ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" DIN IAŞI (S.N.)
MATEMATICĂ, Tomul LIII, 2007, Supliment

MANIFOLDS ENDOWED WITH SEVERAL COMPLEX STRUCTURES

BY

ILIE BURDUJAN

Dedicated to Academician Radu Miron at his 80th anniversary

Abstract. On an almost Clifford manifold (M^{8n}, Q) we study the integrability of an almost complex structure which is compatible with the almost Clifford structure Q. The obtained results extend the results of Alekseevsky, Marchiafava and Pontecorvo [1]. More exactly, we prove that the existence of three compatible complex structures I_1, I_2, I_3 such that $I_1 \neq \pm I_2$, $I_1 \neq \pm I_3$, $I_j \neq I_k \circ I_\ell$ for all $(j,k,\ell) \in S_3$ where S_3 denote the symmetric group with three elements, forces (M^{8n}, Q) to be a Clifford manifold.

Mathematics Subject Classification 2000: 53C10, 32C10.

Key words: almost hypercomplex structure, almost Clifford structure, Clifford structure.

1. Introduction. An *almost hypercomplex structure* on a 8n-dimensional manifold M is a six-tuple $H = (J_1, J_2, ..., J_6)$ of almost complex structures $J_\alpha : TM \to TM$, $\alpha = 1, 2, ..., 6$ which behave under composition like the corresponding labelled elements of the canonical basis of the Clifford algebra $\mathcal{C}\ell_{03}$. When each J_α is a complex structure, H is said to be a *hypercomplex* structure on M. This Definition is formally obtained changing "quaternion algebra" with "Clifford algebra" in the synonym notion defined on 4n-dimensional manifolds (see [1], [2]).

An *almost Clifford structure* on M is a rank-6 subbundle $Q \subset End(TM)$ which is locally spanned by almost hypercomplex structures $H = (J_\alpha)$; such a locally defined six-tuple H will be called an *admissible basis* of Q and we will write $Q = \langle H \rangle$. We say that Q is a *Clifford structure* when there is a torsionless connection ∇ on TM preserving Q, i.e., $\nabla_X \sigma \in \Gamma(Q)$ for all vector fields X and smooth sections $\sigma \in \Gamma(Q)$. Briefly, we also say that (M^{8n}, Q) is a (*an almost*) *Clifford manifold* and ∇ is a (*an almost*) *Clifford connection*.

A first consequence of the definition of an almost Clifford manifold is that the bundle Q has structure group $SU(2) \oplus SU(2)/\mathbb{Z}_2 \oplus \mathbb{Z}_2 \equiv SO(3) \oplus SO(3)$. We have then a natural inner product $\langle , \rangle$ on Q by taking each admissible basis $H = (J_a)$, $a = 1, 2, ..., 6$ to be an orthonormal basis.

We say that an almost complex structure on (an open subset of) M is *compatible* with the almost Clifford structure Q if it can be written as $I = \sum_{a=1}^{6} c_a J_a$ with respect to any admissible basis $H = (J_a)$ for suitable functions c_a (i. e. it is a (local) section of the twistor space Z associated with Q).

2. Clifford algebra $\mathcal{C}\ell_{03}$. Recall that $\mathcal{C}\ell_{03}$ denotes the Clifford algebra with three generators $\{e_1, e_2, e_3\}$. It is a real unital associative 8-dimensional algebra for which there exists a special basis $(e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7)$ such that

$$\begin{array}{ll}
e_0 e_i = e_i e_0 = e_i, & i = 0, 1, ..., 7, \\
e_i^2 = -e_0, \;\; e_7^2 = e_0, & i = 1, 2, ..., 6, \\
e_i e_j + e_j e_i = 0, & i \neq j, \;\; i, j = 1, 2, ..., 6, \;\; i + j \neq 7, \\
e_i e_j = e_j e_i, & i = 0, 1, ..., 7, \;\; i \neq j, \;\; i + j = 7, \\
e_1 e_2 = e_4, & e_1 e_3 = e_5, \;\; e_2 e_3 = e_6.
\end{array}$$

For our comfort, we denote $\mathcal{O} = \mathcal{C}\ell_{03}$ and name its elements *octons*. The before introduced basis $\mathcal{B}_c = (e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7)$ is called the *canonical* (or, *natural*) basis of $\mathcal{O}$. The center of $\mathcal{O}$ is $Z(\mathcal{O}) = \mathbb{R}e_0 + \mathbb{R}e_7$ (see [4]). It must be remarked that $Z(\mathcal{O}) \simeq \mathcal{D}$ where $\mathcal{D}$ denotes the real (associative and commutative) algebra of the so-called *double numbers*. A new basis $\mathcal{B}_a = (E_0, E_1, E_2, E_3, E_4, E_5, E_6, E_7)$, defined by

$$E_0 = \frac{1}{2}(e_0 + e_7)\,,\; E_1 = \frac{1}{2}(e_1 - e_6)\,,\; E_2 = -\frac{1}{2}(e_2 + e_5)\,,\; E_3 = \frac{1}{2}(e_3 - e_4)\,,$$

$$E_4 = \frac{1}{2}(e_0 - e_7)\,,\; E_5 = \frac{1}{2}(e_1 + e_6)\,,\; E_6 = \frac{1}{2}(e_2 - e_5)\,,\; E_7 = \frac{1}{2}(e_3 + e_4)\,,$$

is associated with the canonical basis $\mathcal{B}_c$; it is named the *adapted basis* corresponding to $\mathcal{B}_c$. The multiplication table for $\mathcal{O}$, in the adapted basis

$\mathcal{B}_a$, is:

$$\begin{array}{ll} E_0E_i = E_iE_0 = E_i,\ \ i \in \{1,2,3\}, & E_4E_\alpha = E_\alpha E_0 = E_\alpha,\ \ \alpha \in \{5,6,7\}, \\ E_i^2 = -E_0,\ \ i \in \{1,2,3\}, & E_\alpha^1 = -E_4,\ \ \alpha \in \{5,6,7\}, \\ E_iE_j = -E_jE_i = E_k, & E_\alpha E_\beta = -E_\beta E_\alpha = E_\gamma, \end{array}$$

where (i,j,k) and (α,β,γ) are cyclic permutations of $\{1,2,3\}$ and $\{5,6,7\}$, respectively. Consequently, $\mathcal{O}$ can be naturally identified with $\mathbb{H} \oplus \mathbb{H}$.

Thus, the conjugation in $\mathbb{H}$ induces a conjugation in $\mathcal{O}$ defined by:

$$\begin{aligned} \mathcal{O} \ni a = a_0e_0 + a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6 + a_7e_7 \to \\ \to \overline{a} = a_0e_0 - a_1e_1 - a_2e_2 - a_3e_3 - a_4e_4 - a_5e_5 - a_6e_6 + a_7e_7 \in \mathcal{O}. \end{aligned}$$

Since $\overline{(\overline{a})} = a$ and $\overline{a \cdot b} = \overline{b} \cdot \overline{a}$, it results that this conjugation is an anti-involution. Moreover,

$$a\overline{a} = \left(\sum_{i=0}^{7} a_i^2\right) e_0 + (a_0a_7 - a_1a_6 + a_2a_5 - a_3a_4)\, e_7 \in \mathcal{D}, \tag{2.1}$$

so that the following two quadratic forms $h_1, h_2 : \mathcal{O} \to \mathbb{R}$ arise naturally

$$h_1(a) = \sum_{i=0}^{7} a_i^2\ ,\ \ h_2(a) = a_0a_7 - a_1a_6 + a_2a_5 - a_3a_4, \quad \forall a \in \mathcal{O}.$$

h_1 is a positive defined quadratic form, while h_2 is a nonsingular one having the signature (4,4). The linear group preserving both these quadratic forms is isomorphic to $\mathbf{O(4,\mathbb{R})} \times \mathbf{O(4,\mathbb{R})}$.

The presence of a natural conjugation on $\mathcal{O}$ suggests the possibility to define an (quasi-)inner product on it. Actually, (2.1) claims to consider a structure of $\mathcal{D}$-module on M. Such a structure arise naturally since every element $a = a_0e_0 + a_1e_1 + ... + a_7e_7 \in M$ has the form

$$a = (a_0e_0{+}a_7e_7)e_0{+}(a_1e_0{-}a_6e_7)e_1{+}(a_2e_0{+}a_5e_7)e_2{+}(a_3e_0{-}a_4e_7)e_3. \tag{2.2}$$

We define now the following bilinear $\mathcal{D}$-form on the $\mathcal{D}$-module $\mathcal{O}$ by

$$\begin{array}{l} \mathcal{O} \times \mathcal{O} \ni (a,b) \to \langle a,b \rangle \in \mathcal{D}, \\ \langle a,b \rangle = \frac{1}{2}(a\cdot \bar{b} + b\cdot \bar{a}) \in \mathcal{D}, \quad \forall a,b \in \mathcal{O}. \end{array} \tag{2.3}$$

Since it is a $\mathcal{D}$-bilinear symmetric $\mathcal{D}$-form satisfying the condition

$$\langle a,a \rangle = 0 \Longleftrightarrow a = 0$$

it will be called a *quasi-inner product*.

The set $G_{\mathcal{O}}$ consisting in all regular elements of $\mathcal{O}$ is a group. It is the product of two subgroups, namely $G_{\mathcal{O}} = \mathcal{O}(1) \cdot \mathcal{D}^*$, where $\mathcal{O}(1) = \{a \in \mathcal{O} | a \cdot \overline{a} = e_0\}$ and $\mathcal{D}^*$ is the set of all invertible elements from $Z(\mathcal{O}) \cong \mathcal{D}$. $\mathcal{O}(1)$ and $\mathcal{D}^*$ are normal subgroups of $G_{\mathcal{O}}$ with $\mathcal{O}(1) \cap \mathcal{D}^* = \{\pm e_0, \pm e_7\}$.

The $\mathcal{D}$-module $\mathcal{O}^m$ can be endowed with an quasi-inner product defined by

$$\begin{aligned} &\langle p, q\rangle = \frac{1}{2}\sum_{i=1}^{m}\left(p_i\cdot \bar{q}_i + q_i\cdot \bar{p}_i\right), \text{ for all} \\ &p = (p_1, p_2, ..., p_m), q = (q_1, q_2, ..., q_m) \in \mathcal{O}^m. \end{aligned} \tag{2.4}$$

As it is usual, we define the group $Op(m)$ as being the group consisting in all matrices $\sigma \in \mathcal{M}_m(\mathcal{O})$ such that $\langle \sigma p, \sigma q\rangle = \langle p, q\rangle$, for all $p, q \in \mathcal{O}^m$. It is easily to prove that $\mathcal{O}(1)$ can be identified, via an isomorphism, with $Op(1)$.

The Lie algebra $\mathcal{O}^-$ associated by means of the usual bracket to the associative algebra $\mathcal{O}$ is isomorphic to $su(2) \oplus su(2) \oplus \mathcal{D}^-$.

3. Almost Clifford structures. Let (M^{8n}, Q) be an almost Clifford manifold. The tensor fields (J_a) and (J'_a) with $a \in \{1, 2, ..., 6\}$ defining canonical basis of Q on coordinate neighborhoods U, U', respectively, are related on $U \cap U'$ by:

$$J'_a = \sum_{b=1}^{6} s_{ab} J_b \tag{3.1}$$

where $[s_{ab}]$ belongs to a subgroup of $SO(6)$ isomorphic with $SO(3) \times SO(3)$. The structural group of (M^{8n}, Q) is $G = GL(n, \mathcal{O}) \cdot Sp(1)$, where $GL(n, \mathcal{O})$ is the real representation of the linear group of square nondegenerate matrices of order n with entries octons; it can be identified with a subgroup of $GL(8n, \mathbb{R})$ (see [3]).

In this section we consider the situation when there exists three compatible (almost) complex structures I_1, I_2, I_3 such that $I_1 \neq \pm I_2$, $I_1 \neq \pm I_3$, $I_j \neq I_k \circ I_\ell$ for all $(j, k, \ell) \in S_3$ (here S_3 denote the symmetric group with three elements) which are globally defined on M. We can always choose a basis $H = (J_\alpha), \alpha = 1, 2, ..., 6$ such that $I_1 = J_1$, $I_2 = a_1J_1 + a_2J_2 + a_3J_3 + a_4J_4 + a_5J_5 + a_6J_6$, $I_3 = b_1J_1 + b_2J_2 + b_3J_3 + b_4J_4 + b_5J_5 + b_6J_6$, where

$\sum_{i=1}^{6} a_i^2 = 1$, $\sum_{i=1}^{6} b_i^2 = 1$ and $a_1a_6 - a_2a_5 + a_3a_4 = 0$, $b_1b_6 - b_2b_5 + b_3b_4 = 0$. Then, we have $\langle I_1, I_2\rangle = a_1$, $\langle I_1, I_3\rangle = b_1$, $\langle I_2, I_3\rangle = a_1b_1 + ... + a_6b_6$. We are interested in the globally defined angle functions $a_1, b_1 : M \to [-1, 1]$ given by $a_1(p) = \langle I_1, I_2\rangle(p)$, $b_1(p) = \langle I_1, I_3\rangle(p)$, respectively.

Remark 1. (1) If $a_1(p) = \pm 1$ then $I_1 = \pm I_2$ at p and I_1, I_2 commute at p; conversely, if $a_1 \neq 0$ and I_1, I_2 commute at p, then necessarily $a_1 \pm 1$ and $I_1 = \pm I_2$.

(2) The anticommutator $\{I_1, I_2\} = \frac{1}{2}(I_1I_2 + I_2I_1)$ satisfies the following identity

$$\{I_1, I_2\} = -a_1 id + (a_4 - a_5 + a_6)J_7.$$

Therefore $a_1(p) = 0$ and $a_6(p) = -a_4 + a_5$ if and only if I_1 and I_2 anti-commute in p, or equivalently, $I_2 = a_2(J_2 + \lambda J_3) + a_4(J_4 + \lambda J_5 + (\lambda - 1)J_6)$. Similar result holds regarding the anti-commutator $\{I_1, I_3\}$. Actually, the smooth functions $a_1, b_1 : M \to [-1, 1]$ measure the angles of I_1 with I_2 and I_3, respectively.

(3) The commutator of two endomorphisms P_1, P_2 is defined by $[P_1, P_2] = \frac{1}{2}(P_1P_2 - P_2P_1)$. By a straightforward computation we get:

$$\begin{array}{rcl} [I_1, I_2] & = & -a_4J_2 - a_5J_3 + a_2J_4 + a_3J_5 \\ [I_1, I_3] & = & -b_4J_2 - b_5J_3 + b_2J_4 + b_3J_5 \\ [I_1, [I_1, I_2]] & = & -a_2J_2 - a_3J_3 - a_4J_4 - a_5J_5 \\ [I_1, [I_1, I_3]] & = & -b_2J_2 - b_3J_3 - b_4J_4 - b_5J_5. \end{array}$$

They belong to Q and therefore, at each point where $I_1 \neq \pm I_2$, $I_1 \neq \pm I_3$, we can find

$$\begin{array}{l} J_2 = \alpha_1[I_1, I_2] + \alpha_2[I_1, I_3] + \alpha_3 I_1 \circ [I_1, I_2] + \alpha_4 I_1 \circ [I_1, I_3], \\ J_3 = \beta_1[I_1, I_2] + \beta_2[I_1, I_3] + \beta_3 I_1 \circ [I_1, I_2] + \beta_4 I_1 \circ [I_1, I_3]. \end{array} \tag{3.2}$$

Further, we get

$$J_4 = I_1 \circ J_2, \quad J_5 = I_1 \circ J_3, \quad J_6 = J_2 \circ J_3, \quad J_7 = I_1 \circ J_6. \tag{3.3}$$

Conversely, given the complex structures I_1, I_2, I_3 as before, we consider $J_1 = I_1$ and define J_2 and J_3 by formula (3.2) with appropriately locally defined functions α_i, β_i, $i = 1, 2, 3, 4$; then J_1, J_2, J_3 are two by two anti-commuting and $J_1^2 = J_2^2 = J_3^2 = -Id$.

We recall some definitions and results on the relation between hyper-complex and Clifford structures.

Let $H=(J_\alpha)$ be an almost hypercomplex structure on M. Recall that the Nijenhuis tensor $N(A,B)$ of two endomorphims A, B is defined by

$$\begin{aligned}N(A,B)(X,Y) &= N(AX,BY)-AN(BX,Y)\\ &-BN(X,AY)+N(BX,Ay)\\ &-BN(AX,Y)-AN(X,BY)+(AB+BA)[X,Y]\end{aligned}$$

for any vector fields X and Y. We denote $N_{ab}=N(J_a,J_b)$ for $a,b=1,2,...,6$.

Let us define the *structure tensor* N of H by

$$N=\sum_{a=1}^{6}N_{aa},$$

and the *structure 1-forms* α_a $(a=1,2,...,6)$ by

$$\alpha_a(X)=\frac{1}{8n-2}trace(Y\to J_aN(X,Y)).$$

Setting $N_XY:=N(X,Y)$ we get

$$N_X=\sum_{a=1}^{6}J_aN_XJ_aY, \tag{3.4}$$

It must be remarked that

$$\sum_{a=1}^{6}\alpha_a(J_aX)=0.$$

The Nijenhuis tensor of an almost complex structure J is of particular importance because, by the Newlander-Niremberg Theorem, $N(J,J)=0$ is a necessary and sufficient condition for the integrability of J.

Taking into account that the ordered triples (J_1,J_2,J_4), (J_1,J_3,J_5), (J_2,J_3,J_6) and (J_4,J_5,J_6) generate subalgebras of $\mathcal{O}$ isomorphic to the real algebra of quaternions and by using the well known results of K. Yano &M. Ako on the integrability of the almost quaternionic manifolds (see [8]) we get the following result.

Proposition 3.1. *Let* (M,Q) *be an almost Clifford manifold. Then*

a) *if two of the six Nijenhuis tensor fields* $N_{11},N_{22},N_{44},N_{12},N_{14},N_{24}$ *vanish, then the others vanish, too;*

b) *if two of the six Nijenhuis tensor fields* $N_{11}, N_{33}, N_{55}, N_{13}, N_{15}, N_{35}$ *vanish, then the others vanish, too;*

c) *if two of the six Nijenhuis tensor fields* $N_{22}, N_{33}, N_{66}, N_{23}, N_{26}, N_{36}$ *vanish, then the others vanish, too;*

d) *if two of the six Nijenhuis tensor fields* $N_{44}, N_{55}, N_{66}, N_{45}, N_{46}, N_{56}$ *vanish, then the others vanish, too.*

Then we can prove the Theorem.

Theorem 3.2. *If* $N_{11} = N_{22} = N_{33} = 0$, *then* $N_{ij} = 0$ *for* $i, j \in \{1, 2, ..., 6\}$.

This result can be improved relative to some three vanishing tensor fields N_{ab} that are not belong to one of the four series a), b), c), d).

Theorem 3.3. *In order that there exists in an almost Clifford manifold a symmetric affine connection such that* $\nabla J_1 = \nabla J_2 = \nabla J_3 = 0$, *it is necessary and sufficient that* $N_{11} = N_{22} = N_{33} = 0$.

Following the argumentation used in [1], one obtains that the structure tensor T^Q of a Clifford structure Q which is locally generated by $H = (J_a)$ is given by

$$T^Q = N + \sum_{a=1}^{6} \partial(\alpha_a \otimes J_a) \tag{3.5}$$

where ∂ denotes the (Spencer's) operator of alternation. Obviously, T^Q depends only on Q. Furthermore, $T^Q = 0$ if and only if Q is a Clifford structure, i.e. there is a torsionless connection preserving Q. In a similar way as the proof of Prop. 2.3 [1] we get:

Proposition 3.4. *The almost Clifford structure* Q *on* M^{8n} *is a Clifford structure if and only if in a neighborhood of any point there exists a local admissible basis* $H = (J_a)$ *such that*

$$N(X, Y) = \sum_{a=1}^{6} (\alpha_a(X) J_a(Y) - \alpha_a(Y) J_a(X)). \tag{3.6}$$

This result was used in proving of the main result of this Note.

Theorem 3.5. *Let Q be an almost Clifford structure on M_{8n}. If there exist three compatible complex structure I_1, I_2, I_3 such that $I_1 \neq \pm I_2$, $I_1 \neq \pm I_3$, $I_j \neq I_k \circ I_\ell$ for all $(j,k,\ell) \in S_3$ (here S_3 denote the symmetric group with three elements), then Q is a Clifford structure.*

A similar situation had been studied by Alekseevski, Marchiafava and Pontecorvo in [1] and by Tricerri in [6] and [7].

REFERENCES

1. Alekseevski, D.V.; Marchiafava, S.; Pontecorvo, M. – *Compatible complex structures on almost quaternionic manifolds*, Trans. Amer. Math. Soc., v. 351, No. 3 (1999), 997–1014.
2. Boyer, Ch.P.– *A note on hyper-hermitian four-manifolds*, Proc. Amer. Math. Soc., 102 (1988), no. 1, 157–164.
3. Burdujan, I.,– *On* Clifford-*like structure*, Bull. Math. Soc. Math. Roumanie, Tome 45(93), no.3-4, 2002, 145–170.
4. Chevalley, Cl.– *Theory of* Lie *Groups*, I, Princeton University Press, Princeton, 1946.
5. Ishihara, S.– *Quaternion -Kahlerian manifolds*, J. Diff. Geom. 9(1974), 483–500.
6. Tricerri, F.– *Sulle varietà dotate di due strutture quasi complesse linearmente independenti*, Riv. Mat. Univ. Parma, v. 3, 1974, 349–358.
7. Tricerri, F.– *Connessioni lineari e metrche Hermitiene sopra varietà dotate di due strutture quasi complesse*, Riv. Mat. Univ. Parma, v. 4, 1975, 177–186.
8. Yano, K.; Ako, M.– *Integrability conditions for almost quaternion structures*, Hokkaido Math. J., v. 1, 1972, 63–86.

Received: 15.X.2007

Department of Mathematics,
University of Agricultural and Veterinary Medicine, Iaşi,
ROMÂNIA
burdujan_ilie@yahoo.com
burdi@univagro-iasi.ro