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FRACTIONAL DYNAMICAL SYSTEMS ON FRACTIONAL JET BUNDLE AND APPLICATIONS

BY

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Dedicated to Academician Radu Miron at his 80th anniversary

Abstract. Using the fractional integration and differentiation on $\mathbb{R}$ we build the fractional jet fibre bundle on a differentiable manifold and we emphasize some important geometrical objects. Euler-Lagrange fractional equations are described. Some significant examples from mechanics and economics are presented.

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Key words: fractional derivatives, fractional bundle, Euler-Lagrange fractional equations.

1. Introduction. The operators of fractional differentiation have been introduced by Leibnitz, Liouville, Riemann, Grunwal and Letnikov [7]. The fractional derivatives and integrals are used in the description of some models in mechanics, physics [7], economics [5] and medicine [12]. The fractional variational calculus [1] is an important instrument in the analysis of such models. The Euler-Lagrange equations are non-autonomous fractional differential equations in those models.

In this paper we present the fractional jet fibre bundle of order k on a differentiable manifold as being $J^{\alpha k}(\mathbb{R}, M) = \mathbb{R} \times Osc^{\alpha k}(M)$, $\alpha \in (0, 1)$, $k \in \mathbb{N}^*$. The fibre bundle $J^{\alpha k}$ is built in a similar way as the fibre bundle E^k by R. Miron [10]. Among the geometrical structures defined on $J^{\alpha}(\mathbb{R}, M)$ we consider the dynamical fractional connection and the fractional Euler-Lagrange equations associated with a function defined on $J^{\alpha k}(\mathbb{R}, M)$.

In section 2 we describe the fractional operators on $\mathbb{R}$ and some of their properties which are used in the paper. In section 3 we describe the

fractional osculator bundle of order k. In section 4 the fractional jet fibre bundle $J^\alpha(\mathbb{R}, M)$ is defined, the fractional dynamical connection is built and the fractional Euler-Lagrange equations are established using the notion of fractional extremal value and classical extremal value on $J^{\alpha k}(\mathbb{R}, M)$. In section 5 we consider some examples and applications.

A more exhaustive presentation of the ideas from this paper may be found in [4].

2. Elements of fractional integration and differentiation on $\mathbb{R}$. Let $f : [a, b] \to \mathbb{R}$ be an integrable function and $\alpha \in (0, 1)$. The left-sided (right-sided) fractional derivative of f is the function

$$
\begin{aligned}
({}_-D_t^\alpha f)(t) &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{f(s) - f(a)}{(t-s)^\alpha} ds \\
({}_+D_t^\alpha f)(t) &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^b \frac{f(b) - f(s)}{(s-t)^\alpha} ds,
\end{aligned} \tag{2.1}
$$

where $t \in [a,\ b)$ and Γ is Euler's gamma function.

If $\{\alpha_n\}_{n\geq 0}$ is a real number sequence with $\lim\limits_{n\to\infty} \alpha_n = 1$ then

$$
\lim_{n\to\infty} ({}_-D_t^{\alpha_n} f)(t) = ({}_-D_t^1 f)(t) = \frac{d}{dt} f(t). \tag{2.2}
$$

The physical and geometrical interpretation of the fractional derivative on $\mathbb{R}$ is suggested by the interpretation of the Stieltjes integral, because the integral used in the definition of the fractional derivative is a Riemann-Stieltjes integral [11].

By definition, the left-sided (right-sided) fractional derivative of f, of order α, $m = [\alpha] + 1$, is the function

$$
\begin{aligned}
D_t^\alpha f(t) &= \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dt}\right)^m \int_{-\infty}^t \frac{f(s) - f(0)}{(t-s)^\alpha} ds,\ 0 \in (-\infty, t) \\
{}^*D_t^\alpha f(t) &= \frac{1}{\Gamma(m-\alpha)} \left(-\frac{d}{dt}\right)^m \int_t^\infty \frac{f(s) - f(0)}{(s-t)^\alpha} ds,\ 0 \in (t, \infty).
\end{aligned} \tag{2.3}
$$

If $\overline{\text{supp} f} \subset [a, b]$, then $D_t^\alpha f = {}_-D_t^\alpha f$, ${}^*D_t^\alpha f = {}_+D_t^\alpha f$.

Let us consider the seminorms

$$
\begin{aligned}
|x|_{J_L^\alpha(\mathbb{R})} &= \|D_t^\alpha x\|_{L^2(\mathbb{R})} \\
|x|_{J_R^\alpha(\mathbb{R})} &= \|{}^*D_t^\alpha x\|_{L^2(\mathbb{R})},
\end{aligned}
$$

and the norms

$$\|x\|_{J_L^\alpha(\mathbb{R})} = \left(\|x\|^2_{L^2(\mathbb{R})} + |x|^2_{J_L^\alpha(\mathbb{R})}\right)^{1/2}$$
$$\|x\|_{J_R^\alpha(\mathbb{R})} = \left(\|x\|^2_{L^2(\mathbb{R})} + |x|^2_{J_R^\alpha(\mathbb{R})}\right)^{1/2},$$

and $J^\alpha_{0L}(\mathbb{R})$, $J^\alpha_{0R}(\mathbb{R})$ the closures of $C_0^\infty(\mathbb{R})$ with respect to the two norms from above, respectively.

Proposition 1. *Let $I \subset \mathbb{R}$ and let $J^\alpha_{0L}(I)$ and $J^\alpha_{0R}(I)$ be the closures of $C_0^\infty(I)$ with respect to the norms from above. For any $x \in J^\beta_{0L}(I)$, $0 < \alpha < \beta$, the following relation holds:*

$$D_t^\beta x(t) = D_t^\alpha D_t^{\beta-\alpha} x(t).$$

For any $x \in J^\beta_{0R}(I)$, $0 < \alpha < \beta$, it also holds

$${}^*D_t^\beta x(t) = {}^*D_t^{\alpha *} D_t^{\beta-\alpha} x(t).$$

3. The fractional osculator bundle of order k on a differentiable manifold. Let $\alpha \in (0, 1]$ be fixed and M a differentiable manifold of dimension n. Two curves $\rho,\ \sigma : I \to \mathbb{R}$, with $\rho(0) = \sigma(0) = x_0 \in M$, $0 \in I$, have a fractional contact α of order $k \in \mathbb{N}^*$ in x_0, if for any $f \in \mathcal{F}(U)$, $x_0 \in U$, U a chart on M, it holds

$$D_t^{\alpha a}(f \circ \rho)\,|_{t=0} = D_t^{\alpha a}(f \circ \sigma)\,|_{t=0} \tag{3.1}$$

where $a = \overline{1,\ k}$. The relation (3.1) is an equivalence relation. The equivalence class $[\rho]^{\alpha k}_{x_0}$ is called the fractional k-osculator space of M in x_0 and it will be denoted by $Osc^{\alpha k}_{x_0}(M)$. If the curve $\rho : I \to M$ is given by $x^i = x^i(t)$, $t \in I$, $i = \overline{1,\ n}$, then, the class $[\rho]^{\alpha k}_{x_0}$, may be written as

$$x^i(t) = x^i(0) + \frac{t^\alpha}{\Gamma(1+\alpha)} D_t^\alpha x^i(t)\,|_{t=0} + ... + \frac{t^{\alpha k}}{\Gamma(1+\alpha k)} D_t^{\alpha k} x^i(t)\,|_{t=0}\ , \tag{3.2}$$

where $t \in (-\varepsilon, \varepsilon)$. We shall use the notation

$$x^i(0) = x^i, \quad y^{i(\alpha a)} = \frac{1}{\Gamma(1+\alpha a)} D_t^{\alpha a} x^i(t)\,|_{t=0}\ , \tag{3.3}$$

for $i = \overline{1, n}$ and $a = \overline{1, k}$.

By definition, the fractional osculator bundle of order k is the fibre bundle $(Osc^{\alpha k}(M), M)$ where $Osc^{\alpha k}(M) = \bigcup_{x_0 \in M} Osc^{\alpha k}_{x_0}(M)$ and $\pi_0^{\alpha k} : Osc^{\alpha k}(M) \to M$ is defined by $\pi_0^{\alpha k}([\rho]^{\alpha k}_{x_0}) = x_0$, $(\forall)[\rho]^{\alpha k}_{x_0} \in Osc^{\alpha k}(M)$.

For $f \in \mathcal{F}(U)$, the fractional derivative of order α, $\alpha \in (0,\ 1)$, with respect to the variable x^i, is defined by

$$(3.4)\qquad \begin{array}{l}(D^{\alpha}_{x^i} f)(x)\\ = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial x^i} \int_{a^i}^{x^i} \frac{f(x^1,...,x^{i-1},s,x^{i+1},...,x^n) - f(x^1,...,x^{i-1},a^i,x^{i+1},...,x^n)}{(x^i - s)^{\alpha}} ds,\end{array}$$

where x^i are the coordinate functions on U, $\frac{\partial}{\partial x^i}$, $i = \overline{1,n}$, is the canonical base of the vector fields on U and $U_{ab} = \{x \in U,\ a^i \leq x^i \leq b^i,\ i = \overline{1,n}\} \subset U$. Let $U,\ U' \subset M$ be two charts on M, $U \cap U' \neq \emptyset$ and consider the change of variable

$$(3.5)\qquad \bar{x}^i = \bar{x}^i(x^1, ..., x^n)$$

with $\det(\frac{\partial \bar{x}^i}{\partial x^j}) \neq 0$.

Using the notation

$$(3.6)\qquad \overset{\alpha}{J}{}^j_i(x,\ \bar{x}) = \frac{1}{\Gamma(1+\alpha)} D^{\alpha}_{\bar{x}^i}(x^j)^{\alpha},$$

we find

(3.7)

$$\begin{array}{l}\frac{\Gamma(\alpha(a-1))}{\Gamma(\alpha)} \bar{y}^{i(\alpha a)} = \Gamma(1+\alpha)\, \overset{\alpha}{J}{}^i_j(\bar{y}^{\alpha(a-1)},\ x) y^{j(\alpha)}\\ + \frac{\Gamma(2\alpha)}{\Gamma(\alpha)} \overset{\alpha}{J}{}^i_j(y^{(\alpha(a-1))},\ y^{\alpha}) y^{j(2\alpha)} + ... + \frac{\Gamma(2\alpha)}{\Gamma(\alpha)} \overset{\alpha}{J}{}^i_j(\bar{y}^{\alpha(a-1)},\ y^{\alpha b}) y^{j((b+1)\alpha)}\\ + ... + \frac{\Gamma(\alpha(a-1))}{\Gamma(\alpha)} y^{i(\alpha a)},\end{array}$$

where $a = \overline{1,k}$.

Proposition 2. (see [2], [6])

a) *The coordinate transformation on* $Osc^{(\alpha k)}(M)$, $(x^i, y^{i(\alpha)}, ..., y^{i(\alpha k)}) \to (\bar{x}^i, \bar{y}^{i(\alpha)}, ..., \bar{y}^{i(\alpha k)})$ *are given by the formulas* (3.5) *and* (3.7).

b) *The operators* $D^{\alpha}_{x^i}$ *and the 1-forms* $d(x^i)^{\alpha}$, $i = \overline{1,n}$, *transform by the formulas*

$$(3.8)\qquad \begin{array}{l} D^{\alpha}_{\bar{x}^i} = \overset{\alpha}{J}{}^i_j(x,\ \bar{x}) D^{\alpha}_{x^j}\\ d(\bar{x}^i)^{\alpha} = \overset{\alpha}{J}{}^i_j(\bar{x},\ x) d(x^j)^{\alpha}.\end{array}$$

4. The fractional jet bundle of order k on a differentiable manifold; geometrical objects. By definition, the k-order fractional jet bundle is the space $J^{\alpha k}(\mathbb{R},\ M) = \mathbb{R} \times Osc^{k\alpha}(M)$. A system of local coordinates on $J^{\alpha k}(\mathbb{R},\ M)$ will be denoted by $(t, x, y^{(\alpha)}, y^{(2\alpha)}, ..., y^{(k\alpha)})$. Consider the projections $\pi_0^{\alpha k} : J^{\alpha k}(\mathbb{R},\ M) \to M$ defined by

$$\pi_0^{\alpha k}(t, x, y^{(\alpha)}, ..., y^{(\alpha k)}) = x. \tag{4.1}$$

On $J^{\alpha}(\mathbb{R},\ M)$ we may define the canonical structures

$$\begin{aligned} &\overset{\alpha}{\theta}_1 = d(t^\alpha) \otimes (D_t^\alpha + y^{i(\alpha)} D_{x^i}^\alpha) \\ &\overset{\alpha}{\theta}_2 = \overset{\alpha}{\theta}{}^i \otimes D_{x^i}^\alpha, \quad \overset{\alpha}{\theta}{}^i = \frac{1}{\Gamma(1+\alpha)}(d(x^i)^\alpha - y^{i(\alpha)} d(t)^\alpha) \\ &\overset{\alpha}{S} = \overset{\alpha}{\theta}{}^i \otimes D_{y^{i(\alpha)}}^\alpha \\ &\overset{\alpha}{V}_i = D_{y^{i(\alpha)}}^\alpha. \end{aligned} \tag{4.2}$$

The structures (4.2) have geometrical character. The space of the operators generated by the operators $\{D_t^\alpha, D_{x^i}^\alpha, D_{y^{i(\alpha)}}^\alpha\}$, $i = \overline{1,\ n}$, will be denoted by $\chi^\alpha((\pi_0^\alpha)^{-1}(U))$. For $\alpha \to 1$ the space of these operators represents the space of the vector fields on $\pi_0^{-1}(U)$.

A vector field $\overset{\alpha}{\Gamma} \in \chi^\alpha((\pi_0^\alpha)^{-1}(U))$ is called *FODE* (fractional ordinary differential equation) iff

$$\begin{aligned} &d(t)^\alpha(\overset{\alpha}{\Gamma}) = 1 \\ &\overset{\alpha}{\theta}{}^i(\overset{\alpha}{\Gamma}) = 0, \end{aligned} \tag{4.3}$$

for $i = \overline{1, n}$. In local coordinates *FODE* is given by

$$\overset{\alpha}{\Gamma} = D_t^\alpha + y^{i(\alpha)} D_{x^i}^\alpha + F^i D_{y^{i(\alpha)}}^\alpha, \tag{4.4}$$

where $F^i \in C^\infty((\pi_0^\alpha)^{-1}(U))$, $i = \overline{1, n}$. The integral curves of the field *FODE* are the solutions of the fractional differential equation (*EDF*)

$$D_t^{2\alpha} x^i(t) = F^i(t, x(t), D_t^\alpha x(t)), \qquad i = \overline{1, n}. \tag{4.5}$$

The fractional dynamical connection on $J^{\alpha}(\mathbb{R},\ M)$ is defined by the

fractional tensor fields $\overset{\alpha}{H}$ of type $(1,1)$ which satisfy the conditions

$$\begin{aligned} &\overset{\alpha}{\theta}_1 \circ \overset{\alpha}{H} = 0 \\ &\overset{\alpha}{\theta}_2 \circ \overset{\alpha}{H} = \overset{\alpha}{\theta}_2 \\ &\overset{\alpha}{H}\Big|_{\overset{\alpha}{V}} = -id\Big|_{\overset{\alpha}{V}}, \end{aligned} \tag{4.6}$$

where $\overset{\alpha}{V}$ is formed by operators generated by $\{D^{\alpha}_{y^{i(\alpha)}}\}_{i=\overline{1,n}}$.

The tensor field $\overset{\alpha}{H}$ has a geometrical character, and is called a d^{α}-tensor field. We get

Proposition 3. a) *The fractional dynamical connection $\overset{\alpha}{H}$, in the chart $(\pi_0^{\alpha})^{-1}(U)$, is given by*

$$\begin{aligned} \overset{\alpha}{H} = \frac{1}{\Gamma(1+\alpha)}&[(-y^{i(\alpha)}D^{\alpha}_{x^i} + H^i D^{\alpha}_{y^{i(\alpha)}}) \otimes d(t)^{\alpha} \\ &+(D^{\alpha}_{x^i} + H^j_i D^{\alpha}_{y^{j(\alpha)}}) \otimes d(x^i)^{\alpha} - D^{\alpha}_{y^{i(\alpha)}} \otimes d(y^{i(\alpha)})^{\alpha}]. \end{aligned} \tag{4.7}$$

b) *The fractional dynamical connection $\overset{\alpha}{H}$ defines a $f(3,-1)$ fractional structure on $J^{\alpha}(\mathbb{R},\ M)$, i.e., $\left(\overset{\alpha}{H}\right)^3 = \overset{\alpha}{H}$.*

c) *The fractional tensor fields $\overset{\alpha}{l}$ and $\overset{\alpha}{m}$ which are defined by*

$$\begin{aligned} &\overset{\alpha}{l} = \overset{\alpha}{H} \circ \overset{\alpha}{H} \\ &\overset{\alpha}{m} = -\overset{\alpha}{H} \circ \overset{\alpha}{H} + I, \end{aligned} \tag{4.8}$$

where I is the identity map, satisfy the relations

$$\begin{aligned} &\overset{\alpha}{l} \circ \overset{\alpha}{l} = \overset{\alpha}{l}, \ \ \overset{\alpha}{m} \circ \overset{\alpha}{m} = \overset{\alpha}{m} \circ \overset{\alpha}{l}, \ \ \overset{\alpha}{l} + \overset{\alpha}{m} = I \\ &\overset{\alpha}{l}(D^{\alpha}_t) = -y^{i(\alpha)} D_{x^i} - (y^{i(\alpha)} \overset{\alpha}{H^j_i} + \overset{\alpha}{H^j}) D_{y^{i(\alpha)}} \\ &\overset{\alpha}{l}(D^{\alpha}_{x^i}) = D^{\alpha}_{x^i}, \ \ \overset{\alpha}{l}(D^{\alpha}_{y^{i(\alpha)}}) = D^{\alpha}_{y^{i(\alpha)}} \\ &\overset{\alpha}{m}(D^{\alpha}_t) = D^{\alpha}_t + y^{i(\alpha)} D^{\alpha}_{x^i} + (y^{i(\alpha)} \overset{\alpha}{H^j_i} + \overset{\alpha}{H^j}) D^{\alpha}_{y^{i(\alpha)}} \\ &\overset{\alpha}{m}(D^{\alpha}_{x^i}) = 0, \ \ \overset{\alpha}{m}(D^{\alpha}_{y^{i(\alpha)}}) = 0. \end{aligned} \tag{4.9}$$

Let $L \in C^{\infty}(J^{\alpha}(\mathbb{R},\ M))$ be a fractional Lagrange function. By definition, the Cartan fractional 1-form is the 1-form $\overset{\alpha}{\theta}_L$ given by

$$\overset{\alpha}{\theta}_L = L d(t)^{\alpha} + \overset{\alpha}{S}(L). \tag{4.10}$$

We call the Cartan fractional 2-form, the 2-form $\overset{\alpha}{\omega}_L$ given by

$$\overset{\alpha}{\omega}_L = d^\alpha \overset{\alpha}{\theta}_L \tag{4.11}$$

where d^α is the fractional exterior differential:

$$d^\alpha = d(t)^\alpha D_t^\alpha + d(x^i)^\alpha D_{x^i}^\alpha + d(y^{i(\alpha)})^\alpha D_{y^{i(\alpha)}}^\alpha. \tag{4.12}$$

In the chart $(\pi_0^\alpha)^{-1}(U)$, $\overset{\alpha}{\theta}_L$ and $\overset{\alpha}{\omega}_L$ are given by

$$\begin{aligned}
\overset{\alpha}{\theta}_L &= \left(L - \frac{1}{\Gamma(1+\alpha)} y^{i(\alpha)} D_{y^{i(\alpha)}}^\alpha(L)\right) d(t)^\alpha \\
&\quad + \frac{1}{\Gamma(1+\alpha)} D_{y^{i(\alpha)}}^\alpha(L) d(x^i)^\alpha \\
\overset{\alpha}{\omega}_L &= A_i d(t)^\alpha \wedge d(x^i)^\alpha + B_i d(t^\alpha) \wedge d(y^{i(\alpha)})^\alpha \\
&\quad + A_{ij} d(x^i)^\alpha \wedge d(x^j)^\alpha + B_{ij} d(x^i)^\alpha \wedge d(y^{j(\alpha)})^\alpha,
\end{aligned} \tag{4.13}$$

where

$$\begin{aligned}
A_i &= \frac{1}{\Gamma(1+\alpha)} D_t^\alpha D_{y^{i(\alpha)}}^\alpha(L) \\
&\quad + \frac{1}{\Gamma(1+\alpha)} y^{j(\alpha)} D_{x^i}^\alpha D_{y^{j(\alpha)}}^\alpha(L) - D_{x^i}^\alpha(L) \\
B_i &= \frac{1}{\Gamma(1+\alpha)} D_{y^{i(\alpha)}}^\alpha (y^{j(\alpha)} D_j^\alpha(\alpha)(L)) \\
A_{ij} &= D_{x^i}^\alpha D_{y^{i(\alpha)}}^\alpha(L), \quad B_{ij} = -D_{y^{j(\alpha)}}^\alpha D_{y^{i(\alpha)}}^\alpha(L).
\end{aligned} \tag{4.14}$$

Proposition 4. *If L is regular (i.e., $\det(\frac{\partial^2 L}{\partial y^{i(\alpha)} \partial y^{j(\alpha)}}) \neq 0$) then there exists a fractional field FODE $\overset{\alpha}{\Gamma}_L$ such that $i_{\overset{\alpha}{\Gamma}_L} \overset{\alpha}{\omega}_L = 0$. In the chart $(\pi_0^\alpha)^{-1}(U)$ we have*

$$\overset{\alpha}{\Gamma}_L = D_t^\alpha + y^{i(\alpha)} D_{x^i} + \overset{\alpha}{M}{}^i D_{y^{i(\alpha)}}^\alpha, \tag{4.15}$$

where

$$\begin{aligned}
&\overset{\alpha}{M}{}^i = g^{ik}(D_k^\alpha(L) - d_t^\alpha \left(\frac{\partial^\alpha L}{\partial y^{k(\alpha)}}\right) \\
&d_t^\alpha = D_t^\alpha + y^{i(\alpha)} D_{x^i}^\alpha \\
&(g^{ik}) = (D_{y^{i(\alpha)}}^\alpha D_{y^{k(\alpha)}}^\alpha(L))^{-1}.
\end{aligned} \tag{4.16}$$

Proposition 5. (see [4]) a) *If a fractional extremal value is attained then a necessary condition is that the fractional Euler-Lagrange equations must be satisfied:*

$$D^{\alpha}_{x^i}L+\sum_{a=1}^{k}(-1)^a d_t^{\alpha a}(D^{\alpha}_{y^{i(\alpha a)}}L)=0, \tag{4.17}$$

where

$$d_t^{\alpha a}=D_t^{\alpha}+y^{i(\alpha)}D^{\alpha}_{x^i}+y^{i(2\alpha)}D^{\alpha}_{y^{i(\alpha)}}+...+y^{i(\alpha a)}D^{\alpha}_{y^{i(\alpha(a-1))}}, \tag{4.18}$$

and $i=\overline{1,n}$.

b) *If an extremal value is attained then a necessary condition is that the Euler-Lagrange equations must be satisfied:*

$$\frac{\partial L}{\partial x^i}+\sum_{a=1}^{k}(-1)^a d_t^{a}(D^{\alpha}_{y^{i(\alpha a)}}L)=0, \tag{4.19}$$

where

$$d_t^{a}=D_t^{1}+y^{i(\alpha)}D^{1}_{x^i}+...+y^{i(\alpha a)}D^{1}_{y^{i(\alpha(a-1))}}. \tag{4.20}$$

Example. Consider the fractional differential equation

$$\frac{c\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)}x^{\gamma-\alpha}(t)f(t)+a_1\Gamma(1+2\alpha)y^{(2\alpha)}+a_2\Gamma(1+3\alpha)y^{(3\alpha)}=0. \tag{4.21}$$

The equation (4.21) is the fractional Euler-Lagrange equation (4.17) for the function

$$L=\frac{c}{1+\gamma-\alpha}x^{\gamma}f(t)-a_1\Gamma(1+2\alpha)(y^{\alpha})^{\alpha}+a_2\Gamma(1+3\alpha)(y^{2\alpha})^{\alpha}.$$

The equation (4.21) is the fractional Euler-Lagrange equation (4.19) for the function

$$L=\frac{c\Gamma(1+\gamma)x^{\gamma-\alpha+1}}{\Gamma(1+\gamma-\alpha)^{(1+\gamma-\alpha)}}f(t)-\frac{a_1}{2}\Gamma(1+2\alpha)(y^{\alpha})^2+\frac{a_2}{2}\Gamma(1+3\alpha)(y^{2\alpha})^2.$$

5. Examples and applications

1. The nonhomogeneous Bagley-Torvik equation. The dynamics of a flat rigid body embedded in a Newton fluid is described by the equation

$$aD_t^2x(t) + bD_t^{3/2}x(t) + cx(t) - f(t) = 0, \tag{5.1}$$

where $a, b, c \in \mathbb{R}$ and the initial conditions are $x(0) = 0$, $D_t^1x(0) = 0$. The equation (5.1) is a fractional differential equation on the bundle $J^\alpha(\mathbb{R},\ \mathbb{R})$ for $\alpha = \frac{1}{4}$.

Proposition 6. *The equation* (5.1) *represents a Euler-Lagrange equation on the bundle* $J^{4\alpha}(\mathbb{R},\ \mathbb{R})$ *for* $\alpha = \frac{1}{4}$*, with the Lagrange function given by*

(5.2)
$$L(t, x, y^{(3/2)}, y^{(2)}) = \frac{1}{2}cx^2 - f(t)x - \frac{b}{2}\Gamma(5/2)(y^{(3/2)})^2 + \frac{a}{2}\Gamma(3)(y^{(2)})^2.$$

2. Differential equations of order one, two and three which admit fractional Lagrangians. The following differential equations don't have classical Lagrangians such that the Euler-Lagrange equation represents the given equation:

$$\dot{x}(t) + V_1(t, x) = 0,\ \ V_1(t, x) = \frac{\partial U_1(t, x)}{\partial x}, \tag{5.3}$$

$$\ddot{x}(t) + a_1\dot{x}(t) + V_2(t, x) = 0,\ \ V_2(t, x) = \frac{\partial U_2(t, x)}{\partial x}, \tag{5.4}$$

$$\dddot{x}(t) + a_2\ddot{x}(t) + a_1\dot{x}(t) + V_3(t, x) = 0,\ \ V_3(t, x) = \frac{\partial U_3(t, x)}{\partial x}. \tag{5.5}$$

For $\alpha = \frac{1}{2}$ we obtain the fractional Lagrangians that describe the equations (5.3), (5.4), (5.5), respectively

$$\begin{aligned}
&L(t, x, y^{(1/2)}) = U_1(t, x) - \frac{1}{2}\Gamma(2)(y^{(1/2)})^2\\
&L(t, x, y^{(1/2)}, y^{(1)}) = U_2(t, x) - \frac{1}{2}a_1\Gamma(2)(y^{(1/2)})^2 + \frac{1}{2}\Gamma(3)(y^{(1)})^2\\
&L(t, x, y^{(1/2)}, y^{(1)}, y^{(3/2)}) = U_3(t, x) - \frac{a_1}{2}\Gamma(2)(y^{(1/2)})^2 +\\
&\frac{a_2}{2}\Gamma(2)(y^{(1)})^2 - \frac{1}{2}\Gamma(4)(y^{(3/2)})^2.
\end{aligned} \tag{5.6}$$

In the category of the equations (5.4) and (5.5) there are:
a) the nonhomogeneous classical friction equation

$$m\ddot{x}(t) + \gamma\dot{x}(t) - \frac{\partial U(t,x)}{\partial x} = 0, \tag{5.7}$$

b) the nonhomogeneous model of Phillips [9]

$$\ddot{x}(t) + a_1\dot{x}(t) + b_1x(t) + f(t) = 0, \tag{5.8}$$

c) the nonhomogeneous business cycle with innovation [9]

$$\dddot{y}(t) + a_2\ddot{y}(t) + a_1\dot{y}(t) + b_1x(t) + f(t) = 0. \tag{5.9}$$

Conclusions. The paper presents the main differentiable structures on $J^\alpha(\mathbb{R},\ M)$, in order to describe fractional differential equations and ordinary differential equations, using Lagrange functions defined on $J^\alpha(\mathbb{R},\ M)$.

With the help of the methods shown, there may be analyzed other models, such as those found in [5] and [12].

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