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CONSTANT MEAN CURVATURE SUBMANIFOLDS IN $(\alpha, \beta)-$FINSLER SPACES

BY

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Dedicated to Academician Radu Miron at his 80th anniversary

Abstract. The mean curvature form is explicitely determined for certain Randers and Kropina spaces with non-constant 1-form. This extended framework includes for constant 1-form the Randers and Kropina subcases, and extends the results from [7] and [2, 1], respectively.

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1. Introduction. Within the framework of Finsler submanifolds, a significant advance has been established by SHEN ([6]) by deriving the Busemann-Hausdorff mean curvature of a Finsler submanifold. Further, for (α, β)-Finsler submanifolds with constant β-form, SOUZA and TENENBLAT ([7]) and the author ([1, 2]) have investigated Randers minimal hypersurfaces, and constant mean curvature (in particular, minimal) Kropina hypersurfaces, respectively.

In the present work, we obtain the explicit formulas for the mean curvature form in the more general case, when the 1-form β is non-constant. These lead, in the constant β case, to the results mentioned above.

Let $(\tilde{M}, \tilde{F})$ be a Finsler structure, and let $\varphi : (M, F) \to (\tilde{M}, \tilde{F})$ be an isometric immersion, with F induced by $\tilde{F}$. Then the following result holds true ([6]):

Theorem 1. *The mean curvature of M is given by* ([6, (57), p. 563])

$$\tilde{H}_\varphi(X) = \frac{1}{G}\left(G_{;x^i} - G_{;z^i_a z^j_b}\varphi^j_{;u^a u^b} - G_{;x^j z^i_a}\varphi^j_{;u^a}\right)X^i, \tag{1.1}$$

where lower indices stand for corresponding partial derivatives and:

- $(u^a, v^b)_{a,b\in\overline{1,n}}$ *are local coordinates in* TM *(*$\dim M = n$*);*
- $(x^i, y^j)_{i,j\in\overline{1,m}}$ *are local coordinates in* $T\tilde{M}$ *(*$\dim \tilde{M} = m$*);*
- z^i_a *are the entries of the Jacobian matrix* $[J(\varphi)] = (\partial\varphi^i/\partial u^a)_{a=\overline{1,n},i=\overline{1,m}}$*;*
- $\varphi_t : M \to \tilde{M}, t \in (-\varepsilon, \varepsilon), \varphi_0 = \varphi$*, is the variation of the surface;*
- X *is the vector field* $X_x = \frac{\partial\varphi_t}{\partial t}\,|_{\,t=0}(x)$ *induced along* φ *attached to the variation;*
- G *is the Finsler induced volume form*

$$G(z) = \frac{vol[B^n]}{vol\{(v^a) \in \mathbb{R}^n \mid \tilde{F}(v^a z^i_a \tilde{e}_i) \le 1\}}, \tag{1.2}$$

where $z = (z^i_a)_{a=\overline{1,n},i=\overline{1,m}} \in GL_{m\times n}(\mathbb{R})$*,* $\tilde{e} = \{\tilde{e}_i\}_{i=\overline{1,m}}$ *is an arbitrary basis in* $\mathbb{R}^m$ *and* $B^n \subset \mathbb{R}^n$ *is the standard Euclidean ball.*

It was proved that the variation of the volume of M reaches a minimum for $H_\varphi = 0$ ([6]).

Let $\tilde{F} : T\tilde{M} \to \mathbb{R}$ be a positive 1-homogeneous locally Minkowski Finsler fundamental function ([5, 4, 3]). In the following we shall study the Finsler hypersurfaces ($\dim \tilde{M} = m = n + 1$) for the following cases:

a) Randers space $(\tilde{M}, \tilde{F})$ with the fundamental function

$$\tilde{F}(x, y) = \sqrt{\delta_{ij}y^i y^j} + b_{n+1}(x)y^{n+1}, \tag{1.3}$$

reducible for $b_{n+1} = const. = b \in [0, 1)$ to the case studied by Souza and Tenenblat ([7]), in which the minimal surfaces of revolution have been characterized.

b) Kropina space $(\tilde{M}, F)$ with the fundamental function

$$\tilde{F}(x, y) = \frac{\delta_{ij}y^i y^j}{b_{n+1}(x)y^{n+1}}, \tag{1.4}$$

reducible for $b_{n+1} = const. = b \in (0, 1)$ to the case studied in [1, 2], where the minimal surfaces of revolution have been explicitly determined.

2. The mean curvature form. Let $M = Im\ \varphi$, $\varphi : D \subset \mathbb{R}^n \to \tilde{M} = \mathbb{R}^{n+1}$ be a simple hypersurface. Using the notations $z^i_\alpha = \frac{\partial \varphi^i}{\partial u^\alpha}$, $u = (u^1, \dots, u^n) \in D$, $h_{\alpha\beta} = \sum_{i=1}^{n+1} z^i_\alpha z^j_\beta$, $b = b_{n+1}(x)$ and

$$B = b(x)^2 z^{n+1}_a z^{n+1}_b h^{ab}, \quad C = \sqrt{\det(h_{\alpha\beta})},$$

we have the following:

Theorem 2. a) *In the Randers case* (1.3)*, the Finsler volume form coefficient is* ([7, p.627])

$$G_R = C \cdot (1 - B)^{(n+1)/2},$$

b) *In the Kropina case* (1.4)*, the volume form coefficient is* ([1, 2])

$$G_K = C \cdot \left(\frac{2}{\sqrt{B}}\right)^n.$$

We note that the mean curvature field (1.1) *rewrites:*

$$\tilde{H}_i = \frac{1}{G}\left(\frac{\partial G}{\partial x^i} - \frac{\partial^2 G}{\partial z^i_\varepsilon z^j_\eta} \cdot \frac{\partial^2 \varphi^j}{\partial u^\varepsilon \partial u^\eta} - \frac{\partial^2 G}{\partial x^j z^i_\varepsilon} \cdot \frac{\partial \varphi^j}{\partial u^\varepsilon}\right), i = \overline{1, n+1}, \tag{2.1}$$

and the volume form of the hypersurface M is

$$dV_F = G du^1 \wedge \cdots \wedge du^n, \quad G \in \{G_R, G_K\}.$$

Moreover, one can derive by tedious computation the explicit expressions of the field (2.1) in both cases, as follows

Theorem 3. a) *The mean curvature field of the hypersurface M in the Randers space $\tilde{M} = \mathbb{R}^{n+1}$ with the fundamental function* (1.3) *has the following expression*

(2.2)

$$\begin{aligned}\tilde{H}_i = & -\frac{(n+1)Bb_i}{b(1-B)} - \frac{1}{C(1-B)^2}\left[\frac{(n^2-1)}{4}\frac{\partial B}{\partial z^i_\varepsilon}\frac{\partial B}{\partial z^j_\eta}C + (1-B)^2\frac{\partial^2 C}{\partial z^i_\varepsilon \partial z^j_\eta}\right.\\ & \left. - \frac{n+1}{2}(1-B)\left(\frac{\partial^2 B}{\partial z^i_\varepsilon \partial z^j_\eta}C + \frac{\partial B}{\partial z^j_\eta}\frac{\partial C}{\partial z^i_\varepsilon} + \frac{\partial B}{\partial z^i_\varepsilon}\frac{\partial C}{\partial z^j_\eta}\right)\right]\frac{\partial^2 \varphi^j}{\partial u^\varepsilon \partial u^\eta}\\ & + \frac{(n+1)b_j}{bC(1-B)}\left[B\frac{\partial C}{\partial z^i_\varepsilon} + \left(1 - \frac{B(n-1)}{2(1-B)}\right)C\frac{\partial B}{\partial z^i_\varepsilon}\right]\frac{\partial \varphi^j}{\partial u^\varepsilon}.\end{aligned}$$

for $i = \overline{1, n+1}$.

b) *The mean curvature field of the hypersurface* M *in the Kropina space* $\tilde{M} = \mathbb{R}^{n+1}$ *with the fundamental function* (1.4) *has the following expression* (2.3)

$$\tilde{H}_i = \frac{nb_j}{bC}\left(\frac{\partial C}{\partial z^i_\varepsilon} - \frac{nC}{2B}\frac{\partial B}{\partial z^i_\varepsilon}\right)\frac{\partial \varphi^j}{\partial u^\varepsilon} - \frac{1}{B^2C}\left[\frac{\partial^2 C}{\partial z^i_\varepsilon \partial z^j_\eta}B^2 + \frac{n(n+2)}{4}C\frac{\partial B}{\partial z^i_\varepsilon}\frac{\partial B}{\partial z^j_\eta} - \frac{nB}{2}\left(\frac{\partial C}{\partial z^i_\varepsilon}\frac{\partial B}{\partial z^j_\eta} + \frac{\partial C}{\partial z^j_\eta}\frac{\partial B}{\partial z^i_\varepsilon} + C\frac{\partial^2 B}{\partial z^i_\varepsilon \partial z^j_\eta}\right)\right]\frac{\partial^2 \varphi^j}{\partial u^\varepsilon \partial u^\eta} - \frac{nb_i}{b},$$

for $i = \overline{1, n+1}$, *where we have denoted* $b_i = \partial b/\partial x^i$. Then the following result characterizes the CMC surfaces in Randers and Kropina spaces:

Corollary 1. *The CMC equation for nonconstant potential* $b_{n+1}(x)$ *in a Randers / Kropina space is* $\tilde{H}_i X^i = const$, *with* $\tilde{H}$ *is given by* (2.2) / (2.3) *respectively, where*

$$X = ||N||^{-1}_{\tilde{F}} \cdot N, \quad N^i = \varepsilon^{ijk}(G_* Z^1)_j (G_* Z^2)_k,$$

$Z^1 = (z^1_1, z^2_1, z^3_1)$, $Z^2 = (z^1_2, z^2_2, z^3_2)$, $G_* v$ *is defined by the equality*

$$(G_* v)(v') = \langle v, v' \rangle_F = \frac{1}{2}\frac{\partial F^2}{\partial y^i \partial y^j} v^i v'^j,$$

"" is the Euclidean Hodge operator and* ε^{ijk} *is the skew-symmetrization symbol.*

3. Particular cases. In Kropina spaces with constant potential, for a CMC Monge patch defined by $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ and parametrized by $\varphi : D \to \mathbb{R}^3$,

$$\varphi(u^1, u^2) = (u^1, u^2, f(u^1, u^2)) = \sum_{i=1}^{3}(u^1\delta_{i1} + u^2\delta_{i2} + f\delta_{i3}),$$

by choosing $X = \varphi_{u^1} \times \varphi_{u^2} = -\delta_{i1}f_1 - \delta_{i2}f_2 + \delta_{i3}$, we infer the CMC Monge patch equation $\tilde{H}_i X^i = const$ for

$$\tilde{H}_i = \frac{4C}{E^3}\left[6E^2\frac{\partial C}{\partial z^i_\varepsilon}\frac{\partial C}{\partial z^j_\eta} + 2C^2\frac{\partial E}{\partial z^i_\varepsilon}\frac{\partial E}{\partial z^j_\eta} - C^2E\frac{\partial^2 E}{\partial z^i_\varepsilon \partial z^j_\eta} - 3CE\left(\frac{\partial E}{\partial z^i_\varepsilon}\frac{\partial C}{\partial z^j_\eta} + \frac{\partial E}{\partial z^j_\eta}\frac{\partial C}{\partial z^i_\varepsilon}\right) + 3E^2C\frac{\partial^2 C}{\partial z^i_\varepsilon \partial z^j_\eta}\right]\frac{\partial^2 \varphi^j}{\partial u^\varepsilon \partial u^\eta}, \tag{3.1}$$

for all $i=\overline{1,n+1}$, where $E=BC^2$. Then we have the following:

Corollary 2. *The coefficients in the equation* (3.1) *have the explicit form:*

$$\begin{aligned}\frac{\partial C}{\partial z^i_\varepsilon}X^i &= f_2C^{-1}[-f_1(\delta_{\varepsilon 1}f_2-\delta_{\varepsilon 2}f_1)+\delta_{\varepsilon 2}]\\ \frac{\partial E}{\partial z^i_\varepsilon}X^i &= 2b^2f_2(-1)^\varepsilon(f_1f_{\tilde\varepsilon}+\delta_{\varepsilon 2}),\end{aligned}$$

where $\tilde\varepsilon=3-\varepsilon,\varepsilon=\overline{1,2}$,

$$\begin{aligned}\frac{\partial E}{\partial z^j_\eta}\frac{\partial^2\varphi^j}{\partial u^\varepsilon\partial u^\eta} &= 2b^2f_2f_{\varepsilon 2};\ \ \tfrac{\partial C}{\partial z^j_\eta}\tfrac{\partial^2\varphi^j}{\partial u^\varepsilon\partial u^\eta}=f_2f_{\varepsilon 2}C^{-1}\\ \frac{\partial^2C^2}{\partial z^i_\varepsilon\partial z^j_\eta}\frac{\partial^2\varphi^j}{\partial u^\varepsilon\partial u^\eta}X^i &= f_{22}-f_2^2f_{11}+f_1f_2f_{12}\\ \frac{\partial^2E}{\partial z^i_\varepsilon\partial z^j_\eta}\frac{\partial^2\varphi^j}{\partial u^\varepsilon\partial u^\eta}X^i &= 2b^2[f_{22}+f_{11}(f_2^2-f_1f_2)+f_1f_{22}],\end{aligned}$$

and where the lower indices denote partial derivatives w.r.t. the corresponding domain-variables.

As well, we note that in the Randers case, one can infer for $b=const.$ the results obtained in [7]. Similarly, in the Kropina case one gets the following existence result [2]:

Corollary 3. *Let* $M=\Sigma=Im\ \varphi$ *be a surface of revolution in a Kropina space* (1.4) *with* $b=const.$*, described by*

$$\varphi(t,\theta)=(f(t)\cos\theta,f(t)\sin\theta,t),\ \ (t,\theta)\in D=\mathbb{R}\times[0,2\pi).$$

Then M *is minimal iff the function* f *satisfies the ODE*

$$1+f'^2=3ff''(1+2f'^2), \tag{3.2}$$

whose general solution is given by

$$f(t)=\tau^{-1}(t+C_2),\qquad \tau(s)=\pm\int\frac{-2ds}{\sqrt{-4-2\cdot L(-2C_1\cdot e^{-2}s^{-2/3})}},$$

where $0<-C_1<<1$ *and* L *is the Lambert function defined implicitly by* $L(x)\cdot e^{L(x)}=x$.

Remark. Since the equation $ye^y = x$ has an infinite number of solutions y for each (non-zero) value of x, out of the infinite number of branches, $L(x)$ is the only one which is analytic at 0 (the principal branch of the Lambert function).

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