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ON THE GEOMETRIZATION OF THE RHEONOMIC FINSLERIAN MECHANICAL SYSTEMS

BY

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Dedicated to Academician Radu Miron at his 80th anniversary

Abstract. Using techniques that are specific to Lagrange geometry, [7], [8], [9] Miron, introduced and investigated some geometric aspects of Finslerian and Lagrangian mechanical systems. In the geometrical theory of mechanical systems an important contribution have Abraham and Marsden [1], De Leon and Rodriguez [5]. The rheonomic Lagrangian mechanical systems, their equations and the associated dynamical systems were studied by author in [4]. In this paper we extend such geometric investigation to rheonomic Finslerian mechanical systems, establishing their evolution equations and their canonical semispray. By means of this important notion we can develop the geometrical theory on the phase space of the considered mechanical system.

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1. Rheonomic Finsler spaces. Preliminaries. Let M be a real differentiable manifold of dimension n, (called the configuration space) and (TM, π, M) its tangent bundle (TM is called the phase space). We consider the manifold $TM\times R$ and we shall use the differentiable structure on $TM\times R$ as product of manifold TM and R.

The manifold $E = TM \times R$ is a $2n+1-$dimensional, real manifold. In a domain of a local chart $U \times (a, b)$, the points $u = (x, y, t) \in E$ have the local coordinates (x^i, y^i, t).

A change of local coordinates on E has the following form:

$$\tilde{x}^i = \tilde{x}^i(x^1, x^2, ..., x^n);\ \tilde{y}^i = \frac{\partial \tilde{x}^i}{\partial x^j} y^j;\quad \tilde{t} = \phi(t) \tag{1.1}$$

with $rank\left(\frac{\partial\tilde{x}^i}{\partial x^j}\right)=n$ and $\phi' := \frac{d\phi}{dt}\not=0$.

Of course, we may take on R only one chart, that is $\tilde{t}=t$ or we may consider the affine change of charts on R, that is $\tilde{t}=at+b$, $a\not=0$, $a,b\in R$.

Definition 1. *A rheonomic Finsler space is a pair $RF^n=(M,F(x,y,t))$, for which $F:TM\times R\to R$ satisfy the following axioms:*

1. *F is a positive scalar function on $E=TM\times R$;*
2. *F is a positive $1-$homogenous with respect to the variables y^i;*
3. *F is differentiable on $\tilde{E}=(TM\setminus\{0\})\times R$ and continous on the nul section of the projection $\pi:TM\to M$;*
4. *The Hessian of F, with the entries:*

$$g_{ij}(x,y,t)=\frac{1}{2}\frac{\partial^2 F^2}{\partial y^i\partial y^j} \tag{1.2}$$

is positively defined on $\tilde{E}$.

F is called the fundamental function and $g_{ij}(x,y,t)$ is the fundamental tensor of space RF^n.

We give some important properties of RF^n.

A consequence of the homogeneity condition is that the energy of a Finsler space coincides with the square of the fundamental function of the space:

$$E_{F^2}(x,y,t)=y^i\frac{\partial F^2}{\partial y^i}-F^2=2F^2-F^2=F^2=g_{ij}(x,y,t)y^iy^j, \tag{1.3}$$

and it is verified the next equality

$$\frac{dF^2}{dt}=-\frac{dx^i}{dt}E_i(F^2)-\frac{\partial F^2}{\partial t}, \tag{1.4}$$

where $E_i(F^2)=\frac{\partial F^2}{\partial x^i}-\frac{d}{dt}\left(\frac{\partial F^2}{\partial y^i}\right)$.

Remark 1. 1. F is a scalar function with respect to (1.1).

2. $g_{ij}(x,y,t)$ is a tensor field with respect to (1.1). It is covariant of order 2, symmetric and nesingular.

3. The pair $(M,L=F^2(x,y,t))$ is a rheonomic Lagrange space, thus we can use the theory of rheonomic Lagrange space [2], [3], [7] for developing the geometry of rheonomic Finsler spaces.

Theorem 1. *For any rheonomic Finsler space, Euler-Lagrange equations are equivalent with the Lorentz equations:*

$$\frac{d^2x^i}{dt^2} + \gamma^i_{jk}(x, \frac{dx}{dt}, t)\frac{dx^j}{dt}\frac{dx^k}{dt} = F^i_j(x, \frac{dx}{dt}, t)\frac{dx^j}{dt}$$

with $\gamma^i_{jk}(x,y,t) = \frac{1}{2}g^{ih}(\frac{\partial g_{hk}}{\partial x^j} + \frac{\partial g_{jh}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^h})$ *Christoffel symbols for the fundamental tensorial field* g_{ij} *and with the electromagnetic tensor field* $F^i_j(x, \frac{dx}{dt}, t)$ $= -g^{ih}\frac{\partial g_{hj}}{\partial t}$

The canonical spray S of RF^n, is as follows

$$S = y^i\frac{\partial}{\partial x^i} - (N^i_0(x,y,t) + N^i_k(x,y,t)y^k)\frac{\partial}{\partial y^i} + \frac{\partial}{\partial t} \text{ with} \tag{1.5}$$

$$N^i_j(x,y,t) = \frac{1}{2}\frac{\partial}{\partial y^j}(\gamma^i_{rs}(x,y,t)y^r y^s); \qquad N^0_j(x,y,t) = \frac{1}{2}\frac{\partial g_{jk}}{\partial t}y^k, \tag{1.6}$$

where γ^i_{jk} are the Christoffel symbols of the fundamental tensor $g_{ij}(x,y,t)$.

The Cartan nonlinear connection N, [6], [7], has the coefficients $(N^i_j(x,y,t),\ N^0_j(x,y,t))$.

Then N is a differentiable distribution on $\tilde{TM} \times R$, supplementary to the vertical distribution V, i.e.:

$$T_u(\tilde{TM} \times R) = N(u) \oplus V(u), \forall u \in \tilde{TM} \times R. \tag{1.7}$$

On the manifold E, the vertical distribution V is generated by $n+1$ local vector fields $\left(\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}, ..., \frac{\partial}{\partial y^n}, \frac{\partial}{\partial t}\right)$

$$V : u \in E \to V_u \subset T_u E \tag{1.8}$$

$$V_u = V_{n,u} \oplus V_{0,n} \quad \forall u \in E,$$

where $V_{0,n}$ =span $(\frac{\partial}{\partial t}\mid_u)$ is an 1-dimensional linear subspace of the tangent space T_uE and the n-dimensional linear space $V_{n,u}$ =span $(\frac{\partial}{\partial y^i}\mid_u)$ is a linear subspace of T_uE.

Let $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial t})_u$ be an adapted basis to decomposition (1.8).

Thus

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i(x,y,t)\frac{\partial}{\partial y^j} - N^0_i(x,y,t)\frac{\partial}{\partial t}. \tag{1.9}$$

The dual adapted basis is $(dx^i, \delta y^i, \delta t)_u$:

$$\delta y^i = dy^i + N^i_j(x,y,t)dx^j; \quad \delta t = dt + N^0_i(x,y,t)dx^i \,. \tag{1.10}$$

In the following we put $t = y^0$ and we introduce the Greek indices $\alpha, \beta, ..$ ranging on the set $\{0, 1, 2, ..., n\}$.

2. Rheonomic Finslerian mechanical systems. Finsler geometry corresponds to the case when the Lagrangians function is second order homogenous with respect to the velocity coordinates. A Finslerian mechanical system is a natural extension of the rheonomic Riemannian one studied by ROMAN in [10].

Definition 2. *A rheonomic Finslerian mechanical system is a triple*

$$\Sigma = (M, F^2(x,y,t), \sigma_i(x,y,t)), \tag{2.1}$$

where $F(x,y,t)$ is the fundamental function of a Finsler space $RF^n = (M, F(x,y,t))$ and $\sigma_i(x,y,t)$ is a d-covector field, (d-means distinguished, [6]) *called the external force of* Σ.

A rheonomic Lagrange space $RL^n = (M, L(x,y,t))$ reduces to a rheonomic Finsler space $RF^n = (M, F(x,y,t))$ if the Lagrangian function is second order homogenous with respect to the velocity coordinates.

The evolution equations of the rheonomic Finslerian mechanical system are given by Lagrange equations for the rheonomic Lagrangian $L(x,y,t) = F^2(x,y,t) = g_{ij}(x,y,t)y^iy^j$, [4]. Then, they are equivalent with the system of second order differential equations

$$\frac{d^2x^i}{dt^2} + 2G^i(x,y,t) + N^i_0(x,y,t) = \frac{1}{2}\sigma^i(x,y,t), \;\; y^i = \frac{dx^i}{dt}, \tag{2.2}$$

where $\sigma^i = g^{ij}\sigma_j$, $\sigma_i(x,y,t)$ being $2-$homogenous with respect to y^i and

$$2G^i = \gamma^i_{rs}(x,y,t)y^ry^s; \quad N^i_0(x,y,t) = \frac{1}{2}g^{ih}\frac{\partial g_{hj}}{\partial t}y^j. \tag{2.3}$$

One proves, [4], that (2.2) has a geometrical meaning.

The system of equations (2.2) locally determine a semispray $\breve{S}$ (or a dynamical system) on the phase space $TM \times R$. If the external force field $\sigma_i(x,y,t)$ is globally defined on $TM \times R$, then $\breve{S}$ is global defined on $\tilde{E}$ and its integral curves are exactly the equations of evolution (2.2) of the rheonomic Finslerian mechanical system.

Remark 2. Generally $\breve{S}$ is a semispray. But in homogenous case it is a spray.

Thus, the geometrical theory of the semispray $\breve{S}$ in the space $TM \times R$ is the geometry of the rheonomic Finslerian mechanical system Σ.

Remarking that the functions

$$\breve{G}^i(x,y,t) = G^i(x,y,t) - \frac{1}{4}\sigma^i(x,y,t) \tag{2.4}$$

with respect to (1.1) transform as

$$2\tilde{\breve{G}}^i(x,y,t) = 2\breve{G}^j(x,y,t)\frac{\partial \tilde{x}^i}{\partial x^j} - \frac{\partial \tilde{y}^i}{\partial x^j}y^j; \quad \tilde{N}_0^i(x,y,t) = N_0^j(x,y,t)\frac{\partial \tilde{x}^i}{\partial x^j}.$$

We can prove:

Theorem 2. a) *$\breve{S}$ given by:*

$$\breve{S} = y^i\frac{\partial}{\partial x^i} - (2\breve{G}^i(x,y,t) + N_0^i(x,y,t))\frac{\partial}{\partial y^i} + \frac{\partial}{\partial t} \tag{2.5}$$

is a semispray on $\tilde{TM} \times R$.

b) *$\breve{S}$ is a dynamical system on $\tilde{TM} \times R$ depending only on the rheonomic Lagrangian mechanical system Σ. We call this semispray* **the evolution semispray** *of the mechanical system Σ .*

c) *The integral curves of $\breve{S}$ are the evolution curves of Σ given by* (2.2).

We can say too:

The geometry of the rheonomic Finslerian mechanical sistem Σ is determined by the geometry of the rheonomic Lagrange space $RL^n = (M, F^2(x,y,t))$ endowed with the evolution semispray $\breve{S}$.

3. Nonlinear connection of a Rheonomic Finslerian mechanical system. The evolution semispray $\breve{S}$ given by (2.5) determines a nonlinear connection $\breve{N}$ depending only by the rheonomic Finslerian mechanical system Σ, which has the local coefficients $(\breve{N}^i_j, N^0_j)$ with

$$\breve{N}^i_j = N^i_j - \frac{1}{4}\frac{\partial \sigma^i}{\partial y^j} \tag{3.1}$$

and (N^i_j, N^0_j) are the coefficients of Cartan nonlinear connection of the rheonomic Finsler space $RF^n = (M, F^2)$. This nonlinear connection is called the **evolution nonlinear connection** of the rheonomic mechanical system $\Sigma = (M, F^2(x,y,t), \sigma_i(x,y,t))$. The evolution nonlinear connection $\breve{N}$ determines a direct decomposition of the tangent space T_uE into horizontal and vertical subspaces

$$T_uE = H_uE \oplus V_uE, \ \forall u \in E = TM \times R \tag{3.2}$$

The adapted basis to this decomposition is given by $\{\frac{\breve{\delta}}{\delta x^i}|_u, \frac{\partial}{\partial y^i}|_u, \frac{\partial}{\partial t}|_u\}$, where

$$\frac{\breve{\delta}}{\delta x^i} = \frac{\delta}{\delta x^i} + \frac{1}{4}\frac{\partial \sigma^j}{\partial y^i}\frac{\partial}{\partial y^j}. \tag{3.3}$$

The dual basis adapted to the decomposition (3.2) is given by $\{dx^i|_u, \breve{\delta} y^i|_u, \breve{\delta} t|_u\}$, where

$$\breve{\delta} y^i|_u = \delta y^i|_u - \frac{1}{4}\frac{\partial \sigma^i}{\partial y^j} dx^j|_u; \breve{\delta} t = \delta t. \tag{3.4}$$

In (3.3) and (3.4), $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial t}\}$ and $\{dx^i, \delta y^i, \delta t\}$ are the adapted bases to the Cartan nonlinear connection of the rheonomic Finsler space RF^n.

As the evolution nonlinear connection is symmetric, then the weak torsion t^α_{jk} vanishes, that is

$$t^\alpha_{jk} = \frac{\partial \breve{N}^\alpha_j}{\partial y^k} - \frac{\partial \breve{N}^\alpha_k}{\partial y^j} = 0.$$

The curvature tensor of the evolution nonlinear connection is given by

$$\breve{R}^\alpha{}_{jk} = \frac{\breve{\delta} \breve{N}^\alpha_j}{\delta x^k} - \frac{\breve{\delta} \breve{N}^\alpha_k}{\delta x^j}. \tag{3.5}$$

We have that the evolution nonlinear connection $\breve{N}$ is integrable if and only if the curvature tensor $\breve{R}^\alpha{}_{jk}$ vanishes.

4. Metric N-linear connection of a rheonomic mechanical system. In the previous sections we have seen that in the presence of an external force field $\sigma_i(x,y,t)$ on a rheonomic Finsler space the geodesic spray and evolution semispray are different, Cartan nonlinear connection of the rheonomic Finsler space and the evolution connection $\breve{N}$ of the mechanical system are different too.

In this section, we determine the canonical metric $\breve{N}-$linear connection of the rheonomic mechanical system $\Sigma = (M, F^2(x,y,t), \sigma_i(x,y,t))$. The canonical metric $\breve{N}-$linear connection $\breve{D}$ with local coefficients $(\breve{F}^i_{jk}, \breve{C}^i_{j\alpha})$ is uniquely determined by the following system of axioms:

a) the nonlinear connection $\breve{N}$ is the evolution nonlinear connection of the rheonomic mechanical system Σ;

b) $g_{ij|\alpha} = 0$ i.e. $\breve{D}$ is h-metric;

c) $\breve{T}^i_{jk} = \breve{F}^i_{jk} - \breve{F}^i_{kj} = 0$ i.e. $\breve{D}$ is $h-$symmetric;

d) $g_{ij}|_k = 0$ i.e. $\breve{D}$ is $v-$ metric;

e) $\breve{S}^i_{jk} = \breve{C}^i_{jk} - \breve{C}^i_{kj} = 0$ i.e. $\breve{D}$ is $v-$symmetric

Theorem 3. *The canonical metric $\breve{N}-$linear connection of the Finslerian mechanical system Σ, denoted $C\Gamma(\breve{N})$, has the coefficients:*

$$\breve{F}^i_{jk} = F^i_{jk} + \frac{1}{4} g^{is}\left(C_{skh}\frac{\partial \sigma^h}{\partial y^j} + C_{jsh}\frac{\partial \sigma^h}{\partial y^k} - C_{jkh}\frac{\partial \sigma^h}{\partial y^s}\right) \tag{4.1}$$

$$\breve{C}^i_{jk} = C^i_{jk};\ \breve{C}^i_{j0} = C^i_{j0}.$$

where $g^{is}C_{sjk} = C^i_{jk}$ and $C\Gamma(N) = (F^i_{jk}, C^i_{jk}, C^i_{j0})$ is the Cartan connection of the rheonomic Finsler space RF^n.

Proof. If we use the Lagrange equations (2.2) and

$$F^i_{jk} = \frac{1}{2} g^{is}\left(\frac{\delta g_{sk}}{\delta x^j} + \frac{\delta g_{js}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^s}\right), \tag{4.2}$$

we obtain expression (4.1). The equalities $\breve{C}^i_{jk} = C^i_{jk}$; $\breve{C}^i_{j0} = C^i_{j0}$ follows from a straightforward calculation. □

In a rheonomic Finsler space we have

$$y^i C_{ijk} = y^i C_{jik} = y^i C_{jki} = 0 \tag{4.3}$$

and if we use it, we observe that

$$\frac{1}{4} y^j g^{is}\left(C_{skh}\frac{\partial \sigma^h}{\partial y^j} + C_{jsh}\frac{\partial \sigma^h}{\partial y^k} - C_{jkh}\frac{\partial \sigma^h}{\partial y^s}\right) = \frac{1}{4} C^i_{kh} y^s \frac{\partial \sigma^h}{\partial y^s}. \tag{4.4}$$

5. Electromagnetic tensors of a rheonomic Finslerian mechanical system. If we consider the evolution nonlinear connection of a mechanical system, we can consider $h-$ and $v-$ deflection tensors:

$$\breve{D}^i_j = y^i_{|_j},\ \breve{d}^i_\alpha = y^i|_\alpha \tag{5.1}$$

and $h-$ and $v-$ electromagnetic tensors

$$\breve{F}_{ij} = \frac{1}{2}(\breve{D}_{ij} - \breve{D}_{ji}),\ \breve{f}_{ij} = \frac{1}{2}(\breve{d}_{ij} - \breve{d}_{ji}) \tag{5.2}$$

with $\breve{D}_{ji} = g_{is}\breve{D}^s_j$ and $\breve{d}_{ij} = g_{is}\breve{d}^s_j$.

The $h-$deflection tensor field has the following expression:

$$\breve{D}^i_j = y^i_{|_j} = D^i_j + \frac{1}{2}C^i_{jh}y^s\frac{\partial\sigma^h}{\partial y^s} + \frac{1}{4}\frac{\partial\sigma^i}{\partial y^j}, \tag{5.3}$$

and the $h-$electromagnetic tensor field $\breve{F}_{ij}$ of the canonical $\breve{N}$-linear connection $\breve{D}$ has the following expression:

$$\breve{F}_{ij} = \frac{1}{2}(\breve{D}_{ij} - \breve{D}_{ji}) = \frac{1}{8}\left(g_{is}\frac{\partial\sigma^s}{\partial y^j} - g_{js}\frac{\partial\sigma^s}{\partial y^i}\right) + F_{ij}; \quad \breve{f}_{ij} = 0, \tag{5.4}$$

with F_{ij} the $h-$electromagnetic tensor field of the canonical N-linear connection of the rheonomic Finsler space RF^n.

We denote by

$$\mathcal{F}_{ij} = \frac{1}{2}\left(\frac{\partial\sigma_j}{\partial y^i} - \frac{\partial\sigma_i}{\partial y^j}\right) \tag{5.5}$$

the **helicoidal tensor** of the mechanical system, [8].

Theorem 4. 1) *The $h-$electromagnetic tensor field $\breve{F}_{ij}$ of the canonical $\breve{N}$-linear connection $\breve{D}$ and the helicoidal tensor $\mathcal{F}_{ij}$ of the mechanical system are related by the following formula*

$$\breve{F}_{ij} = F_{ij} - \frac{1}{4}\mathcal{F}_{ij}. \tag{5.6}$$

2) *The $v-$ electromagnetic tensor $\breve{f}_{ij}$ of the canonical $\breve{N}$-linear connection $\breve{D}$ vanishes.*

Proof. It follows immediately from (5.4) and (5.5). $\square$

For the $h-$electromagnetic tensor field $\breve{F}_{ij}$ we obtain:

Theorem 5. *The electromagnetic tensor $\breve{F}_{ij}$ of a rheonomic Finslerian mechanical system satisfies the following generalized Maxwell equations:*

$$\begin{aligned}\breve{F}_{ij|k} + \breve{F}_{jk|i} + \breve{F}_{ki|j} &= -\frac{1}{4}\Sigma_{(ijk)}\left(\mathcal{F}_{ij|k} + \frac{1}{4}\frac{\partial\sigma^m}{\partial y^k}\frac{\partial\mathcal{F}_{ij}}{\partial y^m}\right)\\ \breve{F}_{ij}|_k + \breve{F}_{jk}|_i + \breve{F}_{ki}|_j &= 0.\end{aligned}$$

where $\Sigma_{(ijk)}$ means the sum of all cyclic permutations of i, j and k.

Remark 3. In particular if the external force field σ_i of the Finslerian mechanical system does not depend on variables y^i, then the elicoidal tensor $\mathcal{F}_{ij}$ vanishes.

6. Almost Hermitian model of a rheonomic Finslerian mechanical system. For a rheonomic Finslerian mechanical system Σ, we consider g_{ij} the metric tensor and the evolution nonlinear connection with the local coefficients $(\breve{N}^i_j, N^0_j)$. Having these geometric objects we can consider the $\breve{N}-$lift G of the metric tensor g_{ij} given by:

$$\mathbb{G} = g_{ij}dx^i \otimes dx^j + g_{ij}\breve{\delta}y^i \otimes \breve{\delta}y^j + \breve{\delta}t \otimes \breve{\delta}t \tag{6.1}$$

The metric $\breve{N}-$lift $\mathbb{G}$ has the following properties:

1) $\mathbb{G}$ depends on the rheonomic Finslerian mechanical system Σ only;

2) $\mathbb{G}$ is a pseudo-Riemannian structure on the manifold $\tilde{TM} \times R$;

The evolution nonlinear connection $\breve{N}$ is characterized by an almost complex structure $\mathbb{F}$ given by:

$$\mathbb{F} = -\frac{\partial}{\partial y^i} \otimes dx^i + \frac{\breve{\delta}}{\delta x^i} \otimes \breve{\delta}y^i$$

Almost complex structure $\mathbb{F}$ has the following properties:

1) $\mathbb{F}$ depends on the rheonomic Finslerian mechanical system Σ only;

2) $\mathbb{F}$ is a tensor field of $(1,1)$-type on the manifold $\tilde{TM} \times R$ and $F \circ F = -Id + \frac{\partial}{\partial t} \otimes dt$;

3) $\mathbb{F}$ is an almost complex structure if and only if the curvature tensor $\breve{R}^{\alpha}{}_{jk}$ of the evolution nonlinear connection vanishes.

Theorem 6. *The pair $(\mathbb{G}, \mathbb{F})$ is an almost Hermitian structure on the differentiable manifold $\tilde{TM} \times R$.*

The almost hermitian model $\left(\tilde{TM} \times R, \mathbb{G}, \mathbb{F}\right)$ gives a good geometric description of the rheonomic Finslerian mechanical system.

We finish the present paper with an example of rheonomic Finslerian mechanical system.

Example 1. Let $\alpha(x, y, t) = \{\gamma_{ij}(x, t) y^i y^j\}^{\frac{1}{2}}$ be a rheonomic Riemannian structure on $M \times R$ and $\beta(x, y, t) = b_i(x, t) y^i$ a covector field $b_i(x, t)$ on $M \times R$. Of course α and β are functions on the phase manifold $TM \times R$.

Now we consider the rheonomic Randers metric

$$F(x, y, t) = \alpha(x, y, t) + \beta(x, y, t). \tag{6.2}$$

Similarly, with the sclerhonomic case of Randers spaces we can see that the pair $RF^n = (M, F(x, y, t))$ is a rheonomic Finsler space.

Now, let us consider the $d-$covector field

$$F_e = \sigma^i(x, y, t) \frac{\partial}{\partial y^i} \tag{6.3}$$

with the components $\sigma_i = g_{ij} \sigma^j = h F^2 \frac{\partial F}{\partial y^i}, \quad h \in R^*$.

The triple $\Sigma_{Randers} = (M, F^2, F_e)$ is a rheonomic Finslerian mechanical system, which have the evolution equations

$$\frac{\partial F^2}{\partial x^i} - \frac{d}{dt} \frac{\partial F^2}{\partial y^i} = \sigma_i; \quad y^i = \frac{dx^i}{dt}. \tag{6.4}$$

This system of second order differential equations is equivalent to system (2.2).

All previous results can be applied for study the geometrical theory on the phase space $TM \times R$ of the rheonomic Randers mechanical systems $\Sigma_{Randers}$.

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