ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" DIN IAŞI (S.N.)
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LAX FORMULATION AND KAHAN'S INTEGRATOR FOR SOME CLASSICAL DYNAMICAL SYSTEMS

BY

ANANIA ARON, IOAN CAŞU, OANA CHIŞ and COSMIN HOGEA

Dedicated to Academician Radu Miron at his 80th anniversary

Abstract. Lax formulation and numerical integration via Kahan's integrator for some classical dynamical systems are introduced and some of their properties are pointed out.

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Key words: Lax formulation, Ishii equation, Bogoyavlensky lattice, Voltera lattice, Kahan integrator, Runge-Kutta integrator.

1. Introduction. In the last time there was a great deal of interest in the geometrical study of concrete dynamical systems.

The goal of our paper is to discuss the Lax formulation, the numerical integration via Kahan's integrator for Ishii equation, 2-dimensional Bogoyavlensky lattice, 2-dimensional Volterra lattice and to point out some of their properties.

2. Ishii's equation. The third order Ishii's equation, see [1] and [3] has the following form:

$$\dddot{x} = \dot{x}x.$$

Written as a system of differential equations it can be expressed as:

$$\left\{\begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = x_1x_2 \end{array}\right. \tag{2.1}$$

Proposition 2.1. ([1]) *The smooth functions* $C_1, C_2 \in C^\infty(\mathbb{R}^3, \mathbb{R})$ *given respectively by*

$$C_1(x_1, x_2, x_3) = x_1x_3 - \frac{1}{2}x_2^2 - \frac{1}{3}x_1^3$$

[*resp.*]

$$C_2(x_1, x_2, x_3) = x_3 - \frac{1}{2}x_1^2$$

are constants of motion for the dynamics (2.1).

A long but straightforward computation or using eventually MATHEMATICA leads us to:

Proposition 2.2. *The dynamics* (2.1) *has the following Lax formulation:*

$$\dot{L} = [L, B],$$

where

$$L = \begin{bmatrix} 0 & 1-m_3 & m_1 & 0 & 0 \\ -1+m_3 & 0 & im_3 & 0 & 0 \\ -m_1 & -im_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1+m_2 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & im_2 & 0 & 0 & 0 \\ -im_2 & 0 & m_2 & 0 & 0 \\ 0 & -m_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_2 & 0 \\ 0 & 0 & 0 & 1-2m_2+m_2^2-m_2 & 0 \end{bmatrix},$$

$i = \sqrt{-1}$.

Remark 2.1. Via the general theory of Lax formulation, we have that

$$Trace(L^2)$$

is a constant of motion. Furthermore,

$$Trace(L^2) = -1 - 4C_2,$$

and then our assertion can be obtained also via the Proposition (2.1).

Let us observe now that Kahan's integrator [4], associated to the dynamics (2.1) has the following form:

$$\left\{\begin{array}{l} x_1^{n+1}-x_1^n=\dfrac{h}{2}\left(x_2^{n+1}+x_2^n\right) \\ x_2^{n+1}-x_2^n=\dfrac{h}{2}\left(x_3^{n+1}+x_3^n\right) \\ x_3^{n+1}-x_3^n=\dfrac{h}{2}\left(x_1^{n+1}x_2^n+x_2^{n+1}x_1^n\right) \end{array}\right. \tag{2.2}$$

Proposition 2.3. (2.2) *has the following properties:*

(i) *It does not preserve the constant of motion* C_1.

(ii) *It does not preserve the constant of motion* C_2.

3. Two-dimensional Bogoyavlensky lattice. The dynamics of the two-dimensional Bogoyavlenski lattice, see [2], is described by the following set of differential equations:

$$\left\{\begin{array}{l} \dot{x}_1=x_1x_2+x_1x_3 \\ \dot{x}_2=x_2x_3-x_1x_2 \\ \dot{x}_3=-x_1x_3-x_2x_3 \end{array}\right. \tag{3.1}$$

Let D be the open set of $\mathbb{R}^3$ given by:

$$D=\{(x_1,x_2,x_3)\in\mathbb{R}^3|x_2=0\}$$

Then we have:

Proposition 3.1. *The smooth functions* $C_1, C_2 \in C^\infty(D,\mathbb{R})$ *given respectively by:*

$$C_1(x_1,x_2,x_3)=x_1+x_2+x_3$$

[*resp.*]

$$C_2(x_1,x_2,x_3)=\frac{x_1x_3}{x_2}$$

are constants of motion for the dynamics (3.1).

A long but straightforward computation or using eventually MATHEMATICA leads us to:

Proposition 3.2. *The dynamics* (3.1) *has the following Lax formulation:*

$$\dot{L} = [L, B],$$

where

$$L = \begin{bmatrix} 0 & ia & 0 & 0 & 0 & 0 & 0 & 0 \\ -ia & 0 & a & 0 & 0 & 0 & 0 & 0 \\ 0 & -a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & ib & 0 & 0 \\ 0 & 0 & 0 & -b & -ib & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 & 0 & 0 & -c & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & -im_1 & 0 & 0 & 0 & 0 & 0 \\ im_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & im_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -im_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & m_1 \end{bmatrix},$$

$i = \sqrt{-1}$, $a = m_2 + m_3$, $\quad b = m_1 + m_2$, $\quad c = m_1 + m_2 + m_3$.

Remark 3.1. Via the general theory of Lax formulation, we have that

$$Trace(L^2)$$

is a constant of motion. Furthermore,

$$Trace(L^2) = -2C_1^2,$$

and then our assertion can be obtained also via the Proposition 2.1.

Let us observe now that Kahan's integrator [4], associated to the dynamics (2.1) has the following form:

$$(3.2) \qquad \begin{cases} x_1^{n+1} - x_1^n = \dfrac{h}{2}\left(x_1^{n+1}x_2^n + x_2^{n+1}x_1^n + x_1^{n+1}x_3^n + x_3^{n+1}x_1^n\right) \\ x_2^{n+1} - x_2^n = \dfrac{h}{2}\left(x_2^{n+1}x_3^n + x_3^{n+1}x_2^n - x_1^{n+1}x_2^n - x_2^{n+1}x_1^n\right) \\ x_3^{n+1} - x_3^n = \dfrac{h}{2}\left(x_1^{n+1}x_3^n + x_3^{n+1}x_1^n + x_2^{n+1}x_3^n + x_3^{n+1}x_2^n\right) \end{cases}$$

Proposition 3.3. *Kahan's integrator* (3.2) *has the following properties:*

(i) *It preserves the constant of motion* C_1.

(ii) *It preserves the constant of motion* C_2.

Proof. The proof can be obtained by direct computation or using eventually MATHEMATICA. □

If we make a comparisson with the 4^{th} order Runge-Kutta integrator see the figures (3.1)-(3.3), we obtain almost the same results. However Kahan's integrator (3.2) has the advantages to be easier to implement.

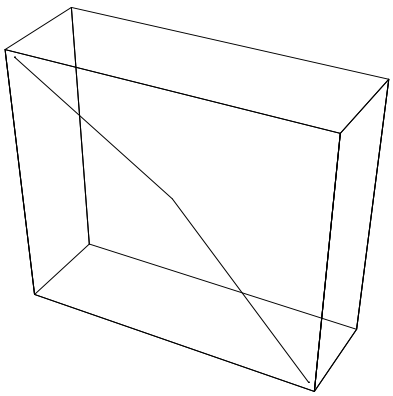

Figure 3.1: the 4th order Runge-Kutta integrator

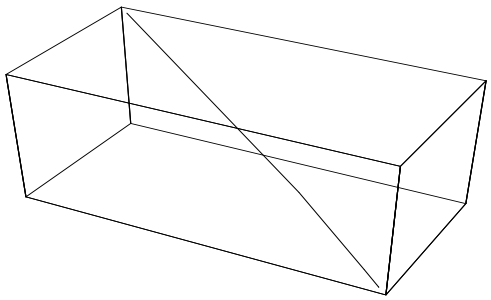

Figure 3.2: Kahan's integrator

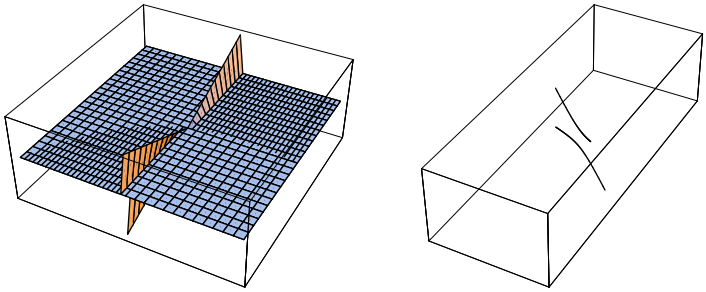

Figure 3.3: The phase curves of the system (3.1)

4. Two-dimensional Volterra lattice. The dynamics of the two-dimensional Volterra lattice, see [5], is described by the following set of differential equations:

$$\left\{\begin{array}{l} \dot{x}_1 = x_1x_2 \\ \dot{x}_2 = x_2x_3 - x_1x_2 \\ \dot{x}_3 = x_3x_4 - x_2x_3 \\ \dot{x}_4 = -x_4x_3 \end{array}\right. \tag{4.1}$$

Proposition 4.1. *The smooth functions $C \in C^\infty(\mathbb{R}^4, \mathbb{R})$ given by:*

$$C(x_1, x_2, x_3) = x_1 + x_2 + x_3 + x_4$$

is a constant of motion for the dynamics (4.1).

A long but straightforward computation or using eventually MATHEMATICA leads us to:

Proposition 4.2. *The dynamics* (4.1) *has the following Lax formulation:*

$$\dot{L} = [L, B],$$

where

$$L = \begin{bmatrix} \sum\limits_{i=1}^{4} x_i & x_3 & 0 & 0 & 0 & 0 \\ 0 & \sum\limits_{i=1}^{4} x_i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_2 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} x_2 - x_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -x_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -x_1 + x_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Remark 4.1. Via the general theory of Lax formulation, we have that

$$Trace(L^2)$$

is a constant of motion. Furthermore,

$$Trace(L^2) = -2C^2,$$

and then our assertion can be obtained also via the Proposition (4.1).

Let us observe now that Kahan's integrator [4], associated to the dynamics (4.1) has the following form:

$$(4.2) \qquad \begin{cases} x_1^{n+1} - x_1^n = \dfrac{h}{2}\left(x_1^{n+1}x_2^n + x_2^{n+1}x_1^n\right) \\ x_2^{n+1} - x_2^n = \dfrac{h}{2}\left(x_2^{n+1}x_3^n + x_3^{n+1}x_2^n - x_1^{n+1}x_2^n - x_2^{n+1}x_1^n\right) \\ x_3^{n+1} - x_3^n = \dfrac{h}{2}\left(x_3^{n+1}x_4^n + x_4^{n+1}x_3^n - x_2^{n+1}x_3^n + x_3^{n+1}x_2^n\right) \\ x_4^{n+1} - x_4^n = -\dfrac{h}{2}\left(x_3^{n+1}x_4^n + x_4^{n+1}x_3^n\right) \end{cases}$$

Proposition 4.3. *Kahan's integrator* (4.2) *preserves the constant of motion C.*

Proof. If we add the all relations (4.2) we have immediately:

$$x_1^{n+1} + x_2^{n+1} + x_3^{n+1} + x_4^{n+1} = x_1^n + x_2^n + x_3^n + x_4^n,$$

and so the desired result. □

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Math Department,
"Politehnica" University of Timişoara,
Piaţa Victoriei nr. 2,
300006-Timişoara,
ROMÂNIA
anania.girban@gmail.com

Department of Mathematics,
West University of Timişoara,
B - dul V. Pârvan nr. 4,
300223 - Timişoara,
ROMÂNIA
chisoana@yahoo.com