ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" DIN IAŞI (S.N.)
MATEMATICĂ, Tomul LIII, 2007, Supliment

THE GEOMETRY OF FRACTIONAL OSCULATOR BUNDLE OF HIGHER ORDER AND APPLICATIONS

BY

ION DORU ALBU, MIHAELA NEAMŢU and DUMITRU OPRIŞ

Dedicated to Academician Radu Miron at his 80th anniversary

Abstract. Using the reviewed Riemann-Liouville fractional derivative we define the bundle $\overset{\alpha k}{E} = Osc^{\alpha k}(M)$ and highlight geometrical structures with a geometrical character.

Mathematics Subject Classification 2000: 26A33, 53C63, 58A05, 58A40.
Key words: fractional derivative, osculator bundle.

1. Introduction. It is known that the operators of integration and derivation have geometrical and physical interpretations and they were used in the modelation of problems from different domains. The use of reviewed Liouville-Riemann integration and derivation operators lead to fractional integration and derivation. The geometrical and physical interpretation is suggested by the Stieltjes integral and it was done by Podlubny [6].

In this paper we present the fractional derivative taking into account the geometrical character, namely the behavior of associated objects under a change of local chart.

The outline of this paper is as follows. In Section 2 we describe the reviewed fractional derivative on $\mathbb{R}$ using [2], [4], the fractional osculator bundle $T^\alpha(M)$ on a differentiable manifold and the behavior of introduced objects under a change of local chart. In Section 3, we define the fractional osculator bundle of k order using the method presented by Miron in [5]. We introduce: the Liouvile fractional vector fields, the αk-fractional spray and the fractional nonlinear connection. We prove that these objects have a geometrical character. Our findings are analogous with R. Miron's results for the fractional case.

The main results from the present paper were used in [3] for the study of some fractional geometrical structures and they will permit the study of other structures of this type.

2. The fractional derivative on R. The fractional osculator bundle on the differentiable manifold

2.1. The fractional derivative on R. Let $f : [a,b] \to \mathbb{R}$ be a derivable function and $\alpha \in \mathbb{R}$, $\alpha > 0$.

The functions:

$$D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dt}\right)^m \int_{-\infty}^t (t-s)^{m-\alpha-1}(f(s)-f(0))ds, \quad 0 \in (-\infty, t)$$

$${}^*D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \left(-\frac{d}{dt}\right)^m \int_t^\infty (s-t)^{m-\alpha-1}(f(s)-f(0))ds, \quad 0 \in (t, \infty)$$

are called the left, respectively the right fractional derivative of α order for function f.

We define the seminorms: $|x|_{J_L^\alpha(R)} = ||D_t^\alpha||_{L^2(R)}$, $|x|_{J_R^\alpha(R)} = ||^*D_t^\alpha||_{L^2(R)}$ and the norms: $||x||_{J_L^\alpha(R)} = (||x||^2_{L^2(R)} + |x|^2_{J_L^\alpha(R)})$, $||x||_{J_R^\alpha(R)} = (||x||^2_{L^2(R)} + |x|^2_{J_L^\alpha(R)})$, where $J_L^\alpha(R)$, respectively $J_R^\alpha(R)$ denotes the closure of $C_0^\infty(R)$ with respect to $||\cdot||_{J_L^\alpha(R)}$, respectively $||\cdot||_{J_R^\alpha(R)}$.

From the above definitions we have the following [2]:

Proposition 2.1. 1. *Let $I \subset R$ and $J^\alpha_{L,0}(I)$, $J^\alpha_{R,0}(I)$ be the closure of $C_0^\infty(I)$ in accordance with the respective norms. Then, for any $f \in J^\beta_{L,0}(I)$, $0 < \alpha < \beta$, respectively for any $f \in J^\beta_{R,0}(I)$, $0 < \alpha < \beta$, the relation $D_t^\beta f(t) = D_t^\alpha D_t^{\beta-\alpha} f(t)$, respectively, the relation ${}^*D_t^\beta f(t) =^* D_t^{\alpha *} D_t^{\beta-\alpha} f(t)$ holds;*

2. *If $\lim_{n\to\infty} \alpha_n = p \in \mathbb{N}^*$, then*

$$\lim_{n\to\infty} (D_t^{\alpha_n} f(t)) = D_t^p f(t), \lim_{n\to\infty} ({}^*D_t^{\alpha_n} f(t)) = {}^*D^p f(t);$$

3. *If f_1, f_2 are analytical functions on $[a,b]$, then:*

$$D_t^\alpha (f_1 f_2)(t) = \sum_{k=0}^\infty \left(\begin{array}{c} \alpha \\ k \end{array}\right) D_t^{\alpha-k} f_1(t) \left(\frac{d}{dt}\right)^k f_2(t),$$

where $\left(\frac{d}{dt}\right)^k = \frac{d}{dt} \circ \ldots \circ \frac{d}{dt}$;

4. *If* $f : [a, b] \to \mathbb{R}$ *is analytical and* $0 \in (a, b)$ *then*

$$f(t) = \sum_{h=0}^{\infty} E_\alpha(t) D_t^{\alpha h} f(t)|_{t=0},$$

where E_α *is the Mittag-Leffler function,* $E_\alpha(t) = \sum_{h=0}^{\infty} \dfrac{t^{\alpha h}}{\Gamma(1 + \alpha h)}$.

2.2. The fractional osculator bundle. Let $\alpha \in (0, 1)$ and M a n-dimensional differentiable manifold. The parameterized curves on M, $c_1, c_2 : I \to M$, with $0 \in I$, $c_1(0) = c_2(0) \in M$ have a fractional contact α in x_0 if the relation

$$D_t^\alpha (f \circ c_1)|_{t=0} = D_t^\alpha (f \circ c_2)|_{t=o} \tag{1}$$

holds, for all $f \in \mathcal{F}(U)$ and $x_0 \in U$, where U is a local chart on M.

Preceding equality (1) defines a relation of equivalence. The classes $[c]_{x_0}^\alpha$ are called the fractional osculator space in x_0, which will be denoted by $Osc_{x_0}^{(\alpha)}(M) = T_{x_0}^\alpha(M)$.

Let $T^\alpha(M) = \bigcup_{x_0 \in M} T_{x_0}^\alpha(M)$ and $\pi^\alpha : T^\alpha(M) \to M$, given by $\pi^\alpha [c]_{x_0}^\alpha = x_0$. There is a differential structure on $T^\alpha(M)$ and $(T^\alpha(M), \pi^\alpha, M)$ is a differentiable bundle space.

If U is a local chart on M with $x_0 \in U$ and $c : I \to M$ is a curve given by $x^i = x^i(t)$, $i = 1..n$, $t \in I$, then $[c]_{x_0}^\alpha$ is characterized by:

$$x^i(t) = x^i(0) + \frac{t^\alpha}{\Gamma(1 + \alpha)} D_t^\alpha x^i|_{t=0}, \quad t \in (-\varepsilon, \varepsilon).$$

The coordinates of $[c]_{x_0}^\alpha$ on $(\pi^\alpha)^{-1}(U) \subset T_{x_0}^\alpha(M)$ are $(x^i, y^{i(\alpha)})$, where

$$x^i = x^i(0), y^{i(\alpha)} = \frac{1}{\Gamma(1 + \alpha)} D_t^\alpha x^i(t)|_{t=0}, \quad i = 1..n.$$

From Proposition 2.1 and and the definition of $T^\alpha(M)$ we have:

Proposition 2.2. 1. *If* $0 < \alpha < \beta$ *then* $T^{(\alpha)}(M) \subset T^{(\beta)}(M)$;
2. *If* $\lim_{n\to\infty} \alpha_n = 1$ *then* $\lim_{n\to\infty} T^{(\alpha_n)}(M) = T(M)$.

Let (x^i), $i = 1..n$ be the coordinate functions on U and $(dx^i)_{i=1..n}$ be the base of 1-forms $\mathcal{D}^1(U)$ and $\left(\frac{\partial}{\partial x^i}\right)_{i=1..n}$ the base of the vector fields $\mathcal{X}(U)$. For $f : U \to \mathbb{R}$ and $\alpha \in (0,1)$, the fractional derivative with respect to x^i is defined by:

(2)

$$D^{\alpha}_{x^i} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial x^i} \cdot$$

$$\cdot \int_{a^i}^{x^i} \frac{f(x^1,\dots,x^{i-1},s,x^{i+1},\dots,x^n) - f(x^1,\dots,x^{i-1},a^i,x^{i+1},\dots,x^n)}{(x^i-s)^{\alpha}} ds$$

where $U_{ab} = \{x \in U, a^i \leq x^i \leq b^i, i = 1..n\} \subset U$.

From (2) we have:

Proposition 2.3. 1. *If* $f_1^i = (x^i)^{\gamma}$ *then* $(D^{\alpha}_{x^i} f_1^i)(x) = \frac{(x^i)^{j-\alpha}\Gamma(1+\alpha)}{\Gamma(1+\gamma-\alpha)}$;
2. *If* $f_2^j = \frac{(x^j)^{\alpha}}{\Gamma(1+\alpha)}$ *then* $(D^{\alpha}_{x^i} f^j)(x) = \delta_i^j$.

We consider the functions $(x^i)^{\alpha} \in \mathcal{F}(U)$ and $d(x^i)^{\alpha} = \alpha(x^i)^{\alpha-1}dx^i \in \mathcal{D}^1(U)$, $i = 1..n$. The fractional exterior derivative is the operator $d^{\alpha} : \mathcal{F}(U) \to \mathcal{D}^1(U)$ given by $d^{\alpha} f = d(x^i)^{\alpha} D^{\alpha}_{x^i}(f)$ ([1]).

Let $D^{\alpha}_{x^i} : \mathcal{D}^1(U) \to \mathcal{D}^1(U)$ be the operator given by:

$$D^{\alpha}_{x^i}(a_j dx^j) = \sum_{k=0}^{\infty} \begin{pmatrix} \alpha \\ k \end{pmatrix} D^{\alpha-k}_{x^i}(a_j) \left(\frac{\partial}{\partial x^i}\right)^k (dx^j). \tag{3}$$

From (3) and $\left(\frac{\partial}{\partial x^i}\right)^k (dx^j) = d(\left(\frac{\partial}{\partial x^i}\right)^k (x^j)) = 0, k \geq 1$ we obtain:

$$D^{\alpha}_{x^i}(a_j dx^j) = D^{\alpha}_{x^i}(a_j) dx^j. \tag{4}$$

Let $d^{\alpha} : \mathcal{D}^1(U) \to \mathcal{D}^2(U)$ be the operator given by:

$$d^{\alpha}(a_j dx^j) = d(x^i)^{\alpha} \wedge D^{\alpha}_{x^i}(a_j dx^j). \tag{5}$$

Proposition 2.4. *Let* U, $\overline{U}$, $U \cap \overline{U} \neq \varnothing$ *be two charts on* M, $x \in U \cap \overline{U}$ *and the change of local chart given by:*

$$\overline{x}^i = \overline{x}^i(x^1,\dots,x^n),\ rang \left(\frac{\partial \overline{x}^i}{\partial x^j}\right) = n. \tag{6}$$

With respect to (6) *the following relations:*

$$d(x^i)^\alpha = \overset{\alpha}{J}{}^i_j(x,\overline{x})d(\overline{x}^j)^\alpha, D^\alpha_{x^i} = \overset{\alpha}{J}{}^j_i(\overline{x},x)D^\alpha_{\overline{x}^j}$$
$$\overset{\alpha}{J}{}^i_j(x,\overline{x}) \cdot \overset{\alpha}{J}{}^j_k(\overline{x},x) = \delta^i_k, \overset{\alpha}{J}{}^i_j(x,\overline{x}) = (x^i)^{\alpha-1}\frac{\partial x^i}{\partial \overline{x}^j}\frac{1}{(\overline{x}^j)^{1-\alpha}},$$

hold, where $\overset{\alpha}{J}{}^i_j(x,\overline{x}) = \frac{1}{\Gamma(1+\alpha)}D^\alpha_{\overline{x}^j}(x^i)^\alpha$.

Let $\mathcal{X}^\alpha(U)$ be the module of the fractional vector fields generated by the operators $\{D^\alpha_{x^i}\}_{i=1..n}$. A fractional field of vectors $\overset{\alpha}{X} \in \mathcal{X}^\alpha(U)$ has the form $\overset{\alpha}{X} = \overset{\alpha}{X}{}^i D^\alpha_{x^i}$, where $\overset{\alpha}{X}{}^i \in \mathcal{F}(U), i = 1..n$. Under a change of local chart it changes by $\overline{\overset{\alpha}{X}}{}^i = \overset{\alpha}{J}{}^i_j(x,\overline{x})X^j$.

The fractional differential equation associated to the fractional field of vectors $\overset{\alpha}{X}$ is:

$$D^\alpha_t x^i(t) = \overset{\alpha}{X}{}^i(x(t)),\ i = 1..n. \tag{7}$$

The fractional differential equation (7) with initial conditions has solutions [2].

3. The fractional osculator bundle of higher order. Geometrical structures

3.1. The fractional k-osculator bundle, $k \geq 1$**.** The parameterized curves on M, $c_1, c_2 : I \to M$, with $0 \in I$, $c_1(0) = c_2(0) = x_0 \in M$ have a fractional contact of k order in x_0 if for any $f \in \mathcal{F}(U)$, the relations $D^{\alpha a}_t(f \circ c_1)|_{t=0} = D^{\alpha a}_t(f \circ c_2)|_{t=0}$, $a = 1..k$ hold, where $x_0 \in U$ and U is a local chart on M.

The classes $([c]^{\alpha a}_{x_0})_{a=1..k}$ are called the fractional osculator space of k order and they will be denoted by $Osc^{\alpha k}_{x_0}(M) = \overset{\alpha k}{E}_{x_0}$.

We consider $\overset{\alpha k}{E} = \bigcup_{x_0\in M}\overset{\alpha k}{E}_{x_0}$ and $\pi^{\alpha k}_0 : \overset{\alpha k}{E} \to M$ given by $\pi^{\alpha k}_0([c]^{\alpha a}_{x_0})_{a=1..k}$ $= x_0$. There is a differentiable structure on $\overset{\alpha k}{E}$ and $(\overset{\alpha k}{E}, \pi^{\alpha k}_0, M)$ is a differentiable bundle.

If U is a local chart on M with $x_0 \in U$ and $c : I \to M$ is a curve given by $x^i = x^i(t)$, $i = 1..n$, $t \in I$, then a class $([c]^{\alpha a}_{x_0})_{a=1..k}$ is given by the curve:

$$x^i(t) = x^i(0) + \sum_{a=1}^{k} \frac{t^{\alpha a}}{\Gamma(1+\alpha k)} D_t^{\alpha a} x^i(t)|_{t=0},\ t \in (-\varepsilon, \varepsilon).$$

In $(\pi_0^{\alpha k})^{-1}(U) \subset \overset{\alpha k}{E}$, the coordinates of $([c]^{\alpha a}_{x_0})_{a=1..k}$ are $(x^i, y^{i(\alpha a)})$, $i = 1..n$, $a = 1..k$, where $x^i = x^i(0)$, $y^{i(\alpha a)} = \frac{1}{\Gamma(1+\alpha k)} D_t^{\alpha a} x^i(t)$, $i = 1..n$, $a = 1..k$.

Proposition 3.1. *Let U, $\overline{U}$, $U \bigcap \overline{U} \neq \emptyset$ be two charts on M and*

$$\overline{x}^i = \overline{x}^i(x^1, \ldots, x^n),\ i = 1..n,\ det\left(\frac{\partial \overline{x}^i}{\partial x^j}\right) \neq 0 \tag{8}$$

the coordinates transformation. The coordinates transformation on $(\pi_0^{\alpha k})^{-1}$ $(U \cap \overline{U})$ are given by:

$$\begin{aligned}
&\overline{y}^{i(\alpha)} = \overset{\alpha}{J}{}^i_j(x, \overline{x}) y^{j(\alpha)}\\
&\frac{\Gamma(\alpha(a-1))}{\Gamma(\alpha)} \overline{y}^{i(\alpha a)} = \Gamma(1+\alpha) \overset{\alpha}{J}{}^i_j(\overline{y}^{\alpha(a-1)}, x) y^{j(\alpha)}\\
&+ \frac{\Gamma(2\alpha)}{\Gamma(\alpha)} \overset{\alpha}{J}{}^i_j(\overline{y}^{\alpha(a-1)}, y^{\alpha}) y^{j(2\alpha)}\\
&+ \ldots + \frac{\Gamma(2\alpha)}{\Gamma(\alpha)} \overset{\alpha}{J}{}^i_j(\overline{y}^{\alpha(a-1)}, y^{\alpha b}) y^{j(\alpha b)}\\
&+ \ldots + \frac{\Gamma(\alpha(a-1))}{\Gamma(\alpha)} y^{i(\alpha a)},\ a = 2\ldots k,\ b = 2\ldots k,\ b \leq a,
\end{aligned} \tag{9}$$

where

$$\begin{aligned}
&\overset{\alpha}{J}{}^i_j(x, \overline{x}) = D^{\alpha}_{\overline{x}^j}(x^i),\ \overset{\alpha}{J}{}^i_j(\overline{y}^{\alpha(a-1)}, x) = D^{\alpha}_{x^j} \overline{y}^{i(\alpha(a-1))},\ i, j = 1\ldots n,\\
&\overset{\alpha}{J}{}^i_j(\overline{y}^{\alpha(a-1)}, y^{\alpha(b-1)}) = D^{\alpha}_{y^{j(\alpha(b-1))}} \overline{y}^{i(\alpha(a-1))},\ a, b = 2\ldots k, b \leq a.
\end{aligned}$$

From the definition of fractional osculator bundle we can deduce that if $\lim_{n\to\infty} \alpha_n = 1$ then $\lim_{n\to\infty} \overset{\alpha_n}{E} = E = Osc^k(M)$. The bundle space (E, π, M) was defined and studied in [5].

3.2. Geometrical structures on $\overset{\alpha k}{E}$. Let $\pi^{\alpha k}_{\alpha h}:\overset{\alpha k}{E}\to\overset{\alpha h}{E}$, $h<k$, be the projections given by: $\pi^{\alpha k}_{\alpha h}(x,y^{(\alpha)},\ldots,y^{(\alpha k)})=(x,y^{(\alpha)},\ldots,y^{(\alpha h)})$ and the operator $d^{\alpha}\pi^{\alpha k}_{\alpha h}:\mathcal{X}^{\alpha}(\overset{\alpha k}{E})\to\mathcal{X}^{\alpha}(\overset{\alpha h}{E})$:

$$d^{\alpha}\pi^{\alpha k}_{\alpha h}=\Gamma(1+\alpha)(d(x^i)^{\alpha}D^{\alpha}_{x^i}+\sum_{a=1}^{h}d(y^{i(\alpha a)})^{\alpha}D^{\alpha}_{y^{i(\alpha a)}}),h<k,$$

where $\mathcal{X}^{\alpha}(\overset{\alpha k}{E})$ is the module of fractional vector fields on $\overset{\alpha k}{E}$.

We consider $\mathcal{V}^{\alpha k}_{\alpha h}=Kerd^{\alpha}\pi^{\alpha k}_{\alpha h}$, $h=0,1,\ldots,k-1$ and its base given by $\{D^{\alpha}_{y^{i(\alpha(k+1))}},\ldots,D^{\alpha}_{y^{i(\alpha k)}}\}$, $i=1..n$. From the definition of $\mathcal{V}^{\alpha k}_{\alpha h}$ we get:

$$\mathcal{V}^{\alpha k}_{\alpha(k-1)}\subset\ldots\subset\mathcal{V}^{\alpha k}_{\alpha}\subset\mathcal{V}^{\alpha k}_{0},d^{\alpha}\pi^{\alpha k}_{\alpha h}(D^{\alpha}_{x^i})=D^{\alpha}_{x^i},\ d^{\alpha}\pi^{\alpha k}_{\alpha h}(D^{\alpha}_{y^{i(b)}})=D^{\alpha}_{y^{i(b)}},\ b=1..h.$$

From Proposition 3.1 we obtain:

Proposition 3.2. *Under the change of local chart* (8), *the operators* $D^{\alpha}_{x^i}$, $D^{\alpha}_{y^{i(\alpha a)}}$, $i=1..n$, $a=1..n$, *change by:*

$$\begin{aligned}D^{\alpha}_{x^i}&=\sum_{a=1}^{k}\overset{\alpha}{J}{}^{j}_{i}(\overline{y}^{(\alpha a)},x)D^{\alpha}_{\overline{y}^{j(\alpha a)}}+\overset{\alpha}{J}{}^{j}_{i}(\overline{x},x)D^{\alpha}_{\overline{x}^j},\\ D^{\alpha}_{y^{i(\alpha a)}}&=\sum_{b=1}^{k}\overset{\alpha}{J}{}^{j}_{i}(\overline{y}^{\alpha b},y^{\alpha a})D^{\alpha}_{\overline{y}^{j(\alpha b)}},a=1..k.\end{aligned}\tag{10}$$

From Proposition 3.2 we can deduce that $\mathcal{V}^{\alpha k}_{\alpha h}$ has geometrical character.

The following fractional fields of vectors:

$$\begin{aligned}&\overset{\alpha}{\Gamma}=y^{i(\alpha)}D^{\alpha}_{y^{i(\alpha k)}},\\ &\overset{2\alpha}{\Gamma}=\Gamma(1+\alpha)y^{i(\alpha)}D^{\alpha}_{y^{i(\alpha(k-1))}}+\frac{\Gamma(2\alpha)}{\Gamma(\alpha)}y^{i(2\alpha)}D^{\alpha}_{y^{i(\alpha k)}},\ldots,\\ &\overset{\alpha k}{\Gamma}=\Gamma(1+\alpha)y^{i(\alpha)}D^{\alpha}_{y^{i(\alpha)}}+\frac{\Gamma(2\alpha)}{\Gamma(\alpha)}y^{i(2\alpha)}D^{\alpha}_{y^{i(2\alpha)}}\\ &\quad+\ldots+\frac{\Gamma(\alpha(k-1))}{\Gamma(\alpha)}y^{i(\alpha k)}D^{\alpha}_{y^{i(\alpha k)}}\end{aligned}\tag{11}$$

are called Liouville fractional fields of vectors.

From (9) and (10) the fields $\overset{\alpha a}{\Gamma}$, $a = 1..k$ have geometrical character and $\overset{\alpha}{\Gamma} \in \mathcal{V}_0^{\alpha k}$, $\overset{2\alpha}{\Gamma} \in \mathcal{V}_\alpha^{\alpha k}$, …, $\overset{\alpha k}{\Gamma} \in \mathcal{V}_{\alpha(k-1)}^{\alpha k}$.

The operators $\overset{\alpha k}{J} : \mathcal{X}^\alpha(\overset{\alpha k}{E}) \to \mathcal{X}^\alpha(\overset{\alpha k}{E})$ with the properties:

$$
\begin{aligned}
&\overset{\alpha k}{J}(D^\alpha_{x^i}) = D^\alpha_{y^{i(\alpha)}},\ \overset{\alpha k}{J}(D^\alpha_{y^{i(\alpha a)}}) = D^\alpha_{y^{i(\alpha(a+1))}},\\
&a = 1..k-1,\ \overset{\alpha k}{J}(D^\alpha_{y^{i(\alpha k)}}) = 0
\end{aligned}
\tag{12}
$$

is called αk fractional tangent structure.

From (10) and (12) we have:

Proposition 3.3. *αk-fractional tangent structure has the following properties:*

1. $\overset{\alpha k}{J}$ *has a geometrical character;*
2. *rang*$(\overset{\alpha k}{J}) = kn,\quad \overset{\alpha k}{J} \circ \ldots \circ \overset{\alpha k}{J} = 0$;
3. $\overset{\alpha k}{J}(\overset{\alpha k}{\Gamma}) = \overset{\alpha(k-1)}{\Gamma}, \ldots, \overset{\alpha k}{J}(\overset{2\alpha}{\Gamma}) = \overset{\alpha}{\Gamma},\ \overset{\alpha k}{J}(\overset{\alpha}{\Gamma}) = 0.$

The fractional field of vectors $\overset{\alpha k}{S} \in \mathcal{X}^\alpha(\overset{\alpha k}{E})$ is called αk-fractional spray if $\overset{\alpha k}{J}(\overset{\alpha k}{S}) = \overset{\alpha k}{\Gamma}$. From (11) and (12) we obtain the form of $\overset{\alpha k}{S}$:

$$
\begin{aligned}
\overset{\alpha k}{S} &= \Gamma(1+\alpha) y^{i(\alpha)} D^\alpha_{x^i} + \frac{\Gamma(2\alpha)}{\Gamma(\alpha)} y^{i(2\alpha)} D^\alpha_{y^{i(\alpha)}}\\
&+ \ldots + \frac{\Gamma(\alpha(k-1))}{\Gamma(\alpha)} y^{i(\alpha k)} D^\alpha_{y^{i(\alpha(k-1))}}\\
&- \frac{\Gamma(\alpha k)}{\Gamma(\alpha)} G^i(x, y^{(\alpha)}, \ldots, y^{(\alpha k)}) D^\alpha_{y^{i(\alpha k)}}.
\end{aligned}
\tag{13}
$$

Proposition 3.4. *The αk-fractional spray uniquely defines the fractional differential equation given by:*

$$
\frac{1}{\Gamma(1+\alpha k)} D_t^{\alpha(k+1)} x^i(t) + G^i(x, D_t^\alpha x, \ldots, \Gamma(1+\alpha(k-1)) D_t^{\alpha k} x) = 0.
$$

Let $\overset{\alpha k}{\mathcal{N}}$ be the submodule of $\mathcal{X}^\alpha(\overset{\alpha k}{E})$ so that $\mathcal{X}^\alpha(\overset{\alpha k}{E})|_{(\pi_0^{\alpha k})^{-1}(U)} = \mathcal{N}^{\alpha k} \oplus \mathcal{V}_0^{\alpha k}|_{(\pi_0^{\alpha k})^{-1}(U)}$. The submodule $\overset{\alpha k}{\mathcal{N}}$ is called fractional nonlinear connection.

We consider $l^{\alpha k} : \mathcal{X}^{(\alpha)}(U) \to \mathcal{X}^{\alpha}(\overset{\alpha k}{E})|_{(\pi_0^{\alpha k})^{-1}(U)}$, $l^{\alpha k}(D^{\alpha}_{x^i}) = \Delta^{\alpha k}_{x^i}$, $i = 1..n$, where

$$\Delta^{\alpha k}_{x^i} = D^{\alpha}_{x^i} - \sum_{a=1}^{k} \overset{\alpha a}{N}{}^{j}_{i} D_{y^{i(\alpha a)}},\ i = 1..n. \tag{14}$$

From (10) and (14) we obtain: $\Delta^{\alpha k}_{\overline{x}^i} = \overset{\alpha}{J}{}^{j}_{i}(x, \overline{x})\Delta^{\alpha k}_{x^i}$.

The functions $(\overset{\alpha a}{N}{}^{j}_{i})_{i,j=1..n;a=1..k}$ are called the coefficients of fractional nonlinear connection.

Let $\overset{\alpha k}{\mathcal{N}}_{\dashv}$ be the vertical submodule given by:

$$\overset{\alpha k}{\mathcal{N}}_0 = \overset{\alpha k}{\mathcal{N}},\ \overset{\alpha k}{\mathcal{N}}_a = \overset{\alpha k}{J}(\mathcal{N}^{\alpha k}_{a-1}),\ a = 1..k-1, \tag{15}$$

where $\overset{\alpha k}{\mathcal{N}}$ is the submodule defined by fractional nonlinear connection.

From (15) we have:

$$\mathcal{X}^{\alpha}(\overset{\alpha k}{E})|_{(\pi_0^{\alpha k})^{-1}(U)} = \overset{\alpha k}{\mathcal{N}}_0 \oplus \ldots \oplus \overset{\alpha k}{\mathcal{N}}_{k-1} \oplus \mathcal{V}^{\alpha k}_0|_{(\pi_0^{\alpha k})^{-1}(U)},\ U \subset M. \tag{16}$$

In what follows we will use a base adapted to the decomposition (16). Then, $\Delta^{\alpha k}_{y^{i(k)}} = \overset{\alpha k}{J}(\Delta^{\alpha k}_{x^i})$, $\Delta^{\alpha k}_{y^{i(\alpha a)}} = \overset{\alpha k}{J}(\Delta^{\alpha k}_{y^{i(a-1)}})$, $a = 2..k$, $D^{\alpha}_{y^{(\alpha)}}$, $i = 1..n$, with

$$\Delta^{\alpha k}_{x^i} \in \overset{\alpha k}{\mathcal{N}}_0,\ \Delta^{\alpha k}_{y^{i(\alpha a)}} \in \overset{\alpha k}{\mathcal{N}}_a,\ a = 2..k-1,\ D^{\alpha}_{y^{i(\alpha)}} \in \mathcal{V}^{\alpha k}_0,\ i = 1..n$$

and

$$\begin{aligned}
&\overset{\alpha k}{\Delta}_{y^{i(\alpha)}} = D^{\alpha}_{y^{i(\alpha)}} - \overset{\alpha}{N}{}^{j}_{i} D^{\alpha}_{y^{i(2\alpha)}} - \ldots - \overset{\alpha(k-1)}{N}{}^{j}_{i} D^{\alpha}_{y^{j(\alpha k)}},\\
&\overset{\alpha k}{\Delta}_{y^{i(2\alpha)}} = D^{\alpha}_{y^{i(2\alpha)}} - \ldots - \overset{\alpha(k-1)}{N}{}^{j}_{i} D^{\alpha}_{y^{j(\alpha k)}}, \ldots,\\
&\overset{\alpha k}{\Delta}_{y^{i(\alpha(k-1))}} = D^{\alpha}_{y^{i(\alpha(k-1))}} - \overset{\alpha}{N}{}^{j}_{i} D^{\alpha}_{y^{j(\alpha k)}}.
\end{aligned} \tag{17}$$

The dual base of (17) is:

$$\begin{aligned}
&\overset{\alpha k}{\delta} y^{i(\alpha)} = d(y^{i(\alpha)})^{\alpha} + \overset{\alpha}{M}{}^{i}_{j} d(x^j)^{\alpha}\\
&\overset{\alpha k}{\delta} y^{i(2\alpha)} = d(y^{i(2\alpha)})^{\alpha} + \overset{\alpha}{M}{}^{i}_{j} d(y^{i(\alpha)})^{\alpha} + \overset{2\alpha}{M}{}^{i}_{j} d(x^j)^{\alpha}, \ldots,\\
&\overset{\alpha k}{\delta} y^{i(\alpha k)} = d(y^{i(\alpha k)})^{\alpha} + \overset{\alpha}{M}{}^{i}_{j} d(y^{i(\alpha(k-1))})^{\alpha} + \cdots + \overset{\alpha k}{M}{}^{i}_{j} d(x^j)^{\alpha},
\end{aligned} \tag{18}$$

where: $\overset{\alpha}{M}{}^i_j = \overset{\alpha}{N}{}^i_j, \ldots, \overset{\alpha k}{M}{}^i_j = \overset{\alpha k}{N}{}^i_j + \overset{\alpha(k-1)}{N_h} \overset{\alpha}{N}{}^h_j + \cdots + \overset{\alpha}{N}{}^i_h \overset{\alpha(k-1)}{N}{}^h_j$. The functions $\overset{\alpha a}{M}{}^i_j$, $a = 1..k$ are called dual coefficients of fractional nonlinear connection.

From (13) and (18) we obtain:

Proposition 3.5. *A αk-fractional spray , $\overset{\alpha k}{S}$, with the components $G^i(x, y^{(\alpha)}, \ldots, y^{(\alpha k)})$, determines a fractional nonlinear connection with the dual coefficients given by:*

$$\begin{aligned}
\overset{\alpha}{M}{}^i_j &= D^\alpha_{y^{i(\alpha)}} G^i,\\
\overset{2\alpha}{M}{}^i_j &= \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} (\overset{\alpha k}{S}(\overset{\alpha}{M}{}^i_j) + \overset{\alpha}{M}{}^i_l \overset{\alpha}{M}{}^l_j), \ldots,\\
\overset{\alpha k}{M}{}^i_j &= \frac{\Gamma(\alpha(k-1))}{\Gamma(\alpha k)} (\overset{\alpha k}{S}(\overset{\alpha(k-1)}{M}{}^i_j) + \overset{\alpha}{M}{}^i_l \overset{\alpha(k-1)}{M}{}^l_j).
\end{aligned}$$

We consider the adapted base (17) and the operator $\overset{\alpha k}{\mathcal{L}}$ defined by:

$$\overset{\alpha k}{\mathcal{L}}_{\Delta^{\alpha b}_{x^i}} (\overset{\alpha k}{\Delta}_{y^{j(\alpha a)}}) = \overset{(\alpha k)}{L}{}^h_{ji} \overset{\alpha k}{\Delta}_{y^{h(\alpha a)}}, \overset{\alpha k}{\mathcal{L}}_{\Delta^{\alpha k}_{y^{i(\alpha b)}}} (\overset{\alpha k}{\Delta}_{y^{j(\alpha a)}}) = \overset{(\alpha b)}{C}{}^h_{ji} \overset{\alpha k}{\Delta}_{y^{h(\alpha a)}}, \tag{19}$$

where $\alpha = 0..k$, $b = 1..k$, $y^{i(0)} = x^i$. The coefficients $(\overset{(\alpha k)}{L}{}^h_{ji}, \overset{(\alpha b)}{C}{}^h_{ji})$ are called the fractional coefficients of linear connection N.

A distinguished fractional tensor field of type $(0, k)$ is given by expression: $\overset{\alpha k}{g} = \overset{\alpha k}{g}_{i_0 i_1 \ldots i_k} d(y^{i_0(0)})^\alpha \otimes \overset{\alpha k}{\delta} y^{i_1(\alpha)} \otimes \cdots \otimes \overset{\alpha k}{\delta} y^{i_k(\alpha k)}$, where $\overset{\alpha k}{\delta} y^{i(\alpha a)}$, $a = 0..k$ are given by (18) and $y^{i(0)} = x^i$.

The covariant derivative with respect to fractional nonlinear connection N of $\overset{\alpha k}{g}$ is defined by:

$$\begin{aligned}
g^{\alpha k}_{i_0 i_1 \ldots i_k | m} &= \overset{\alpha k}{\Delta}_{y^{m(0)}} (g^{\alpha k}_{i_0 i_1 \ldots i_k}) - \overset{(\alpha k)}{L}{}^j_{i_0 m} \overset{\alpha k}{g}_{j i_1 \ldots i_k},\\
g^{\alpha k}_{i_0 i_1 \ldots i_k \overset{(\alpha b)}{|} m} &= \overset{\alpha k}{\Delta}_{y^{m(\alpha b)}} (g^{\alpha k}_{i_0 i_1 \ldots i_k}) - \overset{\alpha k}{C}{}^h_{i_1 m} g^{\alpha k}_{i_0 h \ldots i_k} - \ldots - \overset{\alpha k}{C}{}^h_{i_k m} g^{\alpha k}_{i_0 \ldots h}.
\end{aligned}$$

A fractional metric structure on $\overset{\alpha k}{E}$ is a fractional field of tensors of type $(0, 2)$, $\overset{\alpha k}{g} = \overset{\alpha k}{g}_{ij} d(x^i)^\alpha \otimes d(x^j)^\alpha$, with $\overset{\alpha k}{g}_{ij}(x, y^{(\alpha)}, \ldots, y^{(\alpha k)})$, which is symmetric and positively defined.

The fractional Sasaki lift of $\overset{\alpha k}{g}$ is the fractional field of tensors given by:

$$\overset{\alpha k}{G} = \overset{\alpha k}{g}_{ij} d(x^i)^\alpha \otimes d(x^j)^\alpha + \sum_{a=1}^{k} \overset{\alpha k}{g}_{ij} \delta y^{i(\alpha)} \otimes \delta y^{j(\alpha)}.$$

If the relation $\overset{\alpha k}{g}_{ij|m} = 0$, $\overset{\alpha k}{g}_{ij \overset{\alpha k}{|} m} = 0$ hold, then the fractional linear connection (19) is called metrical.

Proposition 3.6. *On $\overset{\alpha k}{E}$ there is a unique metrical fractional linear connection N with respect to metrical structure $\overset{\alpha k}{G}$ with the property:*

$$\overset{(\alpha k)}{L}{}^i_{jl} = \overset{(\alpha k)}{L}{}^i_{lj}, \ \overset{(\alpha a)}{C}{}^i_{jl} = \overset{(\alpha a)}{C}{}^i_{lj}, \ a = 1..k.$$

The coefficients $\overset{(\alpha k)}{L}{}^i_{jl}$, $\overset{(\alpha k)}{C}{}^i_{jl}$ have the expressions:

$$\begin{aligned}
\overset{(\alpha k)}{L}{}^i_{jl} &= \frac{1}{2} \overset{\alpha k}{g}{}^{is} (\overset{\alpha k}{\Delta}_{x^j} \overset{\alpha k}{g}_{sl} + \overset{\alpha k}{\Delta}_{x^l} \overset{\alpha k}{g}_{js} - \overset{\alpha k}{\Delta}_{x^s} \overset{\alpha k}{g}_{jl}) \\
\overset{(\alpha a)}{C}{}^i_{jl} &= \frac{1}{2} \overset{\alpha k}{g}{}^{is} (\overset{\alpha k}{\Delta}_{y^{j(\alpha a)}} \overset{\alpha k}{g}_{sl} + \overset{\alpha k}{\Delta}_{y^{l(\alpha a)}} \overset{\alpha k}{g}_{js} - \overset{\alpha k}{\Delta}_{y^{s(\alpha a)}} \overset{\alpha k}{g}_{jl}), \ a = 1..k.
\end{aligned}$$

The results from the present paper will permit the study of other geometrical objects which are described with the help of the fractional derivative. Also, we will introduce the fractional osculator Lagrange space of k order and the main structures on it.

REFERENCES

1. Cottrill-Sheperd, K.; Naber, M. – *Fractional differential forms*, J. Math. Phys., 42, 2001, pp 2203-12.
2. Cresson, J. – *Fractional embedding of differential operators and Lagrangian systems*, J. Math. Phys., 38, issue 3, 2007.
3. Ivan, Gh.; Ivan, M.; Opriş, D. – *Fractional dynamics system on fractional Leibnitz algebroids*, to appear.

4. Jumarie, G. – *Lagrangian mechanics of fractional order, Hamilton-Jacobi fractional PDE and Taylor's series of nondifferentiable functions.*, Chaos, Solitons and Fractals, to appear, 2007.

5. Miron, R. – *The geometry of higher order Lagrange spaces. Application in mechanics and physics*, Kluwer Acad. Pub., 82, 1997.

6. Podlubny, I. – *Geometrical and physical interpretation of fractional integration and fractional differentiation.*, Fractional calculus and applied analysis, vol 5, nr. 4, 2002, 367-386.

Received: 15.X.2007

Department of Mathematics,
West University of Timişoara,
ROMÂNIA
albud@math.uvt.ro

Faculty of Economics,
West University of Timişoara,
ROMÂNIA
mihaela.neamtu@fse.uvt.ro

Department of Mathematics,
West University of Timişoara,
opris@math.uvt.ro