally trivial.

A manifold carrying an integrable almost quaternal structure is called a *quaternion manifold*. The typical examples of quaternion manifolds are the quaternion projective spaces.

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TWO-POINT BOUNDARY PROBLEMS IN THE LINEAR THEORY OF RODS

BY

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1. Introduction. In the present paper, by rod we mean a body B comprising a material line and two deformable vectors (called „directors"), attached to every point of line (this mathematical model is called also „generalized rod" [1]). For such a body the basic equations are presented and a strain energy functions is proposed. By using the geometrical and constitutive equations, the displacement equations of equilibrium are written. For some self-adjoint boundary value problems, existence and uniqueness theorems are proved.

2. Notations. Basic equations. Let the particle of the material line of the body B be identified with coordinate S. In a reference configuration of B, let C be the material line $\mathbf{R}(S)$ the position vector of the points of C and $\mathbf{D}_\alpha$ be the values of the directors on C. The coordinate S is the arc length parameter of C. Let c be the present material line of B, $\mathbf{r}$ the position vector of the points of c and $\mathbf{d}_\alpha$ the directors at $\mathbf{r}$. In the present paper the vectors and the tensor are refered to the following base vectors ; the unit principal normal $\mathbf{A}_1$, the unit binormal $\mathbf{A}_2$ and the tangent vector $\mathbf{A}_3=d\mathbf{R}/dS$.

We consider an initial straight elastic rod with variable directors , which are normal to each other and to the straight line C. So that we have

(2.1) $$\mathbf{D}_\alpha=\mathbf{D}_{(\alpha)}\mathbf{A}_\alpha,\ \text{do not sum on } \alpha,\ \alpha=1,2.$$

As $D_{(\alpha)}$, $\alpha=1,2$, depend on S, this mathematical model describes a rod which has nonuniform thickness in the reference configuration.

We present now the basic equations of the theory [1].

Geometrical equations

(2.2) $$\mathbf{r}=\mathbf{R}+\mathbf{u},\ \mathbf{d}_\alpha=\mathbf{D}_\alpha+\bar{\delta}_\alpha,$$

(2.3) $$\begin{cases}\gamma_{\alpha\beta}=D_{(\alpha)}\bar{\delta}_{\beta\alpha}+D_{(\beta)}\bar{\delta}_{\alpha\beta},\ \ \gamma_{\alpha3}=\bar{\delta}_{\alpha3}+D_{(\alpha)}u'_\alpha,\ \ \gamma_{33}=2\bar{\delta}_{33}=2u'_3,\\ k_{\alpha\beta}=D_{(\beta)}\bar{\delta}'_{\alpha\beta}+D'_{(\alpha)}\bar{\delta}_{\beta\alpha},\ \ k_{\alpha3}=\bar{\delta}'_{\alpha3}+D'_{(\alpha)}u'_\alpha,\end{cases}$$

where $\mathbf{u}$, $\bar{\delta}_\alpha$ are the displacement vectors and γ_{ij}, k_{ij} are the deformation measures' denotes the derivative with respect to S.

Constitutive equations

(2.4) $$\begin{cases}=n^3=2\rho_0(\partial\psi/\partial\gamma_{33}),\ \bar{\pi}^\alpha=2(n^\alpha-D'_{(\alpha)}p^{\alpha3})/D_{(\alpha)}=2\rho_0(\partial\psi^*/\partial\gamma_{\alpha3}),\\ \bar{\pi}^{\lambda\mu}=(D_{(\lambda)}\pi^{\lambda\mu}+D_{(\mu)}\pi^{\mu\lambda}-D'_{(\lambda)}p^{\mu\lambda}-D'_{(\lambda)}p^{\mu\lambda\mu}/2D_{(\lambda)}D_{(\mu)}=2\rho_0(\partial\psi^*/\partial\gamma_{\lambda\mu}),\\ p^{\alpha\beta}=\rho_0D_{(\beta)}(\partial\psi^*/\partial k_{\alpha\beta}),\ \ p^{\alpha3}=\rho_0(\partial\psi^*\partial k_{\alpha3}),\ \ n^\alpha=D_{(\alpha)}\pi^{\alpha3}+D'_{(\alpha)}p^{\alpha3}.,\\ D_{(1)}\pi^{12}-D_{(2)}\pi^{21}=D'_{(2)}p^{21}-D'_{(1)}p^{12},\end{cases}$$