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A GENERALIZATION OF MYERS THEOREM

BY

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Dedicated to Academician Radu Miron at his 80th anniversary

Abstract. The Myers theorem extracts some topological properties of a Riemannian manifold (M, g) from the assumptions that its Ricci curvature is uniformly bounded below by a positive constant. The theorem was extended to Finsler manifolds. Proofs of it can be seen in [1], Ch. 7, [3] Ch.7. In 1979, Galloway ([2]) obtains the same topological properties of (M, g) assuming a weaker boundedness hypothesis on the Ricci curvature.

In this paper we show that the version of Myers theorem due to Galloway holds also for Finsler manifolds. So, a positive answer to a problem posed by B. Suceavă in a private communication is provided.

We mention that B. Suceavă proved a Myers type theorem in the spirit of [2] for almost Hermitian manifolds [4].

Our proof is obtained by modifying some points in the proof from [1] and by checking that some facts proved in [2] for Riemannian manifolds hold also for Finsler manifolds.

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1. Preliminaries. We shall use the notations and the terminology from [1] without comments.

Let (M, F) be a Finsler manifold. The Finsler structure F is a function $F : TM \to [0, \infty)$, $(x, y) \to F(x, y)$ which is C^∞ on the slit tangent bundle $TM\backslash 0$, positively homogeneous in y and whose Hessian matrix $g_{ij} := \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^i}$ is positive-definite at every point of $TM\backslash 0$.

The Chern connection is a linear connection in the pull-back bundle $\pi^* TM$ over $TM\backslash 0$, where $\pi : TM \to M$ is the natural projection. It is only h-metrical and it has two curvatures $R_j{}^i{}_{kh}$, $P_j{}^i{}_{kh}$.

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Let be y a non zero element of T_xM. Then $g(x,y) = g_{ij}(x,y)dx^i \otimes dx^j$ is an inner product which is used to measure lengths and angles in T_xM.One calls y a flagpole of the flag (a plane in T_xM) spanned by $l = \frac{y}{F(x,y)}$, and another unit vector V which is orthogonal to the flagpole.

The flag curvature is then given as

$$K(x, y, l \wedge V) := V^i(l^j R_{jikh} l^h)V^k =: V^i R_{ik} V^k. \tag{1.1}$$

The raising and lowering of indices is made by using g^{ij} and g_{ij}, respectively. Sometimes, the flag curvature is denoted simply $K(l, V)$. If V is not a unit vector, then we have $g_{(x,y)}(V, V)K(l, V) = V^i R_{ik} V^i$. Let $\{l, e_\alpha, \alpha = 1, \ldots, n-1\}$ be a g-orthonormal basis for the fiber of π^*TM over the point $(x, y) \in TM\backslash 0$. With respect to it one has $K(x, y, l \wedge e_\alpha) = R_{\alpha\alpha}$. The Ricci scalar denoted by Ric is

$$Ric := \sum_{\alpha=1}^{n-1} K(x, y, l \wedge e_\alpha) = \sum_{\alpha=1}^{n-1} R_{\alpha\alpha}. \tag{1.2}$$

In any basis one gets

$$Ric = g^{ik} R_{ik}. \tag{1.3}$$

The Ricci tensor is defined as follows

$$Ric_{jk} = \frac{1}{2} \frac{\partial^2(F^2 Ric)}{\partial y^j \partial y^k} \tag{1.4}$$

and one shows that

$$Ric = l^j l^k Ric_{jk}. \tag{1.5}$$

Equivalently,

$$Ric(x, y) = \frac{1}{F^2(x, y)}[y^i y^j Ric_{ij}]. \tag{1.6}$$

If (M, F) has constant flag curvature c, then

$$Ric = (n-1)c, \quad Ric_{jk} = (n-1)c g_{jk}. \tag{1.7}$$

Let $\sigma(t), 0 \leq t \leq L$, be a unit geodesic with velocity field T. One abreviates $g_{(\sigma,T)}$ by g_T.

For a vector field $W(t) := W^i(t)\frac{\partial}{\partial x^i}$ along σ, the expression,

$$D_T W = \left[\frac{dW^i}{dt} + W^j T^k (\Gamma^i_{jk}(G,T))\right] \frac{\partial}{\partial x^i} \tag{1.8}$$

is called covariant derivative with reference vector T. The formula 1.8 can be stated for any curve but for geodesics one has

$$\frac{d}{dt} g_T(V,W) = g_T(D_T V, W) + g_T(V, D_T W) \tag{1.9}$$

for any vector fields V, W along σ.

Note that (1.9) holds for any curve if V or W is proportional to T.

The constant speed geodesics are solutions of $D_T T = 0$, with reference vector T.

One says that W is parallel long σ if $D_T W$, with reference vector T. Parallel transport (with reference vector T) one defines on the standard way. By (1.9) the parallel transport preserves g_T-lengths and angles.

For two continuous and piecewise C^∞ vector fields V and W along σ the index form is

$$I(V,W) = \int_0^L [g_T(D_T V, D_T W) - g_T(R(V,T)T, W)]dt. \tag{1.10}$$

Here all D_T are calculated with reference vector T and

$$R(V,T)T := (T^j R^i_{jkh} T^h) V^k \frac{\partial}{\partial x^i}$$

is evaluated at the point (σ, T).

The index form is bilinear and symmetric. We quote from [1] the following facts

Proposition 1.1 [1, p.174] *Let $\sigma(t) = \exp_p(tT)$, $0 \le t \le r$ be a constant speed geodesic from $p = \sigma(0)$ to $q = \sigma(r)$.*

The following five statements are mutally equivalent:

(a) *The point q is not conjugate to p along σ.*

(b) *Any Jacobi field that vanishes as both points p and q must be identically zero along σ.*

(c) *Take the variation field of any variation of σ by geodesics. If it vanishes at p and q, then it must be identically zero along σ.*

(d) Given any $v \in T_pM$ and $w \in T_qM$, there exits a unique Jacobi field J along σ that equals v at p and w at q.

(e) The derivative $\exp_{p*}$ of the exponent map $\exp_p$ is nonsingular at the location rT in T_pM.

Proposition 1.2 ([1,p.182]). *Let $\sigma(t), 0 \le t \le r$ be a geodesic in a Finsler manifold (M, F). Suppose no point $\sigma(t), 0 < t \le r$ is conjugate to $p := \sigma(0)$. Let W be any piecewise C^∞ vector field along σ and let J denote the unique Jacobi field along σ that has the same boundary values as W. That is, $J(0) = W(0)$ and $J(r) = W(r)$. Then*

$$I(W, W) > I(J, J). \tag{1.11}$$

Equality holds if and only if W is actually a Jacobi field, in which case the said J coincides with W.

We close this Section by quoting, for the sake of comparison, the Bonnet-Myers theorem from [1], p. 194:

Let (M, F) be a forward geodesically complete connected Finsler manifold of dimension n. Suppose its Ricci scalar has the following uniform positive lower bound

$$Ric \ge (n-1)\lambda > 0.$$

Equivalently, suppose its Ricci tensor satisfies $y^i y^i Ric_{ij}(x, y) \ge (n-1)\lambda F^2(x, y)$ with $\lambda > 0$. Then:

(1) *Along every geodesic the distance between any two successive conjugate points is at most $\frac{\pi}{\sqrt{\lambda}}$. In other words, every geodesic with length $\frac{\pi}{\sqrt{\lambda}}$ or longer must contain conjugate points.*

(2) *The diameter of M is at most $\frac{\pi}{\sqrt{\lambda}}$.*

(3) *M is in fact compact.*

(4) *The fundamental group $\pi(M, x)$ is finite.*

2. A generalization of Bonnet-Myers theorem. Looking over the proof of Bonnet-Myers theorem given in [1], p.194-198 it comes out that essential is a proof of its first statement.

Thus we give a more general form of this statement as follows:

Lemma 1. *Let* $\sigma(t)$, $0 \le t \le L$ *be a unit speed geodesic with velocity field* T. *If*

$$Ric(T,T) \ge a + \frac{df}{dt}, \text{ for a constant } a > 0 \tag{2.1}$$

and some function f *with* $|f(t)| \le C, C \ge 0$, *and*

$$L \ge \frac{\pi}{a}(c + \sqrt{c^2 + a(n-1)}), \tag{2.2}$$

then σ *must contain conjugate points.*

Remarks.

(i) For $c = 0$ and $a = (n-1)\lambda$, Lemma 2.1 reduces to the assertion (1) of the Bonnet-Myers theorem.

(ii) The condition (2.1) on Ricci allows and negative values of $Ric(T,T)$ along σ.

Proof. Using the parallel transport with reference vector T one construct a moving frame $\{e_i(t)\}$ along σ such that

(i) Each e_i is parallel along σ, that is $D_T e_i = 0$,

(ii) $\{e_i(t)\}$ is a g_T-ortonormal frame,

(iii) $e_n = T$.

Define $W_\alpha(t) = f_\alpha(t)e_\alpha(t)$ for some smooth functions f_α, $\alpha = 1, 2, \ldots, n-1$.

Fix a positive $r \le L$ and consider the index from I for $\sigma(t), 0 \le t \le r$. By (1.10) we have

$$I(W_\alpha, W_\alpha) = \int_0^r [\|D_T W_\alpha\|^2 - \|W_\alpha\| K(T, W_\alpha)]dt,$$

where the abbreviation $\|V\| := g_T(V,V)$ was used and $K(T, W_\alpha)$ is the flag curvature evaluated at the point $(\sigma(t), T) \in TM \backslash 0$.

As $D_T W_\alpha = \frac{df_\alpha}{dt} e_\alpha$, it results $\|D_T W_\alpha\|^2 = |f_\alpha(t)|^2$. It is known that the flag curvature does not depend on vectors spanning the flag. Thus we have $K(T, W_\alpha) = K(T, e_\alpha)$.

Using these facts, $I(W_\alpha, W_\alpha)$ takes the form

$$I(W_\alpha, W_\alpha) = \int_0^r \left[\left(\frac{df_\alpha}{dt}\right)^2 - f_\alpha^2 K(T, e_\alpha)\right] dt.$$

We take $f_\alpha(t) = \sin\frac{\pi t}{r}$ and we get

$$I(W_\alpha, W_\alpha) = \frac{\pi^2}{2r} - \int_0^L \sin^2\frac{\pi t}{r} K(T, e_\alpha) dt.$$

Summing over α one obtains

$$\sum_\alpha I(W_\alpha, W_\alpha) = (n-1)\frac{\pi^2}{2r} - \int_0^r Ric(T,T)\sin^2\frac{\pi t}{r} dt.$$

By hypotheses, $-Ric(T,T) \leq -a - \frac{df}{dt}$. Hence

$$\sum_\alpha I(W_\alpha, W_\alpha) \leq (n-1)\frac{\pi^2}{2r} - \int_0^r (a + \frac{df}{dt})\sin^2\frac{\pi t}{r} dt.$$

An integration by parts gives first

$$\sum_\alpha I(W_\alpha, W_\alpha) \leq (n-1)\frac{\pi^2}{2r} - \frac{ar}{2} + \frac{\pi}{r}\int_0^r f(t)\sin\frac{2\pi t}{r} dt,$$

and then using $|\sin u| \leq 1$ and $\|f(t)| \leq c$, one finds

$$\sum_\alpha I(W_\alpha, W_\alpha) \leq (n+1)\frac{\pi^2}{2r} - \frac{ar}{2} + \pi c$$

and we have $\sum_\alpha I(W_\alpha, W_\alpha) \leq o$ if $r \geq \frac{\pi}{a}(c + \sqrt{c^2 + a(n-1)})$ an inequality that holds for $r = L$ by hypothesis. It follows that some $I(W_\alpha, W_\alpha)$ must be nonpositive and let denote that W_α by W.

We proceed by contradiction. Suppose that $\sigma(t), 0 \leq t \leq r = \frac{\pi}{a}(c + \sqrt{c^2 + a(n-1)})$ contains no conjugate points.

By Proposition 1.1, the vector field W, with $W(0) = W(r) = 0$, can not be a Jacobi field since is nowhere zero on $(0, r)$. And by the same Proposition 1.1 the unique Jacobi field which vanishes at the endpoints of $\sigma(t)$, $0 \leq t \leq r$ is identically zero field. By Proposition 1.2 we have $0 = I(J,J) < I(W,W) \leq 0$, which is a contradiction and lemma is proved. In combination with Theorem 7.5.1 from [1], Lemma 1 tell us that the said geodesic σ minimizes arc length among "nearly" piecewise C^∞ curves from $p = \sigma(0)$ to $q = \sigma(r), r = \frac{\pi}{a}(c + \sqrt{c^2 + a(n-1)})$. The following two consequences of this Lemma cover the content of the Bonnet-Myers theorem.

Theorem 1. *Let (M, F) be a forward geodesically complete connected Finsler manifold. Suppose there exists constants $a > 0$ and $c \geq 0$ such that for every pair of points in M and minimal geodesic σ joining those points having unit tangent vector T, the Ricci curvature satisfies*

$$Ric(T, T) \geq a + \frac{df}{dt} \text{ along } \sigma$$

where f is some function of arclength t satisfying $|f(t)| \leq c$ along σ. Then M is compact and its diam$(M) \leq \frac{\pi}{a}(c + \sqrt{c^2 + a(n-1)})$.

Proof. Since M is forward geodesicaly complete, by the Hopf-Rinow theorem any pair of points in M can be joined by a minimal geodesic. By Lemma 1, such a geodesic must have the length less than or equal with $\frac{\pi}{a}(c + \sqrt{c^2 + a(n-1)})$. Thus diam$(M) \leq \frac{\pi}{a}(c + \sqrt{c^2 + a(n-1)})$ and so M is bounded. Using again the Hopf-Rinow theorem one deduces that M is compact. □

Theorem 2. *Let (M, F) be a forward geodesically complete connected Finsler manifold. Suppose there exist constants $a > 0$ and $c \geq 0$ such that for every pair of points in M (not necessarily distinct) and geodesic σ with unit tangent vector T joining these points, the Ricci curvature satisfies (2.1) where f is some function of the arclength t satisfying $|f(t)| \leq c$ along σ. Then the universal covering manifold of M is compact, with diameter bounded by $\frac{\pi}{a}(c + \sqrt{c^2 + a(n-1)})$, and hence the fundamental group of M is finite.*

Proof. Let $\widetilde{M}$ be the universal covering manifold of M with the universal covering map $p : \widetilde{M} \to M$. In [1] p. 197 one proves that p endows $\widetilde{M}$ with the same local geometry as M. Repeating word by word the proof of Theorem 1.3 from [2] it comes out that $\widetilde{M}$ satisfies the hypothesis of Theorem 2.1, hence it is compact. It follows its closed subset $p^{-1}(x)$ is compact and being discrete is finite. Since $\pi_1(M, x)$ is bijective with $p^{-1}(x)$ it is itself finite. □

REFERENCES

1. BAO, D.; CHEN, S.S.; SHEN, Z. – *An Introduction to Riemannian-Finsler Geometry*, Springer, Grodinate, ... Mathematics, nr. 200, 2000.
2. GALLOWAY, J.G. – *A generalization of Myers theorem and an application relativistic cosmology,* J. Differential Geometry, 14(1979), 105-116.
3. MATSUMOTO, M. – *Foundations of Finsler geometry and special Finsler spaces*, Kaiseisha Preess, 1986.
4. SUCEAVĂ, B. – *A Myers type theorem for almost Hermitian mnaifolds*, Algebra, Geometry and their Applications, Seminar Proceedins, 2002, p. 1-4.

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