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GEOMETRICAL OBJECTS AND STRUCTURES ASSOCIATED TO SOME DYNAMICAL SYSTEMS

BY

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Dedicated to Academician Radu Miron at his 80th anniversary

Abstract. The dynamics of some deformable continuous media can be described by systems of first order partial differential equations. To them we associate, canonically, a geometric structure build by objects of “metrical” nature and of “connection”, and thus, a geometry. The associated structure has some properties as: covariant derivation, parallel transportation and autoparallel manifolds such that the trajectories of the given system to be autoparallels with respect to this structure.

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I Geometrical structures on the jet space $J^1(\mathbf{R}^n, M)$

I.1 Natural frames on $J^1(\mathbf{R}^n, M)$. Let M be a smooth manifold (the configuration space) of finite dimension m and of local coordinates (u^a). Let $J^1(\mathbf{R}^n, M)$ be the evolution space of dimension $n+m+nm$ and $J^1_n(M)$ be the phases space of dimension $m(n+1)$.

A point in $J^1(\mathbf{R}^n, M)$ is locally given by its coordinates (x^i, u^a, u^a_i). Let $\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial u^a}, \frac{\partial}{\partial u^a_i}\right)$ be the natural frame and $\left(dx^i, du^a, du^a_i\right)$ its dual.

We consider a change of local chart on $J^1(\mathbf{R}^n, M)$ given by:

$$\bar{x}^i = x^i,\ \bar{u}^a = \bar{u}^a(u^\alpha),\ \bar{u}^a_i = \frac{\partial \bar{u}^a}{\partial u^\alpha} u^\alpha_i. \tag{1.1}$$

On the set of functions $f(x^h, u^c)$, the partial differential operators (along the solutions) are given by: $\frac{d}{dx^i} = \frac{\partial}{\partial x^i} + u^c_i \frac{\partial}{\partial u^c} : f \to \frac{\partial f}{\partial x^i} + u^c_i(x^h) \frac{\partial f}{\partial u^c}$. In particular, if $f = f(u^a)$, we have $\frac{df}{dx^i} = u^a_i \frac{\partial f}{\partial u^a}$.

Proposition. *The following properties hold:*

$$\begin{aligned}&\frac{\partial \bar{u}^a_i}{\partial u^\alpha_j} = \frac{\partial \bar{u}^a}{\partial u^\alpha}\delta^j_i, \ \frac{\partial \bar{u}^a_i}{\partial u^\alpha} = \frac{\partial^2 \bar{u}^a}{\partial u^\alpha \partial u^\beta}u^\beta_i, \ \frac{d}{dx^i}\left(\frac{\partial \bar{u}^a}{\partial u^\alpha}\right) = \frac{\partial \bar{u}^a_i}{\partial u^\alpha},\\ &\frac{\partial \bar{u}^a_i}{\partial u^\alpha_j}\frac{\partial}{\partial \bar{u}^a_i} = \frac{\partial \bar{u}^a}{\partial u^\alpha}\frac{\partial}{\partial \bar{u}^a_j}, \ \frac{\partial \bar{u}^a_i}{\partial u^\alpha_j}du^\alpha_j = \frac{\partial \bar{u}^a}{\partial u^\alpha}du^\alpha_i.\end{aligned} \tag{1.2}$$

With respect to the above transformations, the natural frame and its dual change, respectively, by the rules:

$$\begin{aligned}&\frac{\partial}{\partial \bar{u}^a} = \frac{\partial u^\alpha}{\partial \bar{u}^a}\frac{\partial}{\partial u^\alpha} + \frac{\partial u^\alpha_j}{\partial \bar{u}^a}\frac{\partial}{\partial u^\alpha_j}, \quad d\bar{u}^a = \frac{\partial \bar{u}^a}{\partial u^\alpha}du^\alpha,\\ &\frac{\partial}{\partial \bar{u}^a_i} = \frac{\partial u^\alpha}{\partial \bar{u}^a}\frac{\partial}{\partial u^\alpha_i}, \quad d\bar{u}^a_i = \frac{\partial \bar{u}^a_i}{\partial u^\alpha}du^\alpha + \frac{\partial \bar{u}^a}{\partial u^\alpha}du^\alpha_i.\end{aligned} \tag{1.3}$$

By (1.3) follows that the vectors $(\frac{\partial}{\partial u^a_i})_y$, in a point $y(u^a, u^a_i)$ of the geometrical zone corresponding to the given map, generate a vectorial subspace V_y of the vectorial space $T(J^1_n(M))$ which has the base $\{(\frac{\partial}{\partial u^a})_y, (\frac{\partial}{\partial u^a_i})_y\}$. This space is called *vertical*. Thus, a vertical distribution is canonically associated (it does not depend on the considered local map).

I.2 Distinguished metrics on M. Let assume that on M are given n (non-degenerated and non-symmetric) "*d-metrics*" of components a^i_{ab} ($i = \overline{1,n}$) and, for any i, we denote by (a^{ab}_i) the inverse matrix of (a^i_{ab}). These components change, on a change of local chart, by the rules:

$$\bar{a}^i_{ab} = \frac{\partial u^\alpha}{\partial \bar{u}^a}\frac{\partial u^\beta}{\partial \bar{u}^b}a^i_{\alpha\beta}, \ \bar{a}^{ab}_i = \frac{\partial \bar{u}^a}{\partial u^\alpha}\frac{\partial \bar{u}^b}{\partial u^\beta}a^{\alpha\beta}_i. \tag{1.4}$$

With them, we build the geometric objects ε^j_i, $i, j = \overline{1,n}$, of components:

$$\varepsilon^{aj}_{ib} = \frac{1}{n}a^{ac}_i a^j_{cb}. \tag{1.5}$$

On a change of local chart, they change by the rules:

$$\bar{\varepsilon}^{aj}_{ib} = \frac{\partial \bar{u}^a}{\partial u^\alpha}\frac{\partial u^\beta}{\partial \bar{u}^b}\varepsilon^{\alpha j}_{i\beta}, \ (\forall i, j = \overline{1,n}). \tag{1.6}$$

Proposition. *The following properties hold:*

1) $det\ \varepsilon_i^j = det(\varepsilon_{ib}^{aj}) \neq 0$, $\forall i, j = \overline{1, n}$;
2) $\varepsilon_{ic}^{ah}\varepsilon_{hb}^{cj} = \varepsilon_{ib}^{aj}$;
3) $\varepsilon_{ic}^{ah}\varepsilon_{hb}^{ci} = \delta_b^a$; $\varepsilon_{ib}^{ai} = \delta_b^a$.

I.3 Connections. Let us consider on the manifold M a geometric object given, in a local map, by a set of functions (a_{ab}).

Definition. *We say that we have a connection N on the manifold M, if the system of local functions a_{ab} change, on a change of local chart, by the rules:*

$$\bar{a}_{ab} = \frac{\partial u^\alpha}{\partial \bar{u}^a}\left(\frac{\partial u^\beta}{\partial \bar{u}^b}a_{\alpha\beta} + \frac{\partial u_i^\beta}{\partial \bar{u}^b}a_{\alpha\beta}^i\right), \tag{1.7}$$

where $a_{\alpha\beta}^i$ are the components of the metrics defined above.

The functions a_{ab}, associated to the local map, are called *Christoffel symbols of first kind* of a connection. They generalize the Christoffel coefficients of first kind of a classical connection ([1], [2]).

With them, by contraction with the given metrics, we build the functions:

$$\underset{i}{M}{}_b^a = \frac{1}{n}a_i^{ac}a_{cb}, \tag{1.8}$$

which are called *Christoffel symbols of second kind*.

Proposition. *On a change of local chart on M, the functions $\underset{i}{M}{}_b^a$ change by the rules:*

$$\underset{i}{\overline{M}}{}_b^a = \frac{\partial \bar{u}^a}{\partial u^\alpha}\left(\frac{\partial u_j^\beta}{\partial \bar{u}^b}\varepsilon_{i\beta}^{\alpha j} + \frac{\partial u^\beta}{\partial \bar{u}^b}\underset{i}{M}{}_\beta^\alpha\right). \tag{1.9}$$

Remark. In the case that all the d-metrics of components a_{ab}^i $(i = \overline{1, n})$ coincide (and, in particular, if $n = 1$) we have $\varepsilon_{jb}^{ai} = \delta_b^a$ and the formulas (1.9) become the rules of change of the components of a nonlinear connection ([3], [4]).

I.4 Differential operators. We define a set of n partial differential operators D_i such that on functions they satisfy $D_i f = \frac{df}{dx^i}$.

Let X be a d-vector field, locally written $X = X^a \frac{\partial}{\partial u^a}$. We build the partial differential operators of the components X^a by the formulas:

$$D_i X^a = \varepsilon_{ib}^{aj} \frac{dX^b}{dx^j} \tag{1.10}$$

and the action of the operators on the vectors $\frac{\partial}{\partial u^a}$ of the natural frame by:

$$D_i \left(\frac{\partial}{\partial u^a} \right) = \underset{i}{M}{}_a^b \frac{\partial}{\partial u^b}, \tag{1.11}$$

where the functions $\underset{i}{M}{}_b^a$ change, on a change of local chart, by the rules (1.9).

Definition. *We call partial covariant derivative of a vector field* $X = X^a \frac{\partial}{\partial u^a}$*, along the direction* i *in* $\mathbf{R}^n$*, the field:*

$$Y_i = D_i X = \left(D_i X^a + \underset{i}{M}{}_b^a X^b \right) \frac{\partial}{\partial u^a}.$$

The functions:

$$\frac{\delta X^a}{\delta x^i} = D_i X^a + \underset{i}{M}{}_b^a X^b = \varepsilon_{ib}^{aj} \frac{dX^b}{dx^j} + \underset{i}{M}{}_b^a X^b$$

are called *partial covariant derivatives* of the components X^a. They are d-vector fields, they change, on a change of local chart, by the rules:

$$\overline{\varepsilon}_{ib}^{aj} \frac{d\overline{X}^b}{dx^j} + \underset{i}{\overline{M}}{}_b^a \overline{X}^b = \frac{\partial \bar{u}^a}{\partial u^\alpha} \left(\varepsilon_{i\beta}^{\alpha j} \frac{dX^\beta}{dx^j} + \underset{i}{M}{}_\beta^\alpha X^\beta \right).$$

Let us consider a submanifold k on M given by the functions: $u^a = u^a(x^h)$, and a submanifold K on $J^1(\mathbf{R}^n, M)$, the lift of the submanifold k, defined by: $u^a = u^a(x^h)$, $u_i^a = \frac{\partial u^a}{\partial x^i}$.

We say that a d-vector field X is *transported by parallelism* on the manifold K, if along this manifold the partial covariant derivatives with respect to the given connection vanish:

$$\varepsilon_{ib}^{aj} \frac{dX^b}{dx^j} + \underset{i}{M}{}_b^a X^b = 0. \tag{1.12}$$

A submanifold $u^a = u^a(x^h)$ is called *autoparallel* if we have:

$$\varepsilon^{aj}_{ib}\frac{\partial^2 u^b}{\partial x^j \partial x^h} + \underset{i}{M}{}^a_b \frac{\partial u^b}{\partial x^h} = 0. \tag{1.13}$$

The operators D_i can be prolonged on 1-forms by the formula: $D_i(f\omega) = D_i(f)\cdot\omega + fD_i(\omega)$, if the values of D_i on the vectors du^a of the dual base are defined:

$$D_i(du^a) = \underset{i}{\overset{*}{M}}{}^a_b du^b.$$

By the compatibility condition follows $\underset{i}{\overset{*}{M}}{}^a_b + \underset{i}{M}{}^a_b = 0$. The covariant derivatives of the components of a 1-form $\omega = a_a du^a$ are given by:

$$\frac{\delta a_b}{\delta x^i} = \varepsilon^{aj}_{ib}\frac{da_a}{dx^j} - \underset{i}{M}{}^a_b a_a.$$

By extension, the covariant derivatives can be defined for any tensor field.

I.5 Adapted frames. Given a connection (locally expressed by the components $\underset{i}{M}{}^a_b$), to any natural frame $(\frac{\partial}{\partial u^a}, \frac{\partial}{\partial u^a_i})$ on $T(J^1_n(M))$, which generates in each point of the tangent space a vectorial space S of dimension $m(n+1)$, and to its dual frame (du^a, du^a_i) we associate adapted frames $(\frac{\delta}{\delta u^a}, \frac{\delta}{\delta u^a_i})$ and $(\delta u^a, \delta u^a_i)$ respectively.

We define the adapted frame $(\delta u^a, \delta u^a_i)$ by:

$$\delta u^a = du^a, \quad \delta u^a_i = \underset{i}{M}{}^a_b du^b + \varepsilon^{aj}_{ib} du^b_j. \tag{1.14}$$

These covectors change, on a change of local chart, by the rules:

$$\delta \bar{u}^a = \frac{\partial \bar{u}^a}{\partial u^\alpha}\delta u^\alpha, \ \delta \bar{u}^a_i = \frac{\partial \bar{u}^a}{\partial u^\alpha}\delta u^\alpha_i. \tag{1.15}$$

We build the adapted frame $(\frac{\delta}{\delta u^a}, \frac{\delta}{\delta u^a_i})$ so that the relations:

$$\delta u^a\left(\frac{\delta}{\delta u^b}\right) = \delta^a_b, \ \delta u^a\left(\frac{\delta}{\delta u^b_j}\right) = 0, \ \delta u^a_i\left(\frac{\delta}{\delta u^b}\right) = 0, \ \delta u^a_i\left(\frac{\delta}{\delta u^b_j}\right) = \delta^a_b\delta^j_i$$

hold. It follows:

Proposition. *The frame given by:*

$$\frac{\delta}{\delta u^a} = \frac{\partial}{\partial u^a} - \underset{j}{N}{}^b_a \frac{\partial}{\partial u^b_j}, \quad \frac{\delta}{\delta u^a_i} = \widetilde{\varepsilon}^{bi}_{ja} \frac{\partial}{\partial u^b_j}, \tag{1.16}$$

where $\widetilde{\varepsilon}^{ah}_{ic}\varepsilon^{cj}_{hb} = \delta^a_b\delta^j_i$, $\underset{i}{N}{}^a_b = \widetilde{\varepsilon}^{ah}_{ic}\underset{h}{M}{}^c_b$, *is dual to the frame* (1.14).

On a change of local chart, we have:

$$\frac{\delta}{\delta \bar{u}^a} = \frac{\partial u^\alpha}{\partial \bar{u}^a}\frac{\delta}{\delta u^\alpha}, \ \frac{\delta}{\delta \bar{u}^a_i} = \frac{\partial u^\alpha}{\partial \bar{u}^a}\frac{\delta}{\delta u^\alpha_i}. \tag{1.17}$$

The vectors $\frac{\delta}{\delta u^a_i}$ generate the vectorial space V and the vectors $\frac{\delta}{\delta u^a}$ generate a complementary space H, called *horizontal space*, such that: $T(J^1_n(M)) = H \otimes V$.

By (1.17) follows that, given a connection $(\underset{i}{M}{}^a_b)$, the vertical distribution V, of dimension nm and generated by the vectors $\frac{\delta}{\delta u^a_i}$, can be decomposed in n vertical distributions V_i, each of dimension m (due to the existence of the metrics a^i_{ab}), such that: $V = \oplus V_i$, and we have a horizontal distribution H, of dimension m, generated by the vectors $\frac{\delta}{\delta u^a}$.

The space S generated by the vectors $(\frac{\delta}{\delta u^a}, \frac{\delta}{\delta u^a_i})$, can be written, due to the existence of the connection, as the direct summation: $S = H \otimes V$. We identify the existence of the connection with this decomposition.

Given the adapted frame (1.16) and the associated Lie brackets $\left[\frac{\delta}{\delta u^a}, \frac{\delta}{\delta u^b}\right]$, the property: $\left[\frac{\delta}{\delta u^a}, \frac{\delta}{\delta u^b}\right] = \underset{j}{R}{}^c_{ab}\frac{\partial}{\partial u^c_j}$, where $\underset{j}{R}{}^c_{ab} = \frac{\delta}{\delta u^b}(\underset{j}{N}{}^c_a) - \frac{\delta}{\delta u^a}(\underset{j}{N}{}^c_b)$, holds.

I.6 Vertical and horizontal operators associated to a connection. Let X be a vector field on $J^1_n(M)$, locally written in a natural map $X = X^a\frac{\partial}{\partial u^a} + X^a_i\frac{\partial}{\partial u^a_i}$. With respect to the adapted frame, X becomes:

$$X = X^a\frac{\delta}{\delta u^a} + (X^a\underset{j}{M}{}^b_a + X^a_i\varepsilon^{bi}_{ja})\frac{\delta}{\delta u^b_j}. \tag{1.18}$$

It has a vertical component vX in V and a horizontal component hX in H such that $X = vX + hX$. We have:

$$vX = (X^a\underset{j}{M}{}^b_a + X^a_i\varepsilon^{bi}_{ja})\frac{\delta}{\delta u^b_j}, \quad hX = X^a\frac{\delta}{\delta u^a}, \tag{1.19}$$

which are written in the natural frame:

$$(1.20) \qquad vX = (X^a \underset{j}{N}{}^b_a + X^b_j)\frac{\partial}{\partial u^b_j}, \quad hX = X^a \frac{\partial}{\partial u^a} - X^a \underset{j}{N}{}^b_a \frac{\partial}{\partial u^b_j}.$$

If X is vertical than $hX = 0$, if X is horizontal than $vX = 0$.

Proposition. *The following properties hold:*

$$vvX = vX, \quad hhX = hX, \quad vhX = 0, \quad hvX = 0.$$

II The study of some elementary dynamical systems

II.1 Systems of implicit first order partial differential equations. Let us consider a smooth manifold M of dimension m, the first order jet space $J^1(\mathbf{R}^n, M)$ and the fibre space $J^1(\mathbf{R}^n, M) \times_M T^*M$.

A function $F : J^1(\mathbf{R}^n, M) \to T^*_u M$ is locally written as:

$$F : (x^i, u^a, u^a_i) \to F(x^i, u^a, u^a_i) = F_a(x^i, u^b, u^b_i)du^a.$$

Its kernel is defined by the conditions:

$$(2.1) \qquad F_a(x^i, u^b, u^b_i) = 0.$$

Definition. *The system* (2.1) *is called system of implicit first order partial differential equations.*

We assume that the system is non-degenerated, that means $det(\frac{\partial F_a}{\partial u^b_i}) \neq 0, \forall i = \overline{1, n}$.

Definition. *A system of functions $u^a = u^a(x^i)$ is called solution of the system* (2.1) *if they, together with there partial derivatives, transform the equations* (2.1) *in identities.*

It follows that a solution defines a submanifold k, whose lift K is in the kernel of the function F, expressed by the equations (2.1).

II.2 Elementary systems of partial differential equations

Definition. *We say that a dynamical system is elementary if its equations are given by:*

$$(2.2) \qquad F_a(u^b, u^b_i) = 0.$$

We associate to the system (2.2) a system of linear forms called *variational forms* and locally written ([5]):

$$M_a^X = a_{ab}^i \frac{d\xi^b}{dx^i} + a_{ab}\xi^b, \tag{2.3}$$

where $X = \xi^a(u^b)\frac{\partial}{\partial u^a}$ is a vector field on M and

$$a_{ab}^i = \frac{\partial F_a}{\partial u_i^b},\ a_{ab} = \frac{\partial F_a}{\partial u^b},\ det(a_{ab}^i) \neq 0,\ \forall i = \overline{1,n}. \tag{2.4}$$

With respect to the transformation rules (1.1), the coefficients of the variational form (2.3) change by the formulas:

$$\begin{aligned} \bar{a}_{ab}^i &= \frac{\partial u^\alpha}{\partial \bar{u}^a}\frac{\partial u^\beta}{\partial \bar{u}^b}a_{\alpha\beta}^i, \\ \bar{a}_{ab} &= \frac{\partial u^\alpha}{\partial \bar{u}^a}\left(\frac{\partial u^\beta}{\partial \bar{u}^b}a_{\alpha\beta} + \frac{\partial u_i^\beta}{\partial \bar{u}^b}a_{\alpha\beta}^i\right) + \frac{\partial^2 u^\gamma}{\partial \bar{u}^a \partial \bar{u}^b}F_\gamma. \end{aligned} \tag{2.5}$$

Proposition. *For any $i = \overline{1,n}$, the coefficients a_{ab}^i are the components of a non-degenerated distinguished tensor on M.*

They represent a set of n "metrics" (as in I.2), which are associated to the system (2.2).

For any i, let (a_i^{ab}) be the inverse of the matrix (a_{ab}^i). With them we can build the coefficients $\varepsilon_{ib}^{aj} = \frac{1}{n}a_i^{ac}a_{cb}^j$, defined by (1.5).

On the manifold K of the solutions of (2.2) ($F_a = 0$), the formulas (2.5_2) become (1.7). The functions (a_{ab}) represent the Christoffel coefficients of first kind of a connection.

We can define the functions $\underset{i}{M}{}_b^a = \frac{1}{n}a_i^{ac}a_{cb}$ as in (1.8).

Proposition. *The coefficients $\underset{i}{M}{}_b^a$ define a connection on K.*

Indeed, on a change of local chart on M, they change on $J^1(\mathbf{R}^n, M)$ by the rules:

$$\overline{\underset{i}{M}}{}_b^a = \frac{\partial \bar{u}^a}{\partial u^\alpha}\left(\frac{\partial u_j^\beta}{\partial \bar{u}^b}\varepsilon_{i\beta}^{\alpha j} + \frac{\partial u^\beta}{\partial \bar{u}^b}\underset{i}{M}{}_\beta^\alpha\right) + \frac{1}{n}\frac{\partial \bar{u}^a}{\partial u^\alpha}\frac{\partial \bar{u}^c}{\partial u^\gamma}\frac{\partial^2 u^\beta}{\partial \bar{u}^b \partial \bar{u}^c}a_i^{\alpha\gamma}F_\beta$$

and on K, these formulas become (1.9). □

This connection is canonically associated to the system (2.2) and it implies the existence of a partial covariant derivation D_i along the direction i defined in the same way as in I.4, and, thus, a derivation along any direction $\Lambda = \lambda^i \frac{\partial}{\partial x^i}$ in $\mathbf{R}^n$: $D = \lambda^i D_i$.

We have:

Theorem. *The lift of an autoparallel submanifold, which has a point on the solutions submanifold is completely situated on that submanifold* (*it is a solution of the equations* (2.2)).

Proof. Let us consider an autoparallel submanifold, satisfying the conditions (1.13), where the coefficients ε^{aj}_{ib} and $\underset{i}{M}{}^{a}_{b}$ are build with the objects a^i_{ab} and a_{ab} given by (2.4).

The system (1.13) is equivalent with: $a^j_{ab} u^b_{jh} + a_{ab} u^b_h = 0$ and by contraction with dx^h, it follows: $\frac{\partial F_a}{\partial u^b_j} du^b_j + \frac{\partial F_a}{\partial u^b} du^b = dF_a = 0.$ $\square$

II.3 Elementary dynamical systems. In the set of systems of first order partial differential equations (2.2), we consider the following relation of equivalence:

Definition. *Two systems* $F_a(u^b, u^b_i) = 0$ *and* $G_a(u^b, u^b_i) = 0$ *are called equivalent if there exists a matrix* (*a d-tensor*) (A^b_a) *with* $det(A^b_a) \neq 0$ *so that* $G_a = A^b_a F_b = 0$.

By this definition follows that all the systems of a class admit the same solutions. We call a class of equivalent systems *dynamical system*.

A property or notion is geometric, if it is related to the elements given by a system of equations, and it is dynamic, if it holds for all the systems of an equivalence class.

Proposition. *The notion of metric and that of Christoffel symbols of the first kind are geometrical notions, they change when we change the representative of the dynamical system. The notion of connection* (*defined by its coefficients*) *is dynamic.*

Indeed, by the relations (2.4) written for the systems $F_a = 0$ and $G_a = 0$, we obtain the coefficients $a^i_{ab} = \frac{\partial F_a}{\partial u^b_i}$, $a_{ab} = \frac{\partial F_a}{\partial u^b}$ and $b^i_{ab} = \frac{\partial G_a}{\partial u^b_i}$, $b_{ab} = \frac{\partial G_a}{\partial u^b}$ respectively.

Let consider the functions $\underset{i}{M}{}^{a}_{b} = a^{ac}_i a_{cb}$ and $\underset{i}{\widehat{M}}{}^{a}_{b} = b^{ac}_i b_{cb}$.

On the solutions manifold we have the geometric properties $\underline{b^i_{ab}} = A^c_a a^i_{cb}$, $b_{ab} = A^c_a a_{cb}$ and the dynamic property $\underset{i}{M}{}^a_b = a^{ac}_i a_{cb} = b^{ac}_i b_{cb} = \widehat{\underset{i}{M}{}^a_b}$, which certify the dynamical character of the connection. □

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