ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" DIN IAŞI (S.N.)
MATEMATICĂ, Tomul LIII, 2007, Supliment

SPECTRAL PROPERTIES OF A CLASS OF HODGE-DE RHAM LIKE OPERATORS

BY

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Dedicated to Academician Radu Miron at his 80th anniversary

Abstract. Let (M, g) be a closed, orientable, n-dimensional smooth Riemannian manifold. To each pointwise nonsingular tensor field $\mathbf{h}$ of type $(1,1)$ on M such that the Nijenhuis derivation corresponding to $\mathbf{h}$ vanishes, one can associate an elliptic self-adjoint second order differential operator $\Delta_{\mathbf{h}}^{(k)}$, which is a generalization of the classical Hodge-de Rham operator $\Delta^{(k)}$ (or k-Laplacian), $k \in \{0, \ldots, n\}$, (see EISEMAN and STONE [8]). In this paper we shall discuss some questions related to the Hodge type decomposition induced by $\Delta_{\mathbf{h}}^{(k)}$ as well as spectral properties of $\Delta_{\mathbf{h}}^{(k)}$.

Mathematics Subject Classification 2000: 58J10, 58J50, 58J35.

Key words: Riemannian metric, Hodge-de Rham like operator, Nijenhuis derivation, generalized Hodge theory, smooth dependence, trace class operator, spectrum, harmonic projector, Green operator, asymptotic expansion.

1. Hodge-de Rham like operators associated to a tensor field of type $(1,1)$ and a generalized Hodge theory. Let (M, g) be a closed (i.e. compact and without boundary), orientable, n-dimensional smooth Riemannian manifold, $C^\infty(M) = A^0(M)$ the real algebra of smooth real functions on M and $\mathfrak{X}(M)$ the $C^\infty(M)$-module of smooth vector fields on M. Recall that a smooth tensor field of type $(1,1)$ on M is a mapping $\mathbf{h} : \mathfrak{X}(M) \to \mathfrak{X}(M)$ that is $C^\infty(M)$-linear. Let $\mathcal{T}_1^1(M)$ be the $C^\infty(M)$-algebra of all tensor fields of type $(1,1)$ on M, $E := A^1(M)$ the $C^\infty(M)$-module of smooth differential 1-forms on M and $L_{C^\infty(M)}(E, E)$ the $C^\infty(M)$-

algebra of all endomorphisms of E. The mapping

$$\mathcal{T}_1^1(M) \to L_{C^\infty(M)}(E,E), \quad \mathbf{h} \mapsto b \circ \mathbf{h} \circ \sharp,$$

where $b : \mathfrak{X}(M) \to A^1(M)$ and $\sharp : A^1(M) \to \mathfrak{X}(M)$ are the musical isomorphisms defined by the Riemannian metric g, is a $C^\infty(M)$-linear isomorphism. In what follows we shall identify these two $C^\infty(M)$-algebras.

Fix a smooth tensor field $\mathbf{h} \in \mathcal{T}_1^1(M) = L_{C^\infty(M)}(E,E)$ and let $A^k(M)$ be the $C^\infty(M)$-module of smooth differential k-forms on M, $0 \leq k \leq n$. Note that $\mathbf{h}$ induces a $C^\infty(M)$-linear endomorphism $\mathbf{h}_k^{(1)} : A^k(M) \to A^k(M)$ for each $k \in \{0,\ldots,n\}$, defined as follows:

$$\mathbf{h}_0^{(1)}(f) := 0 \quad \text{for each} \quad f \in A^0(M),$$

$$\mathbf{h}_k^{(1)}(\omega^1 \wedge \ldots \wedge \omega^k) := \sum_{j=1}^{k} \omega^1 \wedge \ldots \wedge \mathbf{h}(\omega^j) \wedge \ldots \wedge \omega^k, \quad 1 \leq k \leq n,$$

where $\omega^1,\ldots,\omega^k \in E$. In particular, $\mathbf{h}_1^{(1)} = \mathbf{h}$ and

$$\mathbf{h}_n^{(1)}(\omega^1 \wedge \ldots \wedge \omega^n) = Trace(\mathbf{h})\omega^1 \wedge \ldots \wedge \omega^n \quad \text{for any} \quad \omega^1,\ldots,\omega^n \in E.$$

The family $\mathbf{h}^{(1)} := \{\mathbf{h}_k^{(1)} | k \in \{0,\ldots,n\}\}$ defines a graduate derivation of degree 0 of the exterior algebra $\bigoplus_{k=0}^n A^k(M)$ associated to M and let

$$d_{\mathbf{h}} := \mathbf{h}^{(1)} \circ d - d \circ \mathbf{h}^{(1)}$$

be the graduate commutator of $\mathbf{h}^{(1)}$ and d, where d is the graduate derivation of degree $+1$ defined be the exterior differentials. Let

$$(d_{\mathbf{h}})_{\mathbf{h}} := \mathbf{h}^{(1)} \circ d_{\mathbf{h}} - d_{\mathbf{h}} \circ \mathbf{h}^{(1)}$$

be the graduate commutator (for the definition of this concept, see for example Craioveanu [2]) of the graduate derivations $\mathbf{h}^{(1)}$ and $d_{\mathbf{h}}$, and

$$[\mathbf{h},\mathbf{h}] := \frac{1}{2}[d_{\mathbf{h}\circ\mathbf{h}} - (d_{\mathbf{h}})_{\mathbf{h}}]$$

the Nijenhuis derivation associated to $\mathbf{h}$ (for the relation between the Nijenhuis derivation and conservation laws, see Stone [17], [18]).

Lemma 1. (Eiseman and Stone [8])

(i) *If* $[\mathbf{h},\mathbf{h}]=0$, *then* $(A(M),d_{\mathbf{h}}):=(A^k(M),d_{\mathbf{h}}^{(k)})_{0\leq k\leq n}$ *is a complex of cochains;*

(ii) *If* $\mathbf{h}$ *is nonsingular on* M, *then* $d_{\mathbf{h}}$ *satisfies the Poincaré lemma.*

The linear operator $\mathbf{h}_k^{(1)}:A^k(M)\to A^k(M)$ admits an adjoint operator $(\mathbf{h}_t)_k^{(1)}:A^k(M)\to A^k(M)$ with respect to the canonical inner product $\langle,\rangle:A^k(M)\times A^k(M)\to\mathbb{R}$ for each $k\in\{0,\ldots,n\}$.

One can easily check that the linear operator

$$\delta_{\mathbf{h}}^{(k+1)}:A^{k+1}(M)\to A^k(M),\quad \delta_{\mathbf{h}}^{(k+1)}:=\delta^{(k+1)}\circ(\mathbf{h}_t)_{k+1}^{(1)}-(\mathbf{h}_t)_k^{(1)}\circ\delta^{(k+1)},$$

where $\delta^{(k+1)}:A^{k+1}(M)\to A^k(M)$ is the adjoint of the exterior differential $d^{(k)}:A^k(M)\to A^{k+1}(M)$, is the adjoint of $d_{\mathbf{h}}^{(k)}:A^k(M)\to A^{k+1}(M)$ with respect to the canonical inner products on $A^k(M)$ and $A^{k+1}(M)$ respectively, for each $k\in\{0,\ldots,n-1\}$.

Consequently, for each $k\in\{0,\ldots,n\}$, one can define a Hodge-de Rham like operator

$$\Delta_{\mathbf{h}}^{(k)}:A^k(M)\to A^k(M),\quad \Delta_{\mathbf{h}}^{(k)}:=d_{\mathbf{h}}^{(k-1)}\circ\delta_{\mathbf{h}}^{(k)}+\delta_{\mathbf{h}}^{(k+1)}\circ d_{\mathbf{h}}^{(k)},$$

associated to the smooth tensor field $\mathbf{h}$ (see Eiseman and Stone [8]).

Definition 2. *The elements of the real vector space* $Ker(\Delta_{\mathbf{h}}^{(k)})$ [*resp.* $Im(d_{\mathbf{h}}^{(k-1)})$, *resp.* $Im(\delta_{\mathbf{h}}^{(k+1)})$] *are called* $\mathbf{h}$*-harmonic* k*-forms* (*resp.* $\mathbf{h}$*-exact* k*-forms, resp.* $\mathbf{h}$*-coexact* k*-forms*).

The operator $\Delta_{\mathbf{h}}^{(k)}$ is formally self-adjoint and formally positive. If $\mathbf{h}=1_{A^1(M)}$, then $\Delta_{\mathbf{h}}^{(k)}$ coincides with the classical Hodge-de Rham operator (or k- Laplacian) $\Delta^{(k)}$.

Theorem 3. (Eiseman and Stone [8]) *The tensor field* $\mathbf{h}$ *is nonsingular on* M *if and only if the second order differential operator* $\Delta_{\mathbf{h}}^{(k)}$ *is strongly elliptic,* $0\leq k\leq n$. *In particular, if* $\mathbf{h}$ *is nonsingular on* M, *then the real vector space* $Ker(\Delta_{\mathbf{h}}^{(k)}):=\mathcal{K}_{\mathbf{h}}^{(k)}$ *is finite dimensional,* $0\leq k\leq n$, *and the following orthogonal decomposition* (*with respect to* $\langle,\rangle$) *is valid for any* $k\in\{0,\ldots,n\}$:

$$A^k(M)=\mathcal{K}_{\mathbf{h}}^{(k)}\overset{\perp}{\bigoplus}Im(\Delta_{\mathbf{h}}^{(k)}).$$

Note that if the nonsingular tensor field $\mathbf{h}$ is fixed and the real function $M \ni x \mapsto det(\mathbf{h}(x)) \in \mathbb{R}$ is constant, where $det(\mathbf{h}(x))$ denotes the determinant of the linear automorphism $\mathbf{h}(x) : T_x^*M \to T_x^*M$, there exists a smooth Riemannian metric $\tilde{g}$ on M such that $\Delta_{\mathbf{h}}^{(0)} = \tilde{\Delta}$(=the Laplace-Beltrami operator associated to $\tilde{g}$) (see STONE [19]).

Let $W^{s,2}(A^k(M))$, $s \in \mathbb{R}$, be the Sobolev space associated to $A^k(M)$. The linear operator

$$d_{\mathbf{h}}^{(k)} : A^k(M) \to A^{k+1}(M) \text{ [resp. } \delta_{\mathbf{h}}^{(k)} : A^k(M) \to A^{k-1}(M)]$$

extends to the continuous linear operator

$$d_{\mathbf{h}}^{(k)} : W^{1,2}(A^k(M)) \to W^{0,2}(A^{k+1}(M)) := L^2(A^{k+1}(M))$$

$$[\text{resp. } \delta_{\mathbf{h}}^{(k)} : W^{1,2}(A^k(M)) \to W^{0,2}(A^{k-1}(M))].$$

The previous considerations show that if $\mathbf{h}$ is nonsingular on M and the Nijenhuis derivation $[\mathbf{h}, \mathbf{h}]$ associated to $\mathbf{h}$ vanishes, then the complex $(A(M), d_{\mathbf{h}})$ is elliptic. Hence the following $\mathbf{h}$-dependent version of the Hodge decomposition theorem is valid:

Theorem 4. *If* $\mathbf{h}$ *is a nonsingular smooth tensor field of type* $(1,1)$ *on* M *such that* $[\mathbf{h}, \mathbf{h}] = 0$*, then the following statements are true for each* $k \in \{0, \ldots, n\}$*:*

(i) $A^k(M)$ *and* $L^2(A^k(M))$ *admit the following orthogonal decompositions with respect to the canonical inner product:*

$$A^k(M) = \mathcal{K}_{\mathbf{h}}^{(k)} \overset{\perp}{\bigoplus} d_{\mathbf{h}}^{(k-1)}(A^{k-1}(M)) \overset{\perp}{\bigoplus} \delta_{\mathbf{h}}^{(k+1)}(A^{k+1}(M))$$

(*see* EISEMAN *and* STONE [8]) *and*

$$L^2(A^k(M)) = \mathcal{K}_{\mathbf{h}}^{(k)} \overset{\perp}{\bigoplus} d_{\mathbf{h}}^{(k-1)}(W^{1,2}(A^{k-1}(M))) \overset{\perp}{\bigoplus} \delta_{\mathbf{h}}^{(k+1)}(W^{1,2}(A^{k+1}(M)));$$

(ii) The mapping $\mathcal{K}_{\mathbf{h}}^{(k)} \ni \omega \mapsto [\omega][:=$ *the cohomology class in the cochain complex* $(A(M), d_{\mathbf{h}})$ *associated to* $\omega] \in H^k(A(M), d_{\mathbf{h}}) := Ker d_{\mathbf{h}}^{(k)} / Im d_{\mathbf{h}}^{(k-1)}$*, is a* $\mathbb{R}$*-linear isomorphism.*

Corollary 5. *Under the same assumptions concerning* $\mathbf{h}$ *as in Theorem* 4*, the following assertions are true:*

(i) $\mathcal{K}_{\mathbf{h}}^{(k)} \cong H^k(M;\mathbb{R})$ *for each* $k \in \{0,\dots,n\}$. *In particular,* $\dim \mathcal{K}_{\mathbf{h}}^{(k)} = \beta_k(M)$(=*the k-th Betti number of M*)*;*

(ii) $Index(A(M), d_{\mathbf{h}}) := \sum_k (-1)^k \dim H^k(A(M), d_{\mathbf{h}}) = \chi(M)$(=*the Euler-Poincaré characteristic of M*)*;*

(iii) If $Trace(\mathbf{h})$ *is constant, the star Hodge operator induces a* $\mathbb{R}$*- linear isomorphism from* $\mathcal{K}_{\mathbf{h}}^{(k)}$ *onto* $\mathcal{K}_{\mathbf{h}}^{(n-k)}$ *for each* $k \in \{0,\dots,n\}$.

For more details concerning the h-dependent version of Hodge theory, see Eiseman and Stone [8], [9], [10], Craioveanu and Puta [3], Lalescu and Stretcu [14] (for the classical harmonic theory, see Hodge [13]).

Open problem: *It is an open problem to decide whether or not a Weitzenböck formula is valid for* $\Delta_{\mathbf{h}}^{(k)}$*, that is if there exist a connection Laplacian* Δ^{Λ} (*defined via a linear connection* ∇) *on* $A^k(M)$ *and a bundle endomorphism* $R^W : A^k(M) \to End(A^k(M))$ *induced by the curvature tensor field associated to* ∇ *such that*

$$\Delta_{\mathbf{h}}^{(k)}\omega = \Delta^{\Lambda}\omega - R^W\omega \quad \text{for each} \quad \omega \in A^k(M).$$

2. Smooth dependence of the h-Hodge decomposition on the Riemannian metric. From now on let us assume that $\mathbf{h}$ satisfies the assumptions from Theorem 4 and let M be a closed, n-dimensional smooth manifold.

In what follows, we endow the set $\mathfrak{M}(M)$ of all smooth Riemannian metrics on M with the structure of a Fréchet manifold (see Binz, Śniatycki and Fischer [1], Golubitsky and Guillemin [12], pp.74-78), that is $\mathfrak{M}(M)$ is a manifold locally modeled over a Fréchet space which is a complete metrisable locally convex topological vector space. Using the Riesz representation theorem, for any Riemannian metrics $g_o, g \in \mathfrak{M}(M)$, there exists a smooth automorphism of vector bundles $\Phi_{g_o g} : TM \to TM$ such that

$$g(X,Y) = g_o(\Phi_{g_o g} \circ X, \Phi_{g_o g} \circ Y),$$

for each $X, Y \in \mathfrak{X}(M)$. The automorphism $\Phi_{g_o g}$ is uniquely determined modulo an isometry of (M, g_o) and the following mappings

$$\mathfrak{M}(M) \ni g \mapsto \Phi^*_{g_o g} \in L(L^2(A^k(M)))$$

and

$$\mathfrak{M}(M) \ni g \mapsto (\Phi^*_{g_o g})^{-1} \in L(L^2(A^k(M))),$$

induced by $\Phi_{g_o g}$, where $L(L^2(A^k(M)))$ denotes the space of all bounded linear operators of $L^2(A^k(M))$, are smooth, that is they depend smoothly on g in the sense of the Fréchet manifold structure of $\mathfrak{M}(M)$.

Lemma 6. *Let $g_o, g \in \mathfrak{M}(M)$ be arbitrary, but fixed Riemannian metrics and $S^{(k)}_{g_o}$ (resp. $S^{(k)}_{g}$) : $A^k(M) \to A^{n-k}(M)$ the star Hodge operator associated to g_o (resp. g), $0 \leq k \leq n$. Then*

$$\Phi^*_{g_o g} \circ S^{(k)}_{g_o} = S^{(k)}_{g} \circ \Phi^*_{g_o g} : L^2(A^k(M)) \to L^2(A^{n-k}(M))$$

for each $k \in \{0, 1, \ldots, n\}$. In particular, the mapping

$$\mathfrak{M}(M) \ni g \mapsto S^{(k)}_{g} \in L(L^2(A^k(M)), L^2(A^{n-k}(M))),$$

where $L(L^2(A^k(M)), L^2(A^{n-k}(M)))$ denotes the space of all bounded linear operators from $L^2(A^k(M))$ into $L^2(A^{n-k}(M))$, is smooth, for each $k \in \{0, 1, \ldots, n\}$, that is it depends smoothly on $g \in \mathfrak{M}(M)$.

The star Hodge operator $S^{(k)}_{g_o}$(resp. $S^{(k)}_{g}$) associated to the Riemannian metric g_o (resp. g) $\in \mathfrak{M}(M)$ induces the inner product $<,>_{g_o}$ (resp. $<,>_g$) on the Hilbert space $L^2(A^k(M)) = W^{0,2}(A^k(M))$, such that

$$< \omega_1, \omega_2 >_{g_o} := \int_M \omega_1 \wedge S^{(k)}_{g_o}(\omega_2) =: < \omega_1, A^{(k)}_{g g_o}(\omega_2) >_g$$

for any $\omega_1, \omega_2 \in A^k(M)$, where

$$A^{(k)}_{g g_o} : A^k(M) \to A^k(M), \quad A^{(k)}_{g g_o} := (-1)^{k(n-k)} S^{(n-k)}_{g} \circ S^{(k)}_{g_o},$$

for each $k \in \{0, 1, \ldots, n\}$. $A^{(k)}_{g g_o}$ is a $\mathbb{R}$-linear, continuous and formally self-adjoint (symmetric) operator that can be extended to the Hilbert space $L^2(A^k(M)) = W^{0,2}(A^k(M))$. Therefore, all Riemannian metrics induce the same topology on $L^2(A^k(M))$, for each $k \in \{0, 1, \ldots, n\}$. The same property holds for the Sobolev spaces $W^{1,2}(A^k(M))$ and $W^{2,2}(A^k(M))$ for each $k \in \{0, 1, \ldots, n\}$. By Lemma 6 and the definitions of the operators $\delta^{(k)}_{\mathbf{h},g}$: $W^{1,2}(A^k(M)) \to W^{0,2}(A^k(M))$ and $\Delta^{(k)}_{\mathbf{h},g} : W^{2,2}(A^k(M)) \to W^{0,2}(A^k(M))$, we have

Corollary 7. *Let M be a closed, n-dimensional smooth manifold and $W^{s,2}(A^k(M))$ the Sobolev space of class $W^{s,2}$, $s \in \{0, 1, 2\}$, corresponding to*

the prehilbertian vector space $A^k(M)$, $k \in \{0,1,\ldots,n\}$. *Then the following two assertions are valid:*

(*i*) *The mapping*

$$\mathfrak{M}(M) \ni g \mapsto \delta^{(k)}_{\mathbf{h},g} \in L(W^{1,2}(A^k(M)), W^{0,2}(A^{k-1}(M)))$$

is smooth, for each $k \in \{0,1,\ldots,n\}$, *that is the codifferential* $\delta^{(k)}_{\mathbf{h},g}$ *depends smoothly on* $g \in \mathfrak{M}(M)$, *for each* $k \in \{0,1,\ldots,n\}$;

(*ii*) *The mapping*

$$\mathfrak{M}(M) \ni g \mapsto \Delta^{(k)}_{\mathbf{h},g} \in L(W^{2,2}(A^k(M)), W^{0,2}(A^k(M)))$$

is smooth, for each $k \in \{0,1,\ldots,n\}$, *that is the operator* $\delta^{(k)}_{\mathbf{h},g}$ *depends smoothly on* $g \in \mathfrak{M}(M)$, *for each* $k \in \{0,1,\ldots,n\}$.

Also, by Lemma 6, the mapping

$$\mathfrak{M}(M) \ni g \mapsto A^{(k)}_{gg_o} \in L(W^{0,2}(A^k(M)))$$

depends smoothly on $g \in \mathfrak{M}(M)$, for each $k \in \{0,1,\ldots,n\}$.

The next theorem generalizes a result of Wenzelburger [21], [22] (see also Petrişor [15]).

Theorem 8. *Under the above assumptions regarding* M *and* $\mathbf{h}$, *the* $\mathbf{h}$*-Hodge-de Rham decomposition*

$$L^2(A^k(M)) = d^{(k-1)}_{\mathbf{h}}(W^{1,2}(A^{k-1}(M))) \oplus \delta^{(k+1)}_{\mathbf{h},g}(W^{1,2}(A^{k+1}(M))) \oplus Ker(\Delta^{(k)}_{\mathbf{h},g})$$

depends smoothly on the Riemannian metric $g \in \mathfrak{M}(M)$, *for each* $k \in \{0,1,\ldots,n\}$. *This means that the mapping*

$$\begin{aligned}\mathfrak{M}(M) \in g \mapsto \pi^{(k)}_g \{&= \text{the orthogonal projection of } L^2(A^k(M)) \text{ onto}\\ &Im[d^{(k-1)}_{\mathbf{h}} : W^{1,2}(A^{k-1}(M)) \to W^{0,2}(A^k(M)) := L^2(A^k(M))] \text{ with}\\ &Ker(\pi^{(k)}_g) = \mathcal{K}^{(k)}_{\mathbf{h},g} \bigoplus \delta^{(k+1)}_{\mathbf{h},g}(W^{1,2}(A^{k+1}(M)))\} \in L(L^2(A^k(M))),\end{aligned}$$

is smooth.

3. Spectral properties of Hodge-de Rham like operators. In what follows assume that **h** satisfies the same assumptions as in Theorem 4. Since $\Delta_{\mathbf{h}}^{(k)} : A^k(M) \to A^k(M)$ is an elliptic second order differential operator, formally self-adjoint and formally positive, and (M, g) is a closed, smooth Riemannian manifold, the following statements are valid.

Lemma 9. *(i) For each $k \in \{0, 1, \ldots, n\}$, there exists a discrete spectral resolution $\{\omega_{\mathbf{h};j}^{(k)}, \lambda_{\mathbf{h};j}^{(k)}\}_{j\in\mathbb{N}}$ for the operator $\Delta_{\mathbf{h}}^{(k)} : A^k(M) \to A^k(M)$, where $\omega_{\mathbf{h};j}^{(k)} \in A^k(M)$ for any $j \in \mathbb{N}$, that is $\{\omega_{\mathbf{h};j}^{(k)}\}_{j\in\mathbb{N}}$ is a complete orthonormal system in the real Hilbert space $L^2(A^k(M)) = W^{0,2}(A^k(M))$ such that $\Delta_{\mathbf{h}}^{(k)}\omega_{\mathbf{h};j}^{(k)} = \lambda_{\mathbf{h};j}^{(k)}\omega_{\mathbf{h};j}^{(k)}$ for any $j \in \mathbb{N}$. Moreover, $\lambda_{\mathbf{h};j}^{(k)} \in [0, +\infty)$ for any $j \in \mathbb{N}$,each eigenspace of $\Delta_{\mathbf{h}}^{(k)}$ is finite dimensional, and $0 \in \mathbb{R}$ is an eigenvalue of $\Delta_{\mathbf{h}}^{(k)} : A^k(M) \to A^k(M)$ if and only if $\beta_k(M) \neq 0$;*

(ii) If one arrange the eigenvalues of $\Delta_{\mathbf{h}}^{(k)}$ such that

$$0 \leq \lambda_{\mathbf{h};0}^{(k)} \leq \lambda_{\mathbf{h};1}^{(k)} \leq \ldots,$$

then there exist real constants $C(k) > 0$ and $\varepsilon(k) > 0$ such that $\lambda_{\mathbf{h};j}^{(k)} \geq C(k)j^{\varepsilon(k)}$ if $j \geq j_0$ is sufficiently large;

(iii) Let $a_j^{(k)} := \langle \theta, \omega_{\mathbf{h};j}^{(k)} \rangle \in \mathbb{R}$, $j \in \mathbb{N}$, be the Fourier coefficients associated to $\theta \in L^2(A^k(M))$. If $\theta \in A^k(M)$, then

$$\sum_{j\in\mathbb{N}} |a_j^{(k)}|\lambda_{\mathbf{h};j}^{(k)} < +\infty$$

and the series $\sum_{j\in\mathbb{N}} |a_j^{(k)}|\lambda_{\mathbf{h};j}^{(k)}$ tends to θ uniformly with respect to the norm $\|\cdot\|_{\infty,i}$ for any $k \in \{0, 1, \ldots, n\}$, where

$$\|\omega_{\mathbf{h};l}^{(k)}\|_{\infty,i} := \sup_{x\in M} |j_i(\omega_{\mathbf{h};l}^{(k)})_x|_{J^i(\Lambda^k T^*M)_x},$$

$j_i(\omega_{\mathbf{h};l}^{(k)})_x$ denoting the i-jet of the smooth section $\omega_{\mathbf{h};l}^{(k)} \in C^\infty(\Lambda^k T^*M)$ at the point $x \in M$.

For an initial smooth differential k-form $\theta \in A^k(M)$, let us consider the heat equation associated to $\Delta_{\mathbf{h}}^{(k)} : A^k(M) \to A^k(M)$ with the initial condition θ:

$$\left(\frac{\partial}{\partial t} + \Delta_{\mathbf{h}}^{(k)}\right)\omega(x, t) = 0, \tag{3.1}$$

$$\lim_{t\searrow 0}\omega(x,t)=\theta(x), \tag{3.2}$$

where $x\in M$, $t\in(0,+\infty)$. Let $(e^{-t\Delta_{\mathbf{h}}^{(k)}})(\theta)$ be the unique solution of the evolution equation (3.1) that satisfies the initial condition (3.2). The linear operator $e^{-t\Delta_{\mathbf{h}}^{(k)}}:A^k(M)\to A^k(M)$ can be extended to a compact, self-adjoint operator from $L^2(A^k(M))$into $L^2(A^k(M))$, denoted also by $e^{-t\Delta_{\mathbf{h}}^{(k)}}$. Let $\{\omega_{\mathbf{h};j}^{(k)},\lambda_{\mathbf{h};j}^{(k)}\}$ be a discrete spectral resolution for $\Delta_{\mathbf{h}}^{(k)}:A^k(M)\to A^k(M)$ and let us consider the Fourier expansion of $\theta\in A^k(M)$:

$$\theta=\sum_{j\in\mathbb{N}}a_j^{(k)}\omega_{\mathbf{h};j}^{(k)},\quad \text{where}\quad a_j^{(k)}:=\langle\theta,\omega_{\mathbf{h};j}^{(k)}\rangle,\quad j\in\mathbb{N}.$$

Let us define $\omega_{\mathbf{h};j}^{*(k)}\in(A^k(M))^*$ by $\omega_{\mathbf{h};j}^{*(k)}(\varpi):=\langle\varpi,\omega_{\mathbf{h};j}^{(k)}\rangle$, $\varpi\in A^k(M)$ such that $\|\omega_{\mathbf{h};j}^{*(k)}\|=\|\omega_{\mathbf{h};j}^{(k)}\|_0$. With these notations, the following statement is valid.

Theorem 10. *The following two series converge uniformly in the C^l-topology for any $l\in\mathbb{N}$ if $t\geq\delta>0$:*

$$E_{\mathbf{h}}^{(k)}(x,y,t):=\sum_{j\in\mathbb{N}}e^{-t\lambda_{\mathbf{h};j}^{(k)}}\omega_{\mathbf{h};j}^{(k)}(x)\otimes\omega_{\mathbf{h};j}^{*(k)}(y)\in Hom(\Lambda^k(T_y^*M),\Lambda^k(T_x^*M))$$

and

$$(e^{-t\Delta_{\mathbf{h}}^{(k)}}(\theta))(x,t):=\sum_{j\in\mathbb{N}}e^{-t\lambda_{\mathbf{h};j}^{(k)}}a_j^{(k)}\omega_{\mathbf{h};j}^{(k)}(x):=\int_M(E_{\mathbf{h}}^{(k)}(x,y,t))\theta(y)d\mu_g(y),$$

where $x,y\in M$, and μ_g denotes the canonical measure on M associated to g.

Let us define the matrix of Fourier coefficients of the bounded linear operator

$$e^{-t\Delta_{\mathbf{h}}^{(k)}}:L^2(A^k(M))\to L^2(A^k(M))$$

by

$$\begin{aligned}a_{ij}(E_{\mathbf{h}}^{(k)}):&=\langle e^{-t\Delta_{\mathbf{h}}^{(k)}}(\omega_{\mathbf{h};i}^{(k)}),\omega_{\mathbf{h};j}^{(k)}\rangle\\&=\int_{x\in M}\int_{y\in M}(E_{\mathbf{h}}^{(k)}(x,y,t))(\omega_{\mathbf{h};i}^{(k)}(y)|\omega_{\mathbf{h};j}^{(k)}(x))d\mu_g(x)d\mu_g(y),\end{aligned}$$

$i, j \in \mathbb{N}$, $k \in \{0, 1, \ldots, n\}$, where $(|)$ denotes the pointwise inner product on $A^k(M)$ defined by using the fiber metric on $\Lambda^k T^*M$ induced by g.

Theorem 11. (i) *With the previous notations the following equalities*

$$Trace_{L^2}(e^{-t\Delta_{\mathbf{h}}^{(k)}}) := \int_{x\in M} Trace_{\Lambda^k(T_x^*M)}(E_{\mathbf{h}}^{(k)}(x,x,t))d\mu_g(x)$$
$$= \sum_{j\in\mathbb{N}} a_{jj}(E_{\mathbf{h};j}^{(k)}) = \sum_{j\in\mathbb{N}} e^{-t\lambda_{\mathbf{h};j}^{(k)}}$$

are valid for each $k \in \{0, 1, \ldots, n\}$*, that is the continuous linear operator* $e^{-t\Delta_{\mathbf{h}}^{(k)}} : L^2(A^k(M)) \to L^2(A^k(M))$ *is a trace class operator;*

(ii) *If* $(A(M), d_{\mathbf{h}})$ *is the elliptic complex previously defined, then*

$$\sum_{k=0}^{n}(-1)^k Trace_{L^2}(e^{-t\Delta_{\mathbf{h}}^{(k)}}) = Index(A(M), d_{\mathbf{h}}) = \chi(M).$$

The eigenvalues of $\Delta_{\mathbf{h}}^{(k)}$ for various values of k are not completely independent. Thus, if $\lambda_{\mathbf{h};1}^{(k)}$ denotes the first nonvanishing eigenvalue of $\Delta_{\mathbf{h}}^{(k)}$, then $\lambda_{\mathbf{h};1}^{(1)} \leq \lambda_{\mathbf{h};1}^{(0)}$. On the other hand, if $Trace(h)$ is constant, then –via Corollary 5– $\lambda_{\mathbf{h};1}^{(k)} \leq \lambda_{\mathbf{h};1}^{(n-k)}$.

Since $\Delta_{\mathbf{h}}^{(k+1)} \circ d_{\mathbf{h}}^{(k)} = d_{\mathbf{h}}^{(k)} \circ \Delta_{\mathbf{h}}^{(k)}$ and $\Delta_{\mathbf{h}}^{(k-1)} \circ \delta_{\mathbf{h}}^{(k)} = \delta_{\mathbf{h}}^{(k)} \circ \Delta_{\mathbf{h}}^{(k)}$ for each $k \in \{0, 1, \ldots, n\}$, $\Delta_{\mathbf{h}}^{(k)}$ preserves the $\mathbf{h}$-exact (resp. $\mathbf{h}$-coexact) k-forms. For $k \in \{0, 1, \ldots, n\}$ and $j \in \mathbb{N}^*$, let $\lambda_{\mathbf{h};j}^{(k)} \neq 0$ be the j-th eigenvalue (counted without multiplicity) of $\Delta_{\mathbf{h}}^{(k)}$ and $\lambda_{\mathbf{h};j}^{'(k)}$(resp. $\lambda_{\mathbf{h};j}^{''(k)}$) the j-th eigenvalue of the restriction of $\Delta_{\mathbf{h}}^{(k)}$ to the space of $\mathbf{h}$-exact (resp. $\mathbf{h}$-coexact) k-forms. Theorem 4(i) implies the next statements.

Corollary 12. (i) $\lambda_{\mathbf{h};j}^{'(k+1)} = \lambda_{\mathbf{h};j}^{''(k)}$ *for* $k \in \{0, 1, \ldots, n-1\}$*;*
(ii) $\lambda_{\mathbf{h};j}^{(k)} = \min\{\lambda_{\mathbf{h};j}^{'(k)}, \lambda_{\mathbf{h};j}^{''(k+1)}\}$ *for* $k \in \{0, 1, \ldots, n-1\}$*;*
(iii) $\lambda_{\mathbf{h};j}^{(0)} = \lambda_{\mathbf{h};j}^{'(1)}$, $\lambda_{\mathbf{h};j}^{(n)} = \lambda_{\mathbf{h};j}^{'(n-1)}$.

Let us mention other properties of the eigenvalues of $\Delta_{\mathbf{h}}^{(k)}$.

Theorem 13. *For each* $k \in \{0, 1, \ldots, n\}$ *and for each* $j \in \mathbb{N}^*$*, the eigenvalue function*

$$\lambda_{\mathbf{h};j}^{(k)}(M, \cdot) : \mathfrak{M}(M) \ni g \mapsto \lambda_{\mathbf{h};j}^{(k)}(M, g) \in \mathbb{R}$$

[*resp.* $\lambda_{\mathbf{h};j}^{'(k)}(M,\cdot) : \mathfrak{M}(M) \ni g \mapsto \lambda_{\mathbf{h};j}^{'(k)}(M,g) \in \mathbb{R}$, *resp.* $\lambda_{\mathbf{h};j}^{''(k)}(M,\cdot) : \mathfrak{M}(M) \ni g \mapsto \lambda_{\mathbf{h};j}^{''(k)}(M,g) \in \mathbb{R}$] *is continuous with respect to the Whitney* C^∞*-topology on* $\mathfrak{M}(M)$.

Corollary 14. *For each* $k \in \{0,1,\ldots,n\}$ *and for each* $j \in \mathbb{N}^*$, *the multiplicity function*

$$m_{\mathbf{h};j}^{(k)}(M,\cdot) := \sharp\{i \in \mathbb{N}^* | \lambda_{\mathbf{h};i}^{(k)}(M,\cdot) = \lambda_{\mathbf{h};j}^{(k)}(M,\cdot)\} : \mathfrak{M}(M) \ni g \mapsto m_{\mathbf{h};j}^{(k)}(M,g) \in \mathbb{N}^*$$

[*resp.* $m_{\mathbf{h};j}^{'(k)}(M,\cdot) := \sharp\{i \in \mathbb{N}^* | \lambda_{\mathbf{h};i}^{'(k)}(M,\cdot) = \lambda_{\mathbf{h};j}^{'(k)}(M,\cdot)\} : \mathfrak{M}(M) \ni g \mapsto m_{\mathbf{h};j}^{'(k)}(M,g) \in \mathbb{N}^*$, *resp.* $m_{\mathbf{h};j}^{''(k)}(M,\cdot) := \sharp\{i \in \mathbb{N}^* | \lambda_{\mathbf{h};i}^{''(k)}(M,\cdot) = \lambda_{\mathbf{h};j}^{''(k)}(M,\cdot)\} : \mathfrak{M}(M) \ni g \mapsto m_{\mathbf{h};j}^{''(k)}(M,g) \in \mathbb{N}^*$] *of the eigenvalue function* $\lambda_{\mathbf{h};j}^{(k)}(M,\cdot)$ [*resp.* $\lambda_{\mathbf{h};j}^{'(k)}(M,\cdot)$, *resp.* $\lambda_{\mathbf{h};j}^{''(k)}(M,\cdot)$] *depends upper-semicontinuously on* $g \in \mathfrak{M}(M)$*: for each* $g \in \mathfrak{M}(M)$, *there exists* $\varepsilon > 0$ *such that* $d(g,g') < \varepsilon$ *implies*

$$m_{\mathbf{h};j}^{(k)}(M,g') \leq m_{\mathbf{h};j}^{(k)}(M,g)$$

[*resp.* $m_{\mathbf{h};j}^{'(k)}(M,g') \leq m_{\mathbf{h};j}^{'(k)}(M,g)$, *resp.* $m_{\mathbf{h};j}^{''(k)}(M,g') \leq m_{\mathbf{h};j}^{''(k)}(M,g)$], *where* d *denotes the canonical metric on* $\mathfrak{M}(M)$.

For each $k \in \{0,1,\ldots,n\}$ and for each $j \in \mathbb{N}^*$, let $\mathcal{S}_{\mathbf{h};j}^{(k)}(M)$ [resp. $\mathcal{S}_{\mathbf{h};j}^{'(k)}(M)$, resp. $\mathcal{S}_{\mathbf{h};j}^{''(k)}(M)$] be the set of all Riemannian metrics $g \in \mathfrak{M}(M)$ for which the first j eigenvalues of the operator $\Delta_{\mathbf{h},g}^{(k)}$ [resp. of the restriction of $\Delta_{\mathbf{h},g}^{(k)}$ to the space of the h-exact and resp. h-coexact differential k-forms on M] are simple, that is they have multiplicity one, i.e.

$$\mathcal{S}_{\mathbf{h};j}^{(k)}(M) := \{g \in \mathfrak{M}(M) | \lambda_{\mathbf{h};1}^{(k)}(M,g) < \ldots < \lambda_{\mathbf{h};j}^{(k)}(M,g)\}$$

[resp. $\mathcal{S}_{\mathbf{h};j}^{'(k)}(M) := \{g \in \mathfrak{M}(M) | \lambda_{\mathbf{h};1}^{'(k)}(M,g) < \ldots < \lambda_{\mathbf{h};j}^{'(k)}(M,g)\}$, resp. $\mathcal{S}_{\mathbf{h};j}^{''(k)}(M) := \{g \in \mathfrak{M}(M) | \lambda_{\mathbf{h};1}^{''(k)}(M,g) < \ldots < \lambda_{\mathbf{h};j}^{''(k)}(M,g)\}$].

Corollary 15. *For each* $k \in \{0,1,\ldots,n\}$ *and for each* $j \in \mathbb{N}^*$, $\mathcal{S}_{\mathbf{h};j}^{(k)}(M)$ [*resp.* $\mathcal{S}_{\mathbf{h};j}^{'(k)}(M)$, *resp.* $\mathcal{S}_{\mathbf{h};j}^{''(k)}(M)$] *is open in* $\mathfrak{M}(M)$.

Open problem: *It is an open problem to say if it would be possible to prescribe a finite part of the* $\mathbf{h}$*-exact spectrum of* $\Delta_{\mathbf{h}}^{(k)}$, $k \in \{1,\ldots,n-1\}$

(*for a discussion concerning this problem for the classical Hodge-de Rham operator* $\Delta^{(k)}$*, see* CRAIOVEANU *and* PUTA [5], TURMACU [20]).

Now, let us sketch the proof of Theorems 3 and 4(i) using the heat equation method associated to $\Delta_{\mathbf{h}}^{(k)}$. One should remark that this proof provides another way toward a generalized Hodge theory (for the classical case, see, for example, CRAIOVEANU and PUTA [4], CRAIOVEANU, PUTA and RASSIAS [6]). Firstly, let us note that

$$\lim_{t\searrow 0} e^{-t\Delta_{\mathbf{h}}^{(k)}}|_{A^k(M)} = 1_{A^k(M)}$$

for each $k \in \{0, 1, \ldots, n\}$ and recall that $e^{-t\Delta_{\mathbf{h}}^{(k)}} : A^k(M) \to A^k(M)$ is formally self-adjoint and formally positive for any $t \in (0, +\infty)$. Moreover, the unicity of the solution of the evolution equation (3.1) that satisfies the initial condition (3.2) implies the following semigroup property of the family $\{e^{-t\Delta_{\mathbf{h}}^{(k)}} | t \in (0, +\infty)\}$ of bounded linear operators:

$$e^{-s\Delta_{\mathbf{h}}^{(k)}} \circ e^{-t\Delta_{\mathbf{h}}^{(k)}} = e^{-(s+t)\Delta_{\mathbf{h}}^{(k)}}$$

for any $s, t \in (0, +\infty)$.

Lemma 16. *For each fixed smooth differential k-form* $\theta \in A^k(M)$*, the real function*

$$(0, +\infty) \ni t \mapsto \|e^{-t\Delta_{\mathbf{h}}^{(k)}}(\theta)\|^2 \in (0, +\infty)$$

is monotonically decreasing.

Now let us point out the behaviour of the smooth differential k-form $e^{-t\Delta_{\mathbf{h}}^{(k)}}(\theta)$ when t tends to $+\infty$.

Theorem 17. *Let* $\theta \in A^k(M)$ *be a fixed smooth differential k-form. Then the following two statements are valid:*

(i) $e^{-t\Delta_{\mathbf{h}}^{(k)}}(\theta)$ *converges uniformly for* $t \to +\infty$*;*

(ii) $H_{\mathbf{h}}^{(k)}(\theta) := \lim_{t\to+\infty} e^{-t\Delta_{\mathbf{h}}^{(k)}}(\theta) \in A^k(M)$ *is a* $\mathbf{h}$*-harmonic form.*

The operator $H_{\mathbf{h}}^{(k)} : A^k(M) \to A^k(M)$ is called the $\mathbf{h}$-harmonic projector of order $k \in \{0, 1, \ldots, n\}$ associated to $\mathbf{h}$. For $\theta \in A^k(M)$, let

$$(G_{\mathbf{h}}^{(k)}(\theta))(x) := \int_0^\infty (e^{-t\Delta_{\mathbf{h}}^{(k)}}(\theta) - H_{\mathbf{h}}^{(k)}(\theta))(x)dt, \quad x \in M.$$

Lemma 18. *$G^{(k)}_{\mathbf{h}}(\theta)$ is well-defined and $G^{(k)}_{\mathbf{h}}(\theta) \in A^k(M)$ for any $\theta \in A^k(M)$ and $k \in \{0,1,\dots,n\}$.*

The operator $G^{(k)}_{\mathbf{h}} : A^k(M) \to A^k(M)$ is called the Green operator of order $k \in \{0,1,\dots,n\}$ associated to $\mathbf{h}$.

The proofs of Theorems 3 and 4(i) are now an easy consequence of the next computation. Since $H^{(k)}_{\mathbf{h}}(\theta) \in \mathcal{K}^{(k)}_{\mathbf{h}}$ for each $\theta \in A^k(M)$, it follows that

$$\begin{aligned}
\Delta^{(k)}_{\mathbf{h}}(G^{(k)}_{\mathbf{h}}(\theta)) &= \Delta^{(k)}_{\mathbf{h}}\left(\int_0^\infty (e^{-t\Delta^{(k)}_{\mathbf{h}}}(\theta) - H^{(k)}_{\mathbf{h}}(\theta))dt\right)\\
&= \int_0^\infty \Delta^{(k)}_{\mathbf{h}}(e^{-t\Delta^{(k)}_{\mathbf{h}}}(\theta) - H^{(k)}_{\mathbf{h}}(\theta))dt\\
&= \int_0^\infty \Delta^{(k)}_{\mathbf{h}}(e^{-t\Delta^{(k)}_{\mathbf{h}}}(\theta))dt\\
&= -\int_0^\infty \frac{\partial}{\partial t}(e^{-t\Delta^{(k)}_{\mathbf{h}}}(\theta))dt\\
&= \lim_{t\searrow 0} e^{-t\Delta^{(k)}_{\mathbf{h}}}(\theta) - \lim_{t\to\infty} e^{-t\Delta^{(k)}_{\mathbf{h}}}(\theta)\\
&= \theta - H^{(k)}_{\mathbf{h}}(\theta)
\end{aligned}$$

and consequently

$$\theta = H^{(k)}_{\mathbf{h}}(\theta) + \Delta^{(k)}_{\mathbf{h}}(G^{(k)}_{\mathbf{h}}(\theta)) \quad \text{for any} \quad \theta \in A^k(M).$$

Finally, let us discuss the asymptotics of $Trace_{L^2}(e^{-t\Delta^{(k)}_{\mathbf{h}}})$ as $t \searrow 0$ and relate these asymptotics to $Index(A(M), d_{\mathbf{h}})$.

For each $t \in (0,1)$ and $k \in \{0,1,\dots,n\}$, the restriction of $E^{(k)}_{\mathbf{h}}(\cdot,\cdot,t)$ on the diagonal of $M \times M$ admits an asymptotic expansion. More precisely, the following statement is valid.

Lemma 19. *For each $k \in \{0,1,\dots,n\}$ and each $m \in \mathbb{N}$, there exists a smooth mapping $e_m(\cdot, \Delta^{(k)}_{\mathbf{h}}) : M \to End(\Lambda^k(T^*M))$ such that*

(i) $e_m(x, \Delta^{(k)}_{\mathbf{h}}) \in End(\Lambda^k(T^*_xM))$ depends functorially on a finite number of jets of the symbol of the second order differential operator $\Delta^{(k)}_{\mathbf{h}}$;

(ii) $E^{(k)}_{\mathbf{h}}(x,x,t) \sim \sum_m e_m(x, \Delta^{(k)}_{\mathbf{h}}) t^{\frac{m-n}{2}}$ as $t \searrow 0$, for any $x \in M$;

(iii) $e_m(x, \Delta^{(k)}_{\mathbf{h}}) = 0$ for any $x \in M$ and any m odd.

For the proof, we refer to Gilkey [11], Lemma 1.8.2.

Let

$$M \ni x \mapsto a_m(x, \Delta_{\mathbf{h}}^{(k)}) := Trace(e_n(x, \Delta_{\mathbf{h}}^{(k)})) \in \mathbb{R}$$

and

$$a_m(\Delta_{\mathbf{h}}^{(k)}) := \int_{x\in M} a_m(x, \Delta_{\mathbf{h}}^{(k)}) d\mu_g(x) \in \mathbb{R}, \tag{3.3}$$

the invariant scalar functions and the numerical invariants respectively associated to the differential operator $\Delta_{\mathbf{h}}^{(k)}$, $k \in \{0, 1, \ldots, n\}$.

Lemma 20. *(i) The invariants $e_m(x, \Delta_{\mathbf{h}}^{(k)})$ and $a_m(x, \Delta_{\mathbf{h}}^{(k)})$, m even, can be expressed by local formulas that are homogeneous of order m in the jets of the total symbol of $\Delta_{\mathbf{h}}^{(k)}$ for each $k \in \{0, 1, \ldots, n\}$;*

(ii) $a_m(x, \Delta_{\mathbf{h}}^{(k)}) = 0$ for each m odd and any $k \in \{0, 1, \ldots, n\}$;

(iii) $Trace_{L^2}(e^{-t\Delta_{\mathbf{h}}^{(k)}}) \sim \sum_m a_m(\Delta_{\mathbf{h}}^{(k)}) t^{\frac{m-n}{2}}$ as $t \searrow 0$, for each $k \in \{0, 1, \ldots, n\}$.

For the proof of statement *(i)* we refer to Gilkey [11], Lemma 1.8.3(c), while the statement *(ii)* is an immediate consequence of Lemma 19 *(iii)*.

Statement *(iii)* is an immediate consequence of the definition of $Trace_{L^2}(e^{-t\Delta_{\mathbf{h}}^{(k)}})$ [see Theorem 11(*i*)] and of Lemma 19(*ii*).

For each $m \in \mathbb{N}$, let us define the real function

$$M \ni x \mapsto a_m(x, A(M), d_{\mathbf{h}}) := \sum_{k=0}^{n} (-1)^k a_m(x, \Delta_{\mathbf{h}}^{(k)}) \in \mathbb{R}. \tag{3.4}$$

Theorem 21. *Let (M, g) be a smooth closed Riemannian manifold of dimension n and $\mathbf{h}$ a nonsingular smooth tensor field of type $(1, 1)$ on M such that the Nijenhuis derivation corresponding to $\mathbf{h}$ vanishes. Then the following statements are true:*

(i) For each $m \in \mathbb{N}$, the mapping $M \ni x \mapsto a_m(x, A(M), d_{\mathbf{h}}) \in \mathbb{R}$ can be expressed by a local formula in the jets of the symbols of $\Delta_{\mathbf{h}}^{(k)}$, $k \in \{0, 1, \ldots, n\}$, which is homogeneous of order m, that is the invariants $a_m(x, A(M), d_{\mathbf{h}}) \in \mathbb{R}$, $x \in M$, m even [see (ii)], are of local nature;

(ii) $a_m(x, A(M), d_{\mathbf{h}}) = 0$ for each m odd;

(iii) $\int_{x\in M} a_m(x, A(M), d_{\mathbf{h}}) d\mu_g(x) = 0$ for each $m \in \mathbb{N} \setminus \{n\}$;

(iv) $\int_{x\in M} a_n(x, \Lambda E, d_{\mathbf{h}}) d\mu_g(x) = \chi(M)$.

The statement *(i)* [resp. *(ii)*] is an immediate consequence of Lemma 20(*i*) [resp. *(ii)*]. Concerning the statements *(iii)* and *(iv)*, we use Theorem

$11(ii)$, Lemma $20(iii)$, as well as the definitions expressed by (3.3) and (3.4), in order to derive the following relations:

$$\chi(M) = Index(A(M), d_{\mathbf{h}}) := \sum_{k=0}^{n}(-1)^k Trace_{L^2}(e^{-t\Delta_{\mathbf{h}}^{(k)}})$$

$$\sim \sum_m \sum_{k=0}^{n}(-1)^k a_m(\Delta_{\mathbf{h}}^{(k)}) t^{\frac{m-n}{2}}$$

$$= \sum_m \left[\sum_{k=0}^{n}(-1)^k \left(\int_{x\in M} a_m(x, \Delta_{\mathbf{h}}^{(k)}) d\mu_g(x)\right)\right] t^{\frac{m-n}{2}}$$

$$= \sum_m \left(\int_{x\in M} a_m(x, A(M), d_{\mathbf{h}}) d\mu_g(x)\right) t^{\frac{m-n}{2}}.$$

Since the Euler-Poincaré characteristic $\chi(M)$ is independent of the parameter t, the previous asymptotic expansion is independent of t, so that the mentioned statements now follow.

For other details concerning spectral properties of the differential operators $\Delta_{\mathbf{h}}^{(k)}$, $k \in \{0, \ldots, n\}$, see Creţ [7], Lalescu and Stretcu [14], Petrişor [15], Puta [16].

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Received: 15.X.2007

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