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NATURAL 2-π STRUCTURES IN LAGRANGE SPACES

BY

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Dedicated to Academician Radu Miron at his 80th anniversary

Abstract. We investigate the concepts of almost 2-π structure, almost 2-π Riemannian structure, almost symplectical 2-π structure on TM and n-almost contact 2-π structure, n-almost contact Riemannian structure on Osc^2M.

These structures clearly exist in the natural case when they are given by (1.1), (2.6), (3.10), (4.1). The theorems 1.1, 2.2, 3.1 and 4.1 point out some important properties. We investigate also the conditions of normality for these structures.

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Introduction. In the geometry of Lagrange spaces $L^n = (M, L)$ one introduces in a natural way an almost complex structures F induced by a canonical nonlinear connection. Then whenever one considers the Lagrange spaces of order 2, $L^{(2)n} = (M, L)$ the previous structure become more general, that is, an almost contact stucture F , also generated by a canonical nonlinear connection which together with the Sasakian lift G of fundamental metric tensor of space define a natural almost contact metric structure denoted by $H^{3n} = (Osc^2M, G, F)$ which represents, the geometrical model of background space.

In our further considerations, we shall prove that above the kinds of structures can be extend to 2-π structures studied by us in the works [1], [2], [3], [4].

Moreover, in the case of spaces $L^n = (M, L)$ we shall prove that there exists a natural 2-π structure induced by the canonical nonlinear connection

of the space, and the structure together with Sasakian's lift G of the fundamental tensor generates a metrical 2-π structure on the total space of the tangent bundle TM. Also, we shall show that the metric canonical linear connection of the space L^n is compatible with the corresponding structures F and G.

Finally we shall prove that the almost symplectical structure associated is integrable. In general for Lagrange spaces of order 2, $L^{(2)n}$ we shall show that there exists an almost contact natural 2-π structure F generated by the nonlinear canonical connection N.

Afterwards, we prove that any N-linear connection D is compatible with the structure F, that is, $DF = 0$ and by considering the lift of Sasaki's type G for the fundamental tensor of the space, we show that the pair (F, G) is an almost contact metrical 2-π structure. The normality of this structure is characterized.

1. Almost 2-π structure on TM. Let M be a real n-dimensional differentiable manifold of the class C^∞ and (TM, Π, M) its tangent bundle. We denote by $(x^i, y^i)\,(i, j, h = 1, ..., n)$ the canonical coordinates of every point $u = (x, y) \in TM$. The local coordinate transformations $(x, y) \to (\tilde{x}, \tilde{y})$ on TM are usually given by

$$\begin{cases} \tilde{x}^i = \tilde{x}^i(x^1, ..., x^n),\ det \cdot \left\| \dfrac{\partial \tilde{x}^i}{\partial x^j} \right\| \neq 0 \\ \tilde{y}^i = \dfrac{\partial \tilde{x}^i}{\partial x^j} y^j. \end{cases} \tag{1.1}$$

The Liouville vector field Γ is defined by $\Gamma = y^i \frac{\partial}{\partial y^i}$ and the vertical distribution V on TM is locally generated by the system of vector fields $(\frac{\partial}{\partial y^i})$.

It is known that a nonlinear connection N is a regular distribution N on TM such that we have the following direct sum $T_u(TM) = N_u \oplus V_u$ for any point $u \in TM$. There exists a local adapted basis of distribution N and V given by $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}\,(i = 1, ..., n)$ where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j}. \tag{1.2}$$

The system of the functions N_i^j are the coefficients of nonlinear connection N.

Let $\{dx^i, \delta y^i\}\,(i = 1, ..., n)$ be the dual basis of adapted basis. We have

$$\delta y^i = dy^i + N_i^j dx^j. \tag{1.2'}$$

Definition 1.1. [2] *An almost 2-π structure on TM is defined as an $\mathcal{F}(TM)$-linear mapping $\mathbf{F} : \chi(TM) \to \chi(TM)$ with the property*

$$\mathbf{F} \circ \mathbf{F} = -\lambda^2 I, \tag{1.3}$$

where $\lambda \neq 0$ is a complex number, I being a Kronecker tensor.

One says that $\mathbf{F}$ is integrable if the Nijeuhius tensor $N_{\mathbf{F}} = 0$, $N_{\mathbf{F}}$ being given by

$$N_{\mathbf{F}}(X, Y) = \mathbf{F}^2[X, Y] + [\mathbf{F}X, \mathbf{F}Y] - \mathbf{F}[\mathbf{F}X, Y] - \mathbf{F}[X, \mathbf{F}Y]. \tag{1.4}$$

We have

Theorem 1.1 *If N is a nonlinear connection on TM and $\left\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right\}$ is the adapted basis of distribution N then the $\mathcal{F}(TM)$-linear mapping $\mathbf{F} : \chi(TM) \to \chi(TM)$ defined by*

$$\mathbf{F}\left(\frac{\delta}{\delta x^i}\right) = -\lambda\frac{\partial}{\partial y^i},\ \mathbf{F}\left(\frac{\partial}{\partial y^i}\right) = \lambda\frac{\delta}{\delta x^i}\,(i = 1, 2, ..., n) \tag{1.5}$$

is an almost 2-π structure on TM.

Proof. From (1.5) one sees that the structure $\mathbf{F}$ is well defined that this is it does not depend on the transformation of coordinates (1.1) on TM. Then (1.5) implies

$$\mathbf{F}^2\left(\frac{\delta}{\delta x^i}\right) = \lambda^2\frac{\delta}{\delta x^i},\ \mathbf{F}^2\left(\frac{\partial}{\partial y^i}\right) = -\lambda^2\frac{\partial}{\partial y^i}. \tag{1.6}$$

□

Theorem 1.2. *The almost 2-π structure $\mathbf{F}$ given by* (1.5) *is integrable, if and only if, the torsion T^i_{jh} and the curvature tensor R^i_{jh} of the nonlinear connection N are equal with zero, where*

$$T^i_{jh} = \frac{\partial N^i_j}{\partial y^h} - \frac{\partial N^i_h}{\partial y^j},\ R^i_{jh} = \frac{\delta N^i_j}{\delta x^h} - \frac{\delta N^i_h}{\delta x^j}. \tag{1.7}$$

Proof. Since we have $\left[\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^h}\right] = R^i_{jh}\frac{\partial}{\partial y^i}$, $\left[\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^h}\right] = \frac{\partial N^i_j}{\delta y^h}\cdot\frac{\partial}{\partial y^i}$ through the agency of Nijeuhnis tensor's relation (1.4) we obtain

$$
\begin{aligned}
&N_F\left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^h}\right) = -\lambda^2\left(R^i_{jh}\frac{\partial}{\delta y^i} - T^i_{jh}\frac{\delta}{\delta x^i}\right), N_F\left(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^h}\right) \\
&= -\lambda^2\left(T^i_{jh}\frac{\partial}{\partial y^i} - R^i_{jh}\frac{\delta}{\delta x^i}\right), N_F\left(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^h}\right) = -\lambda^2 R^i_{jh}\frac{\partial}{\delta y^i}
\end{aligned} \tag{1.8}
$$

which prove that $T^i_{jh} = 0$, $R^i_{jh} = 0$ implies $N_F = 0$ and reciprocally. □

In all our further consideration the integrable almost 2-π integrable structure will be called 2-π structure.

2. Metrical almost 2-π structure on TM. Let $L^n = (M, L)$ be a Lagrange space. The Lagrangian L is the fundamental function of the space and

$$g_{ij}(x, y) = \frac{1}{2}\cdot\frac{\partial^2 L}{\partial y^i \partial y^j} \tag{2.1}$$

is the fundamental tensor of the space $L^n = (M, L)$ [7]. From the definition of Lagrange space we have that

$$rank\|g_{ij}(x, y)\| = n. \tag{2.2}$$

We can consider the contravariant $g^{ij}(x, y)$ of the fundamental tensor.

We have in [6] the following result:

Theorem 2.1. *For any Lagrange space L^n there exists a nonlinear connection N depending only of the fundamental function L having the coefficients given by*

$$N^i_j = \frac{\partial G^i}{\partial y^j} \tag{2.3}$$

where

$$G^i = \frac{1}{4}g^{ij}\left(\frac{\partial^2 L}{\partial y^i \partial x^h}y^h - \frac{\partial L}{\partial x^j}\right). \tag{2.3′}$$

With these coefficients the connection is called the canonical connection (if the space L^n is a Finsler space F^n, then it coincides with famous Cartan connection).

The adapted basis of distribution N and V is defined by $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\delta y^i}\right)$ with $\frac{\delta}{\delta x^i}$ from (1.2), N_i^j given by (2.3) and (2.3') and the dual basis $(dx^i, \delta y^i)$ defined in (1.2') with the same coefficients N_i^j of canonical nonlinear connection.

Then, the Sasakian lift G of the fundamental tensor has the following expression

$$\mathbf{G} = g_{ij} dx^i \otimes dx^j + g_{ij} \delta y^i \otimes \delta y^j. \tag{2.4}$$

We have the following property:

Theorem 2.2. *The pair* $(\mathbf{F}, \mathbf{G})$ *is an almost Riemannian 2-*π *structure.*

Proof. It is necessary to verify the relation

$$\mathbf{G}(\mathbf{F}X, \mathbf{F}Y) = -\lambda^2 \mathbf{G}(X, Y),\ \forall X, Y \in \chi(TM).$$

We have

$$\mathbf{G}\left(\mathbf{F}\frac{\delta}{\delta x^i}, \mathbf{F}\frac{\delta}{\delta x^j}\right) = \lambda^2 \mathbf{G}\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = -\lambda^2 g_{ij} = -\lambda^2 \mathbf{G}\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right)$$

$$\mathbf{G}\left(\mathbf{F}\frac{\delta}{\delta x^i}, \mathbf{F}\frac{\partial}{\delta y^j}\right) = -\lambda^2 \mathbf{G}\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right) = 0 = -\lambda^2 \mathbf{G}\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right)$$

$$\mathbf{G}\left(\mathbf{F}\frac{\partial}{\partial y^i}, \mathbf{F}\frac{\partial}{\partial x^j}\right) = \lambda^2 \mathbf{G}\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = \lambda^2 g_{ij} = -\lambda^2 \mathbf{G}\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right)$$

□

From the book [6] we have

Theorem 2.3. *Any metrical* N*-connection* D *with* h*- and* v*-metric torsion zero has the coefficients given by the generalized symbols of Christoffel*

$$\begin{cases} L_{jh}^i = \frac{1}{2} g^{im} \left(\frac{\delta g_{mh}}{\delta x^j} + \frac{\delta g_{jm}}{\delta x^h} - \frac{\delta g_{jh}}{\delta x^m} \right) \\ C_{jh}^i = \frac{1}{2} g^{im} \left(\frac{\partial g_{mh}}{\partial y^j} + \frac{\partial g_{jm}}{\delta y^h} - \frac{\partial g_{jh}}{\delta y^m} \right). \end{cases} \tag{2.5}$$

It follows that we have h-and v-covariant derivatives of the fundamental tensor equal with zero $g_{ij|h} = 0$, $g_{ij}|^h = 0$ which implies that $D_X \mathbf{G} = 0$ for all $X \in \chi(TM)$. One check that $D_X \mathbf{F} = 0$.

Now, we consider almost simplectical 2-π structure Θ associated to the structure $(\mathbf{F}, \mathbf{G})$ and defined by

$$\Theta(X, Y) = \mathbf{G}(\mathbf{F}X, Y). \tag{2.6}$$

We have

$$\Theta\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = 0, \Theta\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right) = -\lambda g_{ij}, \Theta\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = 0.$$

Therefore

$$\Theta = -\lambda g_{ij} \delta y^i \otimes dx^j. \tag{2.7}$$

Theorem 2.4. *For any Lagrange space, almost symplectical 2-π structure Θ is integrable.*

Proof. The almost symplectical 2-π structure is integrable, if and only if its exterior differential is equal with zero.

Because, any Lagrange space L^n one has the property $D\Theta_1 = 0$ where $\Theta_1 = g_{ij}\delta y_i \otimes dx^j$ (see [7]) it follows that $\Theta = \lambda\Theta_1$, has the same property.□

Remark 2.1. It is clear that the above theory remains valid whenever the space L^n is replaced by the Finsler space F^n and it can be extended to the study of the general almost 2-π structure on tangent bundle [1],[2],[3],[4].

3. n-almost contact 2-π structure on OSC^2M. The extension of the above theory to the case of the 2-osculator bundle is not immediate because the almost 2-π structures are not specifically for the differentiable manifolds Osc^2M and, in general, their existence is not ensured.

We shall demonstrate in this section that on Osc^2M endowed with a nonlinear connection there exists a natural n-almost contact 2-π structure compatible also with a Sasakian metric in the case of Lagrange spaces of order 2.

First of all, we need some preliminary considerations.

Let us consider again M be a real n-dimensional differentiable manifold and (Osc^2M, Π, M) its 2-osculator bundle.

It is know from [6] that the points of Osc^2M are osculator spaces of order 2 of manifold M.

We denote by $(x^i, y^{(1)i}, y^{(2)i})$ the canonical coordinates for any point $u = (x, y^{(1)}, y^{(2)}) \in Osc^2M$. Then, the canonical coordinate transformations $(x, y^{(1)}, y^{(2)}) \to (\tilde{x}, \tilde{y}^{(1)}, \tilde{y}^{(2)})$ are given by

$$\begin{cases} \tilde{x}^i = \tilde{x}^i(x^1, ..., x^n),\ rank \left\| \dfrac{\partial \tilde{x}^i}{\partial x^j} \right\| = n,\ \tilde{y}^{(1)i} = \dfrac{\partial \tilde{x}^i}{\partial x^j} y^{(1)j} \\ 2\tilde{y}^{(2)i} = \dfrac{\partial \tilde{y}^{(1)i}}{\partial x^j} y^{(1)j} + 2\dfrac{\partial \tilde{y}^{(1)i}}{\partial y^{(1)j}} y^{(2)j} \end{cases} \tag{3.1}$$

and there exists a tangent structure $J : \chi(E) \to \chi(E)$ on the manifold $E = Osc^2M$ defined by

$$J\left(\frac{\delta}{\delta x^i}\right) = \frac{\partial}{\partial y^{(1)i}}, J\left(\frac{\partial}{\partial y^{(1)i}}\right) = \frac{\partial}{\partial y^{(2)i}}, J\left(\frac{\partial}{\partial y^{(2)i}}\right) = 0. \tag{3.2}$$

On Osc^2M there exists the vertical distributions V_1, V_2 with the properties $V_1 \supset V_2$, $dim\, V_1 = 2n$, $dim\, V_2 = n$ [6].

All these distributions are integrable and there exists a number of two Liouville vector fields linearly independent $\overset{1}{\Gamma}, \overset{2}{\Gamma}$.

We have the following properties:

$rank\|J\| = 2n, ImJ = V_1, kerJ = V_2, ImJ|_{V_1} = V_2, ImJ|_{V_2} = 0$ and $J\overset{2}{\Gamma} = \overset{1}{\Gamma}, \overset{0}{\Gamma} = 0$, J is an integrable structure.

A 2-spray on Osc^2M [6] is a vertical field S on Osc^2M with the property $JS = \overset{2}{\Gamma}$.

Then S has form

$$S = y^{(1)i}\frac{\delta}{\delta x^i} + 2y^{(2)i}\frac{\partial}{\partial y^{(1)i}} - 3G^i(x, y^{(1)i}, y^{(2)i})\frac{\partial}{\partial y^{(2)i}}. \tag{3.3}$$

In [6] one proves that the fundamental geometrical objects like as the nonlinear connections, N-linear connections and so on are generated by a 2-spray and any nonlinear connection on Osc^2M is defined now as a regular distribution N_0 on Osc^2M supplementary of the vertical distribution. Writing $N_1 = J(N_0)$, $V_2 = J(N_1)$ for any nonlinear connection N_0 one obtains the decomposition in direct sum of tangent bundle at $E = Osc^2M$ given by

$$T_uE = N_0(u) \oplus N_1(u) \oplus V_2(u). \tag{3.4}$$

All the geometrical objects on Osc^2M are considered with respect to this direct sum and the corresponding locally adapted basis to the decomposition (3.4) is

$$\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, \frac{\delta}{\delta y^{(2)i}}\right), \tag{3.5}$$

where

$$\begin{aligned}\frac{\delta}{\delta x^i} &= \frac{\partial}{\partial x^i} - N^m_{i(1)}\frac{\partial}{\partial y^{(1)m}} - N^m_{i(2)}\frac{\partial}{\partial y^{(2)m}}, \frac{\delta}{\delta y^{(1)i}} \\ &= \frac{\partial}{\partial y^{(1)i}} - N^m_{i(1)}\frac{\partial}{\partial y^{(2)m}}, \frac{\delta}{\delta y^{(2)i}} = \frac{\partial}{\partial y^{(2)i}}\end{aligned} \tag{3.6}$$

in which the system of functions $N^i_{j(1)}, N^i_{j(2)}$ give the coefficients of the nonlinear connection N_0. The dual basis of (3.5) is defined by

$$\{\delta x^i, \delta y^{(1)i}, \delta y^{(2)i}\} \tag{3.7}$$

where

$$\begin{aligned}\delta x^i &= dx^i, \delta y^{(1)i} = dy^{(1)i} + M^i_{j(1)}dx^j, \delta y^{(2)i} \\ &= dy^{(2)i} + M^i_{j(1)}dy^{(1)} + M^i_{j(2)}dx^j.\end{aligned} \tag{3.8}$$

The coefficients $M^i_{j(1)}, M^i_{j(2)}$ are named the dual coefficients of nonlinear connection N_0 and there exists a relationship of duality between the direct coefficients and dual coefficients for any nonlinear connection.

In [6] one indicates a method of calculus for dual coefficients when a 2-spray is given.

We can define now the natural n-almost contact 2-π structure.

Definition 3.1. *A k-almost contact 2-π structure on $E = Osc^2M$ is a system*

$$(\mathbf{F}, \xi_1, ..., \xi_k, \eta^1, ..., \eta^k), \tag{3.9}$$

where $\mathbf{F}$ is a tensor field of type (1.1) on $\tilde{Osc^2}M = Osc^2M \setminus \{0\}$, $\xi_1, ..., \xi_k$ are the linear independent vector fields and $\eta^1, ..., \eta^k$ are 1-form fields, such that we have

$$\begin{cases}\mathbf{F}^2(X) = -\lambda^2 X + \lambda^2 \sum_{a=1}^{k} \eta^a(X)\xi_a, \forall X \in \chi(E) \\ \mathbf{F}(\xi_a) = 0,\ \eta^a(\xi_b) = \delta^a_b,\ (a, b = 1, ..., k)\end{cases} \tag{3.10}$$

where λ is a non zero complex number.

Then it follows that $\xi_1, ..., \xi_k$ belong to the kernel of the mapping $\mathbf{F}$ and we have

$$\mathbf{F}^3 + \lambda^2 \mathbf{F} = 0. \tag{3.11}$$

The Nijeuhnis tensor is given by the relation (1.4) and the normality condition is

$$N_{\mathbf{F}}(X, Y) + \lambda^2 \sum_{a=1}^{n} d\eta^a(X, Y) = 0, \ \ \forall X, Y \in \chi(E). \tag{3.12}$$

We prove that in a Lagrange space $L^{(2)n} = (M, L)$ there exists n-almost contact 2-π structure.

We have

Theorem 3.1. *Let N be a nonlinear connection on $\tilde{E} = Osc^2 M$ having the adapted basis $\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, \frac{\delta}{\delta y^{(2)i}}\right)$. On $\tilde{E}$ there exists an n-almost contact 2-π structure* (3.9) *when $\mathbf{F}$ is the $\mathcal{F}$-linear mapping defined by*

$$\begin{aligned} \mathbf{F}\left(\frac{\delta}{\delta x^i}\right) &= -\lambda \frac{\delta}{\delta y^{(2)i}}, \\ \mathbf{F}\left(\frac{\delta}{\delta y^{(1)i}}\right) &= 0, \\ \mathbf{F}\left(\frac{\delta}{\delta y^{(2)i}}\right) &= \lambda \frac{\delta}{\delta x^i} \end{aligned} \tag{3.13}$$

and

$$\xi_a = \frac{\delta}{\delta y^{(1)a}}, \eta^a = \delta y^{(1)a} \, (a = 1, 2, ..., n). \tag{3.14}$$

Proof. From (3.13) one remark that $\mathbf{F}$ is globally defined on $\tilde{E}$. We verify the relation (3.10) in the base of relations (3.13) and (3.14).

Clearly, the second and the third lines of (3.10) is valid because ξ_a from (3.14) belong to Ker $\mathbf{F}$ and η^a are their duals.

Let us verify the first relation from (3.10). Indeed, we have

$$\mathbf{F}^2\left(\frac{\delta}{\delta x^i}\right) = \mathbf{F}\left(-\lambda \frac{\delta}{\delta y^{(1)i}}\right) = -\lambda^2 \frac{\delta}{\delta x^i}$$

or equivalently

$$\mathbf{F}^2\left(\frac{\delta}{\delta x^i}\right) = -\lambda^2\frac{\delta}{\delta x^i} + \sum_{a=1}^{n}\left[\delta y^a\left(\frac{\delta}{\delta x^i}\right)\xi_a\right]$$

because the last sum is zero.

In the same way one verifies (3.10) for $\chi = \frac{\delta}{\partial y^{(1)i}}$ and $\chi = \frac{\delta}{\partial y^{(2)i}}$ q.e.d.

The n-almost contact 2-π structure will be called natural.

The above theorem leads immediately to. □

Corollary 3.1 *The natural n-almost contact 2-π structure has the following properties:*

1^0. $\mathbf{F}$ *is a tensor field of type* (1.1) *globally defined on* $\tilde{E}$.

2^0. $Ker\mathbf{F} = N_1, Im\mathbf{F} = N_0 \oplus V_2$.

3^0. $rank\|F\| = 2n$

4^0. $\mathbf{F}^3 + \lambda^2\mathbf{F} = 0$

The problem which must solved now is to specify conditions of normality for this structure. First of all we have

Theorem 3.2. *The natural n-almost contact 2-π structure (3.9) is normal if and only if we have*

$$N_{\mathbf{F}}(X,Y) + \lambda^2\sum_{a=1}^{n} d(\delta^{(1)}y^a)(X,Y) = 0. \tag{3.15}$$

Proof. One applies (3.12) in which $\eta^a = \delta y^{(1)a}\,(a = 1, ..., n)$ and consider X and Y equals with the vectors of the adapted basis. □

We obtain

Theorem 3.3. *The almost n-contact 2-π structure (F, ξ_a, η^a) is normal if and only if, the following equations holds*

$$\begin{aligned} &\overset{R}{(12)}{}^i_{jk} = \overset{B}{(11)}{}^i_{jk} = \overset{B}{(12)}{}^i_{jk} = 0; \\ &\overset{B}{(21)}{}^i_{jk} - \overset{B}{(21)}{}^i_{kj} - \overset{R}{(01)}{}^i_{jk} = 0; \overset{B}{(22)}{}^i_{jk} - \overset{B}{(22)}{}^i_{kj} - \overset{R}{(02)}{}^i_{jk} = 0 \end{aligned} \tag{3.16}$$

where

$$
\begin{cases}
\overset{R}{(12)}{}^{i}_{jk} = \dfrac{\delta \overset{N}{(1)}{}^{i}_{j}}{\delta y^{(1)k}} - \dfrac{\delta \overset{N}{(1)}{}^{i}_{k}}{\delta y^{(1)j}}, \overset{R}{(01)}{}^{i}_{jk} = \dfrac{\delta \overset{N}{(1)}{}^{i}_{j}}{\delta x^{k}} - \dfrac{\delta \overset{N}{(1)}{}^{i}_{k}}{\delta x^{j}}; \\
\overset{R}{(02)}{}^{i}_{jk} = \overset{N}{(1)}{}^{i}_{m} \overset{R}{(01)}{}^{i}_{jk} + \dfrac{\delta \overset{N}{(2)}{}^{i}_{j}}{\delta x^{k}} - \dfrac{\delta \overset{N}{(2)}{}^{i}_{k}}{\delta x^{j}} \\
\overset{B}{(11)}{}^{i}_{jk} = \dfrac{\delta \overset{N}{(1)}{}^{i}_{j}}{\delta y^{(1)k}}; \overset{B}{(12)}{}^{i}_{jk} = \overset{N}{(1)}{}^{i}_{m} \overset{B}{(11)}{}^{m}_{jk} + \dfrac{\delta \overset{N}{(2)}{}^{i}_{j}}{\delta y^{(1)k}} - \dfrac{\delta \overset{N}{(2)}{}^{i}_{k}}{\delta y^{(1)j}}; \\
\overset{B}{(21)}{}^{i}_{jk} = \dfrac{\partial \overset{N}{(1)}{}^{i}_{j}}{\delta y^{(2)k}}; \overset{B}{(22)}{}^{i}_{jk} = \overset{N}{(1)}{}^{i}_{m} \overset{B}{(21)}{}^{m}_{jk} + \dfrac{\partial \overset{N}{(2)}{}^{i}_{j}}{\delta y^{(2)k}}.
\end{cases} \tag{3.17}
$$

Our above theoretical considerations prove the validity of the following theorem:

Theorem 3.4. *If the basic manifold M is paracompact, then there exists on the total space of its 2-osculator bundle at least one n-almost contact 2-π structure.*

This theorem allows us to define on Osc^2M the more general n-almost contact 2-π structure and to study them using the methods presented in this section.

4. n-almost contact riemannian 2-π structure. Let us consider a k-almost contact 2-π structure on the Lagrange space $L^{(2)n}$ given by $(F, \xi_1, ..., \xi_k, \eta^1, ..., \eta^k)$ and $\mathbf{G}$ a Riemannian structure on $E = Osc^2M$.

Definition 4.1. *One calls k-almost contact Riemannian 2-π structure any k-almost contact 2-π structure together with a Riemannian structure $\mathbf{G}$ on $\tilde{E}$ for which*

$$
\begin{cases}
G(FX, FY) = -\lambda^2 G(X, Y) - G(X, \lambda^2 \sum_{a=1}^{n} \eta^a(X) \xi_a) \\
\mathbf{G}(X, \xi_a) = \eta^a(X), \ (a = 1, ..., n)
\end{cases} \tag{4.1}
$$

such that the base (η_a) is orthonormal.

Now we consider a Lagrange space in which $L(x, y^{(1)}, y^{(2)})$ is the fundamental function and

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial y^{(1)i}}\partial y^{(2)j} \tag{4.2}$$

is its fundamental tensor.

We have $rank\|g_{ij}\|$ and if we consider its contravariant g^{ij} then the Sasakian lift $\mathbf{G}$ of the fundamental tensor g_{ij} is given by

$$\mathbf{G} = g_{ij}dx^i \otimes dx^j + g_{ij}\delta y^{(1)i} \otimes \delta y^{(1)j} + g_{ij}\delta y^{(2)i} \otimes \delta y^{(2)j}, \tag{4.3}$$

where $dx^i, \delta y^{(1)i}, \delta y^{(2)i}$ is the dual basis of the adapted basis for the nonlinear canonical connection of the space $L^{(2)n}$.

Following the same line as in the usual cases of the Riemannian metric structure in the Lagrange spaces $L^{(1)n}$ presented in section 1 and 2 we obtain the next result:

Theorem 4.1. *The natural n-almost contact 2-π structure* $(F, \xi_1, ..., \xi_n, \eta^1, ..., \eta^n)$ *together with the Sasakian lift* $\mathbf{G}$ *given by* (4.3) *is a n-almost contact Riemannian 2-π structure on* Osc^2M.

Let us consider the metric N-connection with respect to $\mathbf{G}$ which has the coefficients given by generalized Christoffel symbols:

$$\begin{cases} L^m_{ij} = \dfrac{1}{2}g^{ms}\left(\dfrac{\delta g_{is}}{\delta x^j} + \dfrac{\delta g_{sj}}{\delta x^i} - \dfrac{\delta g_{ij}}{\delta x^s}\right) \\ C^m_{ij(\alpha)} = \dfrac{1}{2}g^{ms}\left(\dfrac{\partial g_{is}}{\partial y^{(\alpha)j}} + \dfrac{\partial g_{sj}}{\partial y^{\alpha)i}} - \dfrac{\partial g_{ij}}{\partial y^{(\alpha)s}}\right) \ \alpha = 1, 2. \end{cases} \tag{4.4}$$

Therefore we have

Theorem 4.2. *If D is canonical N-connection with the coefficients* (4.4) *then natural n-almost contact Riemannian 2-π structure* $(\mathbf{F},\mathbf{G})$ *has the property* $D_X\mathbf{F} = 0$ *and* $D_X\mathbf{G} = 0 \forall X \in \chi(Osc^2M)$.

Remark 4.1. The previous theory can be particularized in the case of normal n-almost contact Riemannian 2-π structure.

The previous theory proves that on the differentiable paracompact manifold M and Osc^2M there exists n-almost contact Riemannian 2-π structures and that their general theory can be studied through the agency the specified methods.

Thus in general, the basis problems of the n-almost contact Riemannian 2-π structures in Lagrange spaces of order 2 are elucidated.

All this theory can be applied in the particular case of the Finsler spaces of order 2 by imposing on the Lagrangian certain conditions of homogeneousness.

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