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SOME REMARKS ON (sl(2,$\mathbb{R}$))* AND KAHAN'S INTEGRATOR

BY

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Dedicated to Academician Radu Miron at his 80th anniversary

Abstract. We study some geometrical and dynamical properties of a quadratic and homogeneous Hamilton-Poisson system on the dual of the Lie algebra sl(2,$\mathbb{R}$) with its minus-Lie-Poisson structure. Its numerical integration via Kahan's integrator is also pointed out.

Mathematics Subject Classification 2000: 53D17.

Key words: Numerical integration, Kahan's integrator, Runge Kutta integrator, spectral stability.

1. The geometrical picture of the problem. Let $SL(2,\mathbb{R})$ be the group of real 2×2 matrices with determinant one, i.e.

$$SL(2,\mathbb{R})=\left\{\left[\begin{array}{cc} a & b \\ c & d \end{array}\right] | a,b,c,d\in\mathbb{R},\ ad-bc=1\right\}.$$

It is a closed subgroup of $GL(2,\mathbb{R})$ and so it is a Lie group. Its Lie algebra is:

$$sl(2.\mathbb{R})=\left\{\left[\begin{array}{cc} a & b \\ c & -a \end{array}\right] | a,b,c,\in\mathbb{R}\right\}.$$

Let $\{E_1,E_2,E_3\}$ [resp.$\{e_1,e_2,e_3\}$] be the natural basis of $sl(2,\mathbb{R})$ [resp.$\mathbb{R}^3$], i.e.,

$$E_1=\left[\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right]; E_2=\left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right]; E_3=\left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array}\right];$$

[resp.]

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Let us define a Lie algebra structure $[.,.]$ on $\mathbb{R}^3$, by:

$[.,.]$	e_1	e_2	e_3
e_1	0	$2e_2$	$-2e_3$
e_2	$-2e_2$	0	e_1
e_3	$2e_3$	$-e_1$	0

Then a long but straightforward computation leads us to:

Proposition 1.1. ([3]) *The map* $\Phi : (sl(2,\mathbb{R}), [.,.]) \longrightarrow (\mathbb{R}^3, [.,.])$ *given by:*

$$\begin{cases} \Phi(E_1) =^{def} e_1 \\ \Phi(E_2) =^{def} e_2 \\ \Phi(E_3) =^{def} e_3 \end{cases}$$

is an isomorphism of Lie algebras.

In all that follows we shall identify $sl(2,\mathbb{R}), (sl(2,\mathbb{R}))*$ and $\mathbb{R}^3$, as a vector spaces and then the minus-Lie-Poisson structure on $(sl(2,\mathbb{R}))* \simeq \mathbb{R}^3$ is generated by the matrix:

$$\Pi_- = \begin{bmatrix} 0 & -2x_2 & 2x_3 \\ 2x_2 & 0 & -x_1 \\ -2x_3 & x_1 & 0 \end{bmatrix}.$$

Definition 1.1. *A quadratic and homogeneous Hamilton-Poisson system on* $(sl(2,\mathbb{R}))^* \simeq \mathbb{R}^3$ *is the triple* $(\mathbb{R}^3, \Pi_-, H)$, *where* $H \in C^\infty(\mathbb{R}^3, \mathbb{R})$ *is given by:*

$$H(x_1,x_2,x_3) = \frac{1}{2}(b_1x_1^2 + b_2x_2^2 + b_3x_3^2) + d_1x_2x_3 + d_2x_1x_3 + d_3x_1x_2,$$

$$b_1, b_2, b_3 \in \mathbb{R}, d_1, d_2, d_3 \in \mathbb{R}, b_1^2 + b_2^2 + b_3^2 + d_1^2 + d_2^2 + d_3^2 \neq 0.$$

Its dynamics is described by the following set of differential equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \Pi_- \cdot \nabla H,$$

or equivalent:

$$(1.1)\qquad \begin{cases} \dot{x}_1 = -2x_2(b_2x_2 + d_3x_1) + 2x_3(b_3x_3 + d_2x_1) \\ \dot{x}_2 = 2x_2(b_1x_1 + d_2x_3 + d_3x_2) - x_1(b_3x_3 + d_1x_2 + d_2x_1) \\ \dot{x}_3 = -2x_3(b_1x_1 + d_2x_3 + d_3x_2) + x_1(b_2x_2 + d_1x_3 + d_3x_1) \end{cases}$$

Remark 1.1. ([3]) It is not hard to see that the smooth function $C \in C^\infty(\mathbb{R}^3, \mathbb{R})$ given by:

$$C(x_1, x_2, x_3) = x_1^2 + 4x_2x_3$$

is a Casimir of our Poisson configuration $(\mathbb{R}^3, \Pi_-)$.

Remark 1.2. ([3]) The phase curves of the dynamics (1.1) are intersections of the surfaces $H(x_1, x_2, x_3) = constant$ and $C(x_1, x_2, x_3) = constant$.

2. Numerical integration of the dynamics (1.1) via Kahan's integrator. It is well known that Kahan's integrator [2] for the dynamics (1.1) can be written in the following form:

$$(2.1)\qquad \begin{cases} x_1^{n+1} - x_1^n = h(-d_3(x_1^n x_2^{n+1} + x_2^n x_1^{n+1}) - 2b_2 x_2^n x_2^{n+1} \\ \qquad\qquad +d_2(x_1^n x_3^{n+1} + x_3^n x_1^{n+1}) + 2b_3 x_3^n x_3^{n+1}) \\ x_2^{n+1} - x_2^n = \dfrac{h}{2}(-2d_2 x_1^n x_1^{n+1} + (2b_1 - d_1)(x_1^n x_2^{n+1} + x_2^n x_1^{n+1}) \\ \qquad\qquad +4d_3 x_2^n x_2^{n+1} - b_3(x_1^n x_3^{n+1} + x_3^n x_1^{n+1}) \\ \qquad\qquad +2d_2(x_2^n x_3^{n+1} + x_3^n x_2^{n+1})) \\ x_3^{n+1} - x_3^n = \dfrac{h}{2}(2d_3 x_1^n x_1^{n+1} + b_2(x_1^n x_2^{n+1} + x_2^n x_1^{n+1}) \\ \qquad\qquad +(-2b_1 + d_1)(x_1^n x_3^{n+1} + x_3^n x_1^{n+1}) \\ \qquad\qquad -2d_3(x_2^n x_3^{n+1} + x_3^n x_2^{n+1}) - 4d_2 x_3^n x_3^{n+1}) \end{cases}$$

Proposition 2.1. *Kahan's integrator* (2.1) *is Hamiltonian* (*energy*) *preserving* [1] *if and only if one of the following conditions holds:*

(i)

$$b_3 = \frac{4b_1^2 - 4b_1d_1 + d_1^2}{b_2}; d_2 = 0, d_3 = 0,$$
$$b_2 \in \mathbb{R}, b_2 \neq 0, b_3 \in \mathbb{R}, b_2b_3 > 0, d_1 \in \mathbb{R}.$$

(ii)

$$b_1 = \frac{1}{2}d_1; b_2 = 0, d_2 = 0, d_3 = 0,$$
$$b_3 \in \mathbb{R}, d_1 \in \mathbb{R}.$$

(iii)

$$b_1 = \frac{1}{2}d_1; b_3 = 0, d_2 = 0, d_3 = 0,$$
$$b_2 \in \mathbb{R}, d_1 \in \mathbb{R}.$$

(iv)

$$b_2 = 0; b_3 = 0; d_2 = 0; d_3 = 0,$$
$$b_1 \in \mathbb{R}, d_1 \in \mathbb{R}.$$

(v)

$$b_1 = \frac{-b_2^2d_2 + b_2d_1d_3 + 2d_3^2}{2b_2d_3}; b_3 = \frac{b_2d_2^2}{d_3^2},$$
$$b_2 \in \mathbb{R}, b_2 \neq 0, d_3 \in \mathbb{R}, d_3 \neq 0, d_1 \in \mathbb{R}d_2 \in \mathbb{R}.$$

(vi)

$$b_1 = \frac{b_3d_1 + 3d_2^2}{2b_3}; b_2 = \frac{d_2^4}{b_3^3}; d_3 = -\frac{d_2^3}{b_3^2},$$
$$b_3 \in \mathbb{R}, b_3 \neq 0, d_1 \in \mathbb{R}, d_2 \in \mathbb{R}.$$

(vii)

$$b_1 = \frac{b_3d_1d_2 + 2d_2^3 - b_3^2d_3}{2b_3d_2}; b_2 = \frac{b_3d_3^2}{d_2^2},$$
$$b_3 \in \mathbb{R}, b_3 \neq 0, d_1 \in \mathbb{R}, d_2 \in \mathbb{R}, d_2 \neq 0, d_3 \in \mathbb{R}.$$

(viii)

$$b_1 = \frac{b_2d_1 + 2d_3^2}{2b_2}; b_3 = 0; d_2 = 0$$
$$b_2 \in \mathbb{R}, b_2 \neq 0, d_1 \in \mathbb{R}, d_3 \in \mathbb{R}.$$

(ix)

$$b_1 = \frac{b_3 d_1 + 2d_2^2}{2b_3}; b_2 = 0; d_3 = 0$$

$$b_3 \in \mathbb{R}, b_3 \neq 0, d_1 \in \mathbb{R}, d_2 \in \mathbb{R}, d_3 \in \mathbb{R}.$$

Proof. The proof can be obtained via a long but straightforward computation or it can be obtained using eventually MATHEMATICA. □

Similar arguments lead us to:

Proposition 2.2. ([1]) *Kahan's integrator* (2.1) *is Casimir preserving if and only if one of the conditions* $(i) - (v)$ *and* (ix) *from the Proposition* 2.1 *holds.*

Proposition 2.3. *Kahan's integrator* (2.1) *is Poisson preserving if and only if one of the conditions* (i), (ii), (vii), (viii) *from the Proposition* 2.1 *holds.*

In the cases (iii)-(ix) mentioned above we have sketched the 4th order Runge-Kutta integrator and Kahan's integrator (see the figures 2.1, 2.2, ..., 2.13)

(iii)

$$b_1 = \frac{1}{2} d_1; b_3 = 0, d_2 = 0, d_3 = 0, b_2 \in \mathbb{R}, d_1 \in \mathbb{R}.$$

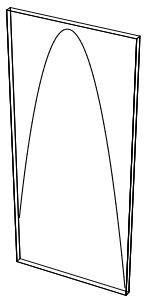

Figure 2.1: 4th order Runge-Kutta integrator

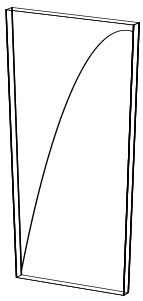

Figure 2.2: Kahan's integrator

(iv)

$$b_2 = 0; b_3 = 0; d_2 = 0; d_3 = 0, b_1 \in \mathbb{R}, d_1 \in \mathbb{R}.$$

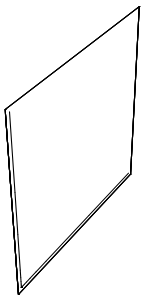

Figure 2.3: 4th order Runge-Kutta integrator

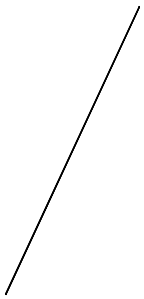

Figure 2.4: Kahan's integrator

(v)

$$b_1 = \frac{-b_2^2 d_2 + b_2 d_1 d_3 + 2d_3^2}{2b_2 d_3}; b_3 = \frac{b_2 d_2^2}{d_3^2},$$
$$b_2 \in \mathbb{R}, b_2 \neq 0, d_3 \in \mathbb{R}, d_3 \neq 0, d_1 \in \mathbb{R} d_2 \in \mathbb{R}.$$

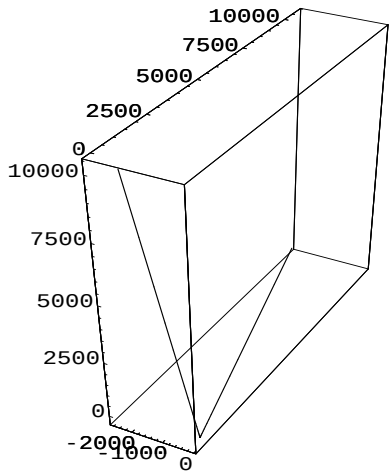

Figure 2.5: 4th order Runge-Kutta integrator

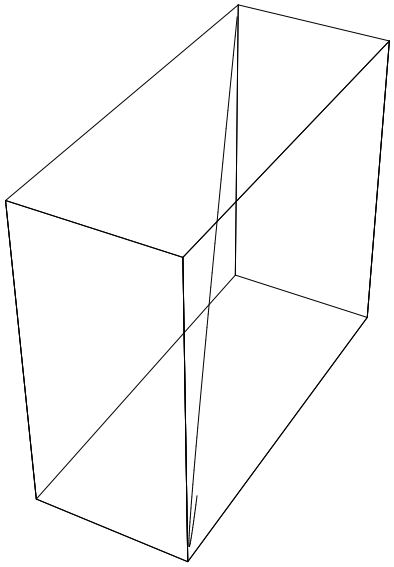

Figure 2.6: Kahan's integrator

(vi)

$$b_1 = \frac{b_3 d_1 + 3d_2^2}{2b_3}; b_2 = \frac{d_2^4}{b_3^3}; d_3 = -\frac{d_2^3}{b_3^2},$$
$$b_3 \in \mathbb{R}, b_3 \neq 0, d_1 \in \mathbb{R}, d_2 \in \mathbb{R}.$$

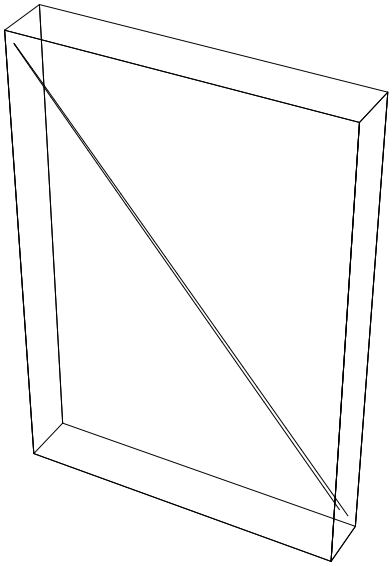

Figure 2.7: 4th order Runge-Kutta integrator

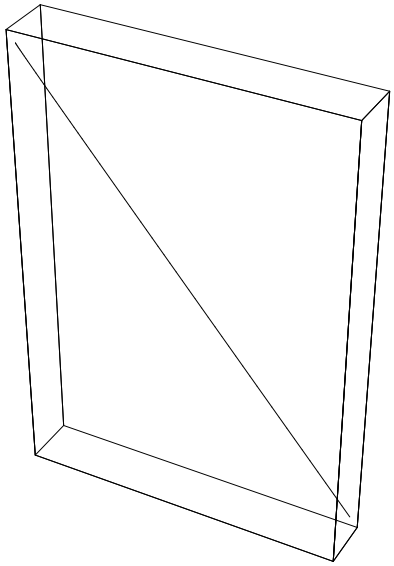

Figure 2.8: Kahan's integrator

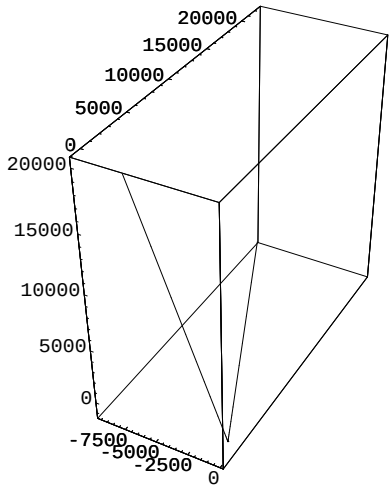

Figure 2.9: 4th order Runge-Kutta integrator

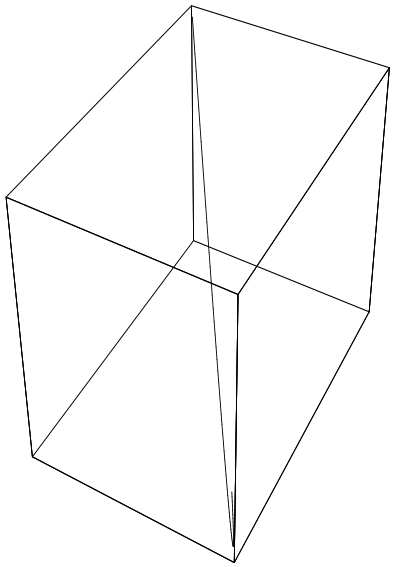

Figure 2.10: Kahan's integrator

(vii)

$$b_1 = \frac{b_3 d_1 d_2 + 2d_2^3 - b_3^2 d_3}{2b_3 d_2}; b_2 = \frac{b_3 d_3^2}{d_2^2},$$

$$b_3 \in \mathbb{R}, b_3 \neq 0, d_1 \in \mathbb{R}, d_2 \in \mathbb{R}, d_2 \neq 0, d_3 \in \mathbb{R}.$$

(viii)

$$b_1 = \frac{b_2 d_1 + 2d_3^2}{2b_2}; b_3 = 0; d_2 = 0$$

$$b_2 \in \mathbb{R}, b_2 \neq 0, d_1 \in \mathbb{R}, d_3 \in \mathbb{R}.$$

In this case the 4th order Runge-Kutta integrator fails.

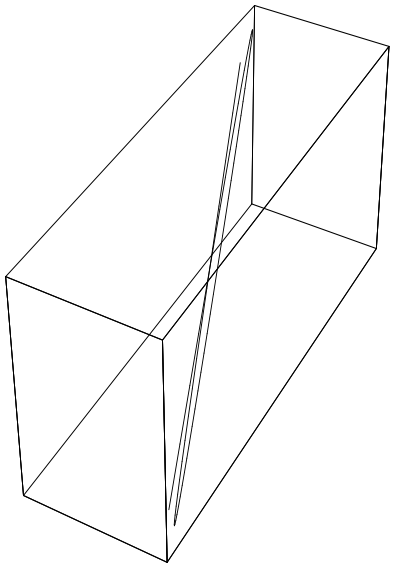

Figure 2.11: Kahan's integrator

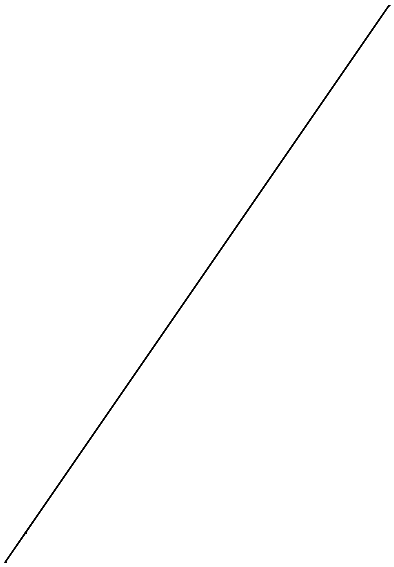

Figure 2.12: 4th order Runge-Kutta integrator

(ix)

$$b_1 = \frac{b_3 d_1 + 2d_2^2}{2b_3}; b_2 = 0; d_3 = 0$$

$$b_3 \in \mathbb{R}, b_3 \neq 0, d_1 \in \mathbb{R}, d_2 \in \mathbb{R}, d_3 \in \mathbb{R}.$$

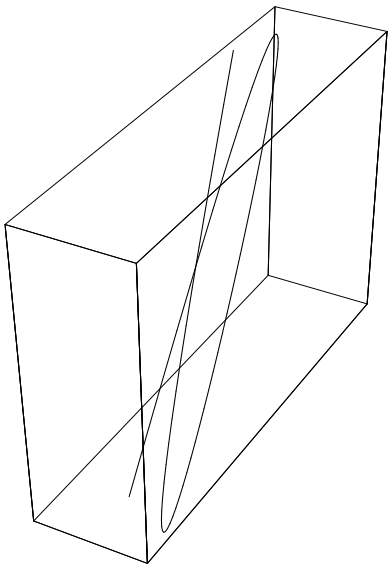

Figure 2.13: Kahan's integrator

We shall discuss the spectral stability of the equilibrium states of the system (1.1) the cases (iii)-(ix) presented in Proposition 2.1.

Case (iii). The equilibrium states of our dynamics (1.1) have the following form:

$$e_{31}^{MN} = (M, 0, N); M, N \in \mathbb{R}.$$

Proposition 3.1. *The equilibrium states* $e_{31}^{MN}, M, N \in \mathbb{R}$ *are spectrally stable.*

Proof. Indeed, the characteristic polynomial of the matrix corresponding to the linear part of our system at the equilibrium e_{31}^{M} has the following form:

$$-x^3$$

and then our assertion follows immediately. □

Case (iv). The equilibrium states of the dynamics (1.1) have the following form:

$$e_{41}^{M} = (M, 0, 0); M \in \mathbb{R}, \quad e_{42}^{MN} = (0, M, N); M, N \in \mathbb{R}.$$

Proposition 3.2. *The equilibrium states e_{41}^M have the following behaviour:*

(i) *If $d_1 \neq 2b_1, M \neq 0$, they are unstable.*

(ii) *If $d_1 = 2b_1 or M = 0$, they are spectrally stable.*

Proof. Indeed, the characteristic polynomial of the matrix corresponding to the linear part of our system at the equlibrium e_1^M has the following form

$$-x(x + 2b_1M - d_1M)(x - 2b_1M + d_1M)$$

and then our assertion follows immediately. □

Similar arguments lead us to:

Proposition 3.3. *The equilibrium states e_{42}^{MN} are spectrally stable.*

Case (v). The equilibrium states of the dynamics (1.1) are:

$$e_{51}^M = \left(-\frac{2b_2d_2M}{d_3^2}, \frac{d_2M}{d_3}, M\right); M \in \mathbb{R}$$

$$e_{52}^M = \left(\frac{2d_3M}{b_2}, \frac{d_2M}{d_3}, M\right); M \in \mathbb{R}$$

$$e_{53}^M = \left(\frac{2d_3M}{b_2}, -\left(\frac{d_2}{d_3} + \frac{2d_3^2}{b_2^2}\right)M, M\right); M \in \mathbb{R}$$

$$e_{54}^M = \left(\frac{2d_3^2M}{b_2d_2}, M, -\frac{(b_2^2d_2d_3 + 2d_3^4)M}{b_2^2d_2^2}\right); M \in \mathbb{R}$$

$$e_{55}^{MN} = \left(-\frac{b_2M}{d_3} - \frac{b_2d_2N}{d_3^2}, M, N\right); M, N \in \mathbb{R}.$$

Proposition 3.4. *The equilibrium states e_{51}^M are spectrally stable.*

Proposition 3.5. *The equilibrium states e_{52}^M have the following behaviour:*

(i) *If $(b_2^2d_2 + d_3^2)/d_3 > 0$ and$M \neq 0$, they are unstable.*

(ii) *If $(b_2^2d_2 + d_3^2)/d_3 \leq 0$ or$M = 0$, they are spectrally stable.*

Proposition 3.6. *The equilibrium states $e_{53}^M, e_{54}^M, e_{55}^M$ are spectrally stable.*

Case (vi). The equilibrium states of the dynamics (1.1) are:

$$e_{61}^{M} = \left(-\frac{2b_3M}{d_2}, -\frac{b_3^2M}{d_2^2}, M\right); M \in \mathbb{R}$$

$$e_{62}^{M} = \left(\frac{2d_2M}{b_3}, M, -\frac{d_2^2M}{b_3^2}\right); M \in \mathbb{R}$$

$$e_{63}^{MN} = \left(\frac{d_2M}{b_3} - \frac{b_3N}{d_2}, M, N\right); M, N \in \mathbb{R}.$$

Proposition 3.7. *The equilibrium states* $e_{61}^{M}, e_{62}^{M}, e_{63}^{MN}$ *are spectrally stable.*

Case (vii). The equilibrium states of the dynamics (1.1) are:

$$e_{71}^{M} = \left(-\frac{2b_3M}{d_2}, \frac{d_2M}{d_3}, M\right); M \in \mathbb{R}$$

$$e_{72}^{M} = \left(\frac{2d_2^2M}{b_3d_3}, \frac{d_2M}{d_3}, M\right); M \in \mathbb{R}$$

$$e_{73}^{M} = \left(\frac{2d_2^2M}{b_3d_3}, -\frac{(2d_2^4 + b_3^2d_2d_3)M}{b_3^2d_3^2}, M\right); M \in \mathbb{R}.$$

$$e_{74}^{M} = \left(\frac{2d_2M}{b_3}, M, -\left(\frac{2d_2^2}{b_3^2} + \frac{d_3}{d_2}\right)M\right); M \in \mathbb{R}.$$

$$e_{75}^{MN} = \left(-\frac{b_3d_3M}{d_2^2} - \frac{b_3N}{d_2}, M, N\right); M, N \in \mathbb{R}$$

Proposition 3.8. *The equilibrium states* e_{71}^{M}, e_{73}^{M} *and* e_{75}^{MN} *are spectrally stable.*

Proposition 3.9. *The equilibrium states* e_{72}^{M} *have the following behaviour:*

(i) *If* $(d_2^3 + b_3^2d_3)/d_2 > 0\, M \neq 0$, *they are unstable.*

(ii) *If* $(d_2^3 + b_3^2d_3)/d_2 \leq 0$ *or* $M = 0$, *they are spectrally stable.*

Remark 3.1. The spectral stability [resp. instability] of the equilibrium states e_{72}^{M} can be also discussed but the results are more complicated and so we shall not give any other details.

Case (viii). The equilibrium states of the dynamics (1.1) are:

$$e_{81}^M = (0,0,M), M \in \mathbb{R}, \quad e_{82}^M = \left(\frac{2d_3M}{b_2}, 0, M\right); M \in \mathbb{R},$$

$$e_{83}^M = \left(\frac{2d_3M}{b_2}, -\frac{2d_3^2M}{b_2^2}, M\right); M \in \mathbb{R}, \quad e_{84}^{MN} = \left(-\frac{b_2M}{d_3}, M, N\right); M, N \in \mathbb{R}.$$

Proposition 3.10. *The equilibrium states e_{81}^M, e_{83}^M and e_{84}^{MN} are spectrally stable.*

Proposition 3.11. *The equilibrium states e_{82}^M have the following behaviour:*

(i) *If $d_3M \neq 0$, they are unstable.*

(ii) *If $d_3M = 0$, they are spectrally stable.*

Case (ix). The equilibrium states of the dynamics (1.1) have the following form:

$$e_{91}^M = \left(\frac{2d_2M}{b_3}, M, 0\right); M \in \mathbb{R}, \quad e_{92}^{MN} = \left(-\frac{b_3N}{d_2}, M, N\right); M, N \in \mathbb{R}.$$

Proposition 3.12. *The equilibrium states e_{91}^M, e_{92}^M are spectrally stable.*

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