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THE INDICATRIX IN FINSLER GEOMETRY

BY

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Dedicated to Academician Radu Miron at his 80th anniversary

Abstract. For the study of Finsler spaces, the indicatrices play an important role. In this paper we review the geometry of the indicatrix in a Finsler space, we establish new properties of it and we give new proofs for some properties of Finsler spaces based on the indicatrices. We find the expression of the curvature for a curve given by the equation $\theta = \theta(F)$ and in the theory of two-dimensional Finsler spaces we obtain that a certain curve with $K = 1$ and the arc-length proportional with θ is the indicatrix of a Finsler space.

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1. Preliminaries. Let (M^n, F) be an $n-$dimensional Finsler space, where M^n is an $n-$dimensional manifold and $F = F(x, y)$ is the Finsler fundamental function. F is a differentiable function of the point $x = (x^i) \in M^n$ and the directional variables $y = (y^i) \in T_x(M^n)$ and is positively homogeneous of degree one with respect to y, where $T_x(M^n)$ is the tangent space of M^n at the point x.

We shall restrict our discussion to the fixed point $x_0 (\in M^n)$ and the corresponding tangent space $T_{x_0}(M^n)$ of M^n at the point x_0. Then the space $V^n = T_{x_0}(M^n) \setminus \{0\}$, where 0 is the origin of $T_{x_0}(M^n)$, may be viewed as an $n-$dimensional Riemannian space with the metric tensor $g_{ij}(x_0, y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$.

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Since the Finsler fundamental function F is positively homogeneous of degree one with respect to y^i, the metric tensor $g_{ij}(y)$ of V^n is positively homogeneous of degree zero with respect to y^i. Accordingly, for

$$C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k} = \frac{1}{4}\frac{\partial^3 F^2}{\partial y^i \partial y^j \partial y^k} \ ,$$

we have

$$C_{ijk}y^i = C_{ijk}y^j = C_{ijk}y^k = 0. \tag{1.1}$$

The Christoffel symbols of the Riemannian connection of V^n are

$$\begin{aligned} \Gamma_j{}^i{}_k &= \frac{1}{2}g^{ih}\left(\frac{\partial g_{jh}}{\partial y^k} + \frac{\partial g_{hk}}{\partial y^j} - \frac{\partial g_{jk}}{\partial y^h}\right) \\ &= g^{ih}C_{hjk} = C^i_{jk}. \end{aligned}$$

Hence, they coincide with the Cartan torsion tensor.

The Riemannian curvature tensor of V^n is $S_j{}^i{}_{kh} = \frac{\partial C^i_{jk}}{\partial y^h} - \frac{\partial C^i_{jh}}{\partial y^k} + C^l_{jk}C^i_{lh} - C^l_{jh}C^i_{lk}$. This is just the $v-$curvature tensor of the Finsler space. It can be put in to the form

$$S^l{}_{ijk} = C^l_{hj}C^h_{ki} - C^l_{hi}C^h_{kj}.$$

and it follows

$$S_{ijkl}y^i = S_{ijkl}y^j = S_{ijkl}y^k = S_{ijkl}y^l = 0, \tag{1.2}$$

for $S_{hijk} = g_{hl}S^l{}_{ijk}$.

The covariant differentiation of the tensor T_{ij} on V^n is the same as the $v-$covariant differentiation:

$$T_{ij}\ |_k = \frac{\partial T_{ij}}{\partial y^k} - T_{lj}C^l_{ik} - T_{il}C^l_{jk} \ .$$

By a direct calculation one gets

$$\begin{aligned} F\ |_i &= l_i, \\ l_i\ |_j &= F^{-1}h_{ij}, \\ h_{ij}\ |_k &= -F^{-1}(h_{ik}l_j + h_{jk}l_i), \end{aligned}$$

where $l_i = \frac{\partial F}{\partial y^i}$ and $h_{ij} = g_{ij} - l_i l_j$.

Moreover, taking into account the positively homogeneousness of degree one of F with respect to y^i, we succesively deduce:

$$\begin{aligned}\frac{\partial}{\partial y^j}\left(\frac{\partial F^2}{\partial y^i}y^i\right) &= 2\frac{\partial F^2}{\partial y^j}\\ \frac{\partial^2 F^2}{\partial y^i \partial y^j}y^i + \frac{\partial F^2}{\partial y^j} &= 2\frac{\partial F^2}{\partial y^j}\\ g_{ij}(x_0,y)y^i y^j &= F^2\end{aligned}$$

2. Indicatrix of a Finsler space. The indicatrix I_{x_0} at $x_0\,(\in M^n)$ is defined as being the hypersurface in $(V^n, g(x_0,y))$ which is given by the following equation

$$F(x_0,y)=1 \quad \text{or} \quad g_{ij}(x_0,y)y^i y^j = 1.$$

It can be locally represented by the equations $y^i = y^i(u^\alpha)$ where $i = 1,2,\ldots,n$ and $\alpha = 1,2,\ldots,n-1$. In order to study the indicatrix I_{x_0} we shall apply the theory of submanifolds in Riemannian manifolds. We shall denote the projection factor $\frac{\partial y^i}{\partial u^\alpha}$ and the unit normal vector to I_{x_0} by B^i_α and N^i, respectively. Then the induced Riemannian metric tensor $g_{\alpha\beta}$ on I_{x_0} is given by

$$g_{\alpha\beta} = g\left(\frac{\partial}{\partial u^\alpha},\frac{\partial}{u^\beta}\right) = g\left(\frac{\partial y^i}{\partial u^\alpha}\frac{\partial}{\partial y^i},\frac{\partial y^j}{u^\beta}\frac{\partial}{\partial y^j}\right) = g_{ij}B^i_\alpha B^j_\beta. \tag{2.3}$$

The Gauss and Weingarten formulae are as follows:

$$\begin{aligned}\text{(G)}\qquad \frac{\partial B^i_\alpha}{\partial u^\beta} &+ C^i_{jk}(x,y)B^j_\alpha B^k_\beta = \Gamma^\gamma_{\alpha\beta}B^i_\gamma + b_{\alpha\beta}N^i,\\ \text{(W)}\qquad \frac{\partial N^i}{\partial u^\beta} &+ C^i_{jk}(x,y)N^j B^k_\beta = -b^\gamma_\beta B^i_\gamma\end{aligned}$$

where Γ's are Christoffel symbols constructed from $g_{\alpha\beta}(u)$ and $b^\gamma_\beta = b_{\beta\sigma}g^{\sigma\gamma}$. If one derives the identity $F(x_0,y^i(u^\alpha))=1$ with respect to u^α, one gets $l_i B^i_\alpha = 0$. Thus the unit normal vector field to I_{x_0} is $N^i = l^i = \frac{y^i}{F}$. It follows that the second term in the left side of the formula (W) vanishes because of $C^i_{jk}y^j = 0$.

We shall denote the D_α the operator of mixt covariant derivative (due to van der Wearden-Bortolotti). For instance, on T^i_β it acts as follows:

$$D_\alpha T^i_\beta = \frac{\partial T^i_\beta}{\partial u^\alpha} + C^i_{jk}T^j_\beta B^k_\alpha - \Gamma^\gamma_{\beta\alpha}T^i_\gamma.$$

By a direct computation one gets

$$D_\alpha g_{ij} = 0, \tag{2.4}$$

$$D_\alpha B^i_\beta = b_{\alpha\beta} l^i, \tag{2.5}$$

$$D_\alpha l^i = -g^{\beta\gamma} b_{\alpha\gamma} B^i_\beta, \tag{2.6}$$

$$D_\gamma g_{\alpha\beta} = 0, \tag{2.7}$$

where the symmetric tensor $b_{\alpha\beta}$ is the second fundamental tensor of I_{x_0} and $g^{\alpha\beta}$ are the contravariant components of $g_{\alpha\beta}$. Moreover, from (1.1) it is clear that

$$D_\alpha y^i = B^i_\alpha. \tag{2.8}$$

We have seen before that $g_{ij} B^i_\alpha y^j = 0$. On applying D_α to this we get

$$g_{ij} B^i_\alpha B^j_\beta + g_{ij} (D_\beta B^i_\alpha) y^j = 0$$

and on substituting (2.3) and (2.5) in this equality, we obtain $g_{\alpha\beta} = -b_{\alpha\beta}$. This equality proves

Theorem 2.1. *The indicatrix I_{x_0} of a Finsler space F^n, $n \geq 3$, is a totally umbilical hypersurface with the mean curvature -1.*

The same equality leads to

$$D_\alpha B^i_\beta = -g_{\alpha\beta} y^i. \tag{2.9}$$

Thus, we have:

Theorem 2.2. *The Gauss and Weingarten derivation formulae of the indicatrix I_{x_0} are as follows:*

$$\begin{aligned} &(G) \quad \frac{\partial B^i_\alpha}{\partial u^\beta} + C^i_{jk}(x,y) B^j_\alpha B^k_\beta = \Gamma^\gamma_{\alpha\beta} B^i_\gamma - g_{\alpha\beta} l^i, \\ &(W) \quad \frac{\partial l^i}{\partial u^\beta} = B^i_\beta \end{aligned}$$

By the Theorem 2.1, I_{x_0} may be regarded as the unit sphere in a Euclidean space.

Let $R_{\alpha\beta\gamma\delta}$ be the curvature tensor of I_{x_0}. From the Gauss equation, we have that

$$R_{\alpha\beta\gamma\delta} = S_{ijkl}B^i_\alpha B^j_\beta B^k_\gamma B^l_\delta - (g_{\alpha\delta}g_{\beta\gamma} - g_{\alpha\gamma}g_{\beta\delta}). \tag{2.10}$$

On the other hand, from the definitions of h_{ij} and $g_{\alpha\beta}$ we have

$$g_{\alpha\beta} = h_{ij}B^i_\alpha B^j_\beta. \tag{2.11}$$

Substituting (2.11) in (2.10), we get

$$R_{\alpha\beta\gamma\delta} = (S_{ijkl} - h_{il}h_{jk} + h_{ik}h_{jl})B^i_\alpha B^j_\beta B^k_\gamma B^l_\delta. \tag{2.12}$$

Contracting (2.12) by $g^{\alpha\delta}$, we obtain

$$R_{\beta\gamma} = (S_{jk} - (n-2)h_{jk})B^j_\beta B^k_\gamma, \tag{2.13}$$

where $R_{\beta\gamma}$ is the Ricci tensor of I_{x_0} and $S_{jk} = g^{il}S_{ijkl}$ is $v-$Ricci tensor. Moreover, contracting (2.13) by $g^{\beta\gamma}$, we have

$$R = S - (n-2)(n-1), \tag{2.14}$$

where R is the scalar curvature of I_{x_0} and $S = g^{jk}S_{jk}$ is $v-$ scalar curvature.

3. On some properties of Finsler spaces based on the indicatrices. Now, we shall represent the hypersurface M^{n-1} with respect to the frame $\left(B^i_\alpha, N^i\right)$ as follows:

$$y^i(u) = v^\beta(u)B^i_\beta + v(u)N^i \equiv v^\beta B^i_\beta + vN^i. \tag{3.15}$$

Then we have

Theorem 3.1 ([4]). *For a hypersurface $M^{n-1} : y^i = y^i(u^\alpha)$ in V^n, the vector v^α and the scalar v on M^{n-1} defined by* (3.15) *satisfy the following equations:*

$$\begin{aligned} &a) \quad D_\alpha v_\beta = g_{\alpha\beta} + b_{\alpha\beta}v, \\ &b) \quad D_\alpha v = -b_{\alpha\beta}v^\beta. \end{aligned} \tag{3.16}$$

Proof. Applying D_α to (3.15), and using (2.5), (2.6) and (2.8), we have

$$(D_\alpha v^\gamma - \delta^\gamma_\alpha - g^{\beta\gamma} b_{\beta\alpha} v) B^i_\gamma + (b_{\alpha\beta} v^\beta + D_\alpha v) N^i = 0.$$

Since B^i_γ and N^i are linearly independent, we have (3.16). □

Theorem 3.2. *The integrability conditions of* (3.16) *are as follows:*

$$[R_{\alpha\beta\gamma\delta} + b_{\alpha\gamma} b_{\beta\delta} - b_{\beta\gamma} b_{\alpha\delta}]\, v^\delta = [D_\beta b_{\alpha\gamma} - D_\alpha b_{\beta\gamma}]\, v, \tag{3.17}$$

$$[D_\beta b_{\alpha\gamma} - D_\alpha b_{\beta\gamma}]\, v^\gamma = 0, \tag{3.18}$$

where $R_{\alpha\beta\gamma\delta} g^{\delta\varepsilon} = R_{\alpha\beta\gamma}{}^{\varepsilon}$ *is the Riemannian curvature tensor of* M^{n-1}.

Proof. This proof clarifies the relation between the equations of Gauss and Codazzi for M^{n-1} and the equations (3.17) and (3.18).

The equations of Gauss and Codazzi are given by

$$R_{\alpha\beta\gamma\delta} = R_{ijkl} B^i_\alpha B^j_\beta B^k_\gamma B^l_\delta + b_{\alpha\delta} b_{\beta\gamma} - b_{\beta\delta} b_{\alpha\gamma}$$

$$R_{ijkl} B^i_\alpha B^j_\beta B^k_\gamma N^l = D_\alpha b_{\beta\gamma} - D_\beta b_{\alpha\gamma}$$

respectively. First, we apply D_α to (3.16)a) and use $D_\alpha D_\gamma v_\beta = D_\gamma D_\alpha v_\beta$ and (2.7) we succesively obtain

$$D_\gamma(b_{\alpha\beta} v) = D_\alpha(b_{\gamma\beta} v)$$

$$(D_\gamma b_{\alpha\beta}) v + b_{\alpha\beta} D_\gamma v = (D_\alpha b_{\gamma\beta}) v + b_{\gamma\beta} D_\alpha v.$$

On substituting (3.16)b) in the last we obtain

$$(D_\gamma b_{\alpha\beta} - D_\alpha b_{\gamma\beta}) v = (b_{\alpha\beta} b_{\gamma\delta} - b_{\gamma\beta} b_{\alpha\delta}) v^\delta$$

Taking into account the equations of Gauss, Codazzi and $y^l = v^\delta B^l_\delta + v N^l$ we deduce:

$$\begin{aligned}(D_\gamma b_{\alpha\beta} - D_\alpha b_{\gamma\beta}) v &= R_{ijkl} B^i_\gamma B^j_\alpha B^k_\beta N^l v \\ &= -R_{ijkl} B^i_\gamma B^j_\alpha B^k_\beta B^l_\delta v^\delta \\ &= [-R_{\gamma\alpha\beta\delta} + b_{\gamma\delta} b_{\alpha\beta} - b_{\alpha\delta} b_{\gamma\beta}] v^\delta\end{aligned}$$

Also, applying D_α to (3.16)b) and using $D_\alpha D_\beta v = D_\gamma D_\alpha v$ we obtain

$$(D_\beta b_{\alpha\delta} - D_\alpha b_{\beta\delta}) v^\delta + b_{\alpha\delta} D_\beta v^\delta - b_{\beta\delta} D_\alpha v^\delta = 0$$

where

$$\begin{aligned} b_{\alpha\delta}D_\beta v^\delta - b_{\beta\delta}D_\alpha v^\delta &= b_{\alpha\delta}D_\beta g^{\delta\gamma}v_\gamma - b_{\beta\delta}D_\alpha g^{\delta\gamma}v_\gamma \\ &= b_{\alpha\delta}g^{\delta\gamma}D_\beta v_\gamma - b_{\beta\delta}g^{\delta\gamma}D_\alpha v_\gamma \\ &= b_{\alpha\delta}g^{\delta\gamma}(g_{\beta\gamma}+b_{\beta\gamma}v) - b_{\beta\delta}g^{\delta\gamma}(g_{\alpha\gamma}+b_{\alpha\gamma}v) \\ &= b_{\alpha\delta}g^{\delta\gamma}b_{\beta\gamma}v - b_{\beta\delta}g^{\delta\gamma}b_{\alpha\gamma}v = 0. \end{aligned}$$

□

A hypersurface of equation $g_{ij}(x_0)y^iy^j = c$ in $T_{x_0}M$ will be called similar to the indicatrix.

Theorem 3.3 ([4]). *Let M^{n-1} be a closed hypersurface in V^n, $y^i = y^i(u^\alpha)$ its parametric equation and $c = \max_{u\in M^{n-1}}\left\{g_{ij}(y(u))y^i(u)y^j(u)\right\}$. If M^{n-1} is totally umbilical: $b_{\alpha\beta} = -\sigma g_{\alpha\beta}$ and satisfies a condition $|\sigma| \leq \frac{1}{\sqrt{c}}$, then M^{n-1} is similar to the indicatrix in V^n.*

Proof. From (3.15) and the assumption, we have

$$\begin{aligned} g_{ij}y^i(u)y^j(u) &= g_{ij}\left(v^\alpha B^i_\alpha + vN^i\right)\left(v^\beta B^j_\beta + vN^j\right) \\ &= g_{\alpha\beta}v^\alpha v^\beta + v^2 \leq c. \end{aligned} \tag{3.19}$$

Accordingly, we get

$$|v| \leq \sqrt{c}. \tag{3.20}$$

On the other hand, applying D_λ to (3.19) and using Theorem 3.1, we get

$$D_\lambda\left(g_{ij}(y(u))y^i(u)y^j(u)\right) = D_\lambda\left(g_{\alpha\beta}v^\alpha v^\beta + v^2\right)$$

$$= 2(g_{\lambda\beta}+vb_{\lambda\beta})v^\beta + 2v(-b_{\lambda\beta}v^\beta)v_\lambda = 2v_\lambda.$$

Moreover, applying D_μ to the above equations, we have

$$D_\mu D_\lambda\left(g_{ij}(y(u))y^i(u)y^j(u)\right) = 2D_\mu v_\lambda = 2(g_{\mu\lambda}+vb_{\mu\lambda}). \tag{3.21}$$

Substitution of the assumption $b_{\alpha\beta} = -\sigma g_{\alpha\beta}$ into (3.21) gives

$$D_\mu D_\lambda\left(g_{ij}(y(u))y^i(u)y^j(u)\right) = 2(1-\sigma v)g_{\mu\lambda}$$

Accordingly, using (3.20) we have

$$g^{\mu\lambda}D_\mu D_\lambda\left(g_{ij}(y(u))y^i(u)y^j(u)\right) = 2(n-1)(1-\sigma v) \geq 0.$$

□

Theorem 3.4 ([4]). *Let M^{n-1} be a closed hypersurface (i.e. boundless and compact) in V^n and H_1 the first mean curvature of M^{n-1} and v the scalar function on M^{n-1} defined by (3.15). If M^{n-1} satisfies an inequality $-1 \leq H_1 v$, then M^{n-1} is similar to the indicatrix in V^n.*

Proof. Contracting $g^{\mu\lambda}$ to (3.21) which is satisfied for a hypersurface in V^n and using the assumption, we get

$$g^{\mu\lambda} D_\mu D_\lambda \left(g_{ij}(y(u)) y^i(u) y^j(u) \right) = 2(n-1)(1 + H_1 v) \geq 0.$$

The remaining part of the proof is similar to the proof of Theorem 3.3. □

Theorem 3.5 ([3]). *Let $M^{n-1} : y^i = y^i(u^\alpha)$, be a hypersurface similar to the indicatrix in $T_{x_0}(M^n)$. Then M^{n-1} satisfies the condition*

$$1 + H_1 v = 0.$$

Proof. Since M^{n-1} is similar to the indicatrix in $T_{x_0}(M^n)$ it is obviously closed and

$$g_{ij}(y(u)) y^i(u) y^j(u) = c \qquad (= \text{constant} > 0).$$

On differentiating this equation with respect to u^α and using (1.1) we obtain $g_{ij}(y(u)) B^i_\alpha y^j(u) = 0$ and hence from (3.15) we have $v_\alpha = 0$ and consequently $y^i(u) = v(u) N^i$. Now from (3.16) we can write $g_{\alpha\beta} + b_{\alpha\beta} v(u) = 0$.

The theorem now follows after contracting with $g_{\alpha\beta}$.

It should be noted that under the assumptions of the Theorem 3.4, the hypersurface M^{n-1} is totally umbilical, $v =$constant $\neq 0$ and $H_1 = -\frac{1}{v}$. □

Theorem 3.6 ([3]). *Let $M^{n-1} : y^i = y^i(u^\alpha)$, be a closed hypersurface in V^n. Then M^{n-1} is similar to the indicatrix in $T_{x_0}(M^n)$ if and only if $D_\alpha v_\beta = 0$ (or $v_\beta = 0$).*

Proof. If we assume that $D_\alpha v_\beta = 0$, then from ((3.16) a)) we have $g_{\alpha\beta} + b_{\alpha\beta} v(u) = 0$, and hence $1 + H_1 v = 0$. Now using Theorem 3.4 we find that M^{n-1} is similar to the indicatrix in $T_{x_0}(M^n)$. The converse follows from the proof of Theorem 3.5. The fact that the conditions $D_\alpha v_\beta = 0$ and $v_\beta = 0$ are equivalent is easy to verify. □

Theorem 3.7. *Let $M^{n-1} : y^i = y^i(u^\alpha)$, be a closed hypersurface in V^n. Then M^{n-1} is similar to the indicatrix in $T_{x_0}(M^n)$ if and only if*

$$1 + 2H_1 v + (n-1)H_1^2 v - (n-2)H_2 v^2 = 0, \tag{3.22}$$

where H_2 is the second mean curvature of M^{n-1}.

Proof. From (3.16) we can write

$$\begin{aligned} g^{\alpha\gamma} g^{\beta\delta}(D_\gamma v_\delta)(D_\alpha v_\beta) &= (D^\alpha v^\beta)(D_\alpha v_\beta) \\ &= (g^{\alpha\beta} + b^{\alpha\beta} v)(g_{\alpha\beta} + b_{\alpha\beta} v) \\ &= (n-1)(1 + 2H_1 v + (n-1)H_1^2 v - (n-2)H_2 v^2). \end{aligned}$$

The result (3.22) now follows from Theorem 3.6. □

Next we shall consider totally umbilical hypersurfaces in V^n. In this case we have from (3.16),

$$(D_\alpha v_\beta + D_\beta v_\alpha) = 2(1 + H_1 v) g_{\alpha\beta}.$$

We shall confine ourselves to totally umbilical hypersurfaces with constant first mean curvature. Such a hypersurface may not necessarily be the hypersurface similar to the indicatrix in $T_{x_0}(M^n)$. For example, in the case where $g_{ij} = \delta_{ij}$ the hypersurfaces are hyperspheres. But the hyperspheres may not necessarily be similar to the indicatrix (the unit hypersphere with the origin as the center) in $T_{x_0}(M^n)$. This is due to the fact that the center of a hypersphere need not coincide with the origin of $T_{x_0}(M^n)$.

Theorem 3.8 ([3]). *Let M^{n-1} be a closed totally umbilical hypersurface with the constant first mean curvature H_1 in V^n, then $H_1 \neq 0$.*

Proof. Let us assume that $H_1 = 0$. Then $b_{\alpha\beta} = 0$ and from ((3.16) b)) we have $D_\alpha v = 0$. Hence v =constant and accordingly $1 + H_1 v$ =constant. In this case, we have $1 + H_1 v \geq 0$ or $1 + H_1 v \leq 0$. Now by Theorem 3.4 this hypersurface M^{n-1} must be similar to the indicatrix in $T_{x_0}(M^n)$ and hence by Theorem 3.5 we must have $1 + H_1 v = 0$. But this is a contradiction. Hence, the result follows. □

Theorem 3.9 ([3]). *Let M^{n-1} be a compact and totally umbilical hypersurface with the constant first mean curvature in V^n. If M^{n-1} satisfies the condition $R_{\alpha\beta} v^\alpha v^\beta \leq 0$, then M^{n-1} is similar to the indicatrix in $T_{x_0}(M^n)$, where $n \geq 3$ and $R_{\alpha\beta}$ is the Ricci tensor of M^{n-1}.*

Proof. Since M^{n-1} is totally umbilical, we have $b_{\alpha\beta} = H_1 g_{\alpha\beta}$. Substituting this relation into (3.17) we obtain

$$\left[R_{\alpha\beta\gamma\delta} + H_1^2(g_{\alpha\gamma}g_{\beta\delta} - g_{\beta\gamma}g_{\alpha\delta})\right] v^\delta = 0. \tag{3.23}$$

On contracting (3.23) with $g^{\alpha\gamma}v^\beta$ we obtain

$$R_{\beta\delta}v^\beta v^\delta + H_1^2\left((n-1)g_{\beta\delta}v^\beta v^\delta - \delta^\alpha_\beta g_{\alpha\delta}v^\beta v^\delta\right) = 0,$$
$$R_{\alpha\beta}v^\alpha v^\beta = (n-2)H_1^2 g_{\alpha\beta}v^\alpha v^\beta \geq 0.$$

Now according to the assumption $R_{\alpha\beta}v^\alpha v^\beta \leq 0$ and Theorem 3.8 it is clear that $v^\alpha = 0$. Thus by Theorem 3.6 the hypersurface M^{n-1} is similar to the indicatrix in $T_{x_0}(M^n)$. □

Theorem 3.10. *Let M^{n-1} be a closed totally umbilical hypersurface with the constant first mean curvature H_1 in V^n. Then M^{n-1} is similar to the indicatrix in $T_{x_0}(M^n)$ if and only if v = constant.*

Proof. Let M^{n-1} be similar to the indicatrix in $T_{x_0}(M^n)$. Then by Theorem 3.5 we have $1 + H_1 v = 0$ and hence v = constant. Conversely, if v =constant then we should have $1 + H_1 v \geq 0$ or $1 + H_1 v \leq 0$. The hypersurface M^{n-1} is clearly similar to the indicatrix in $T_{x_0}(M^n)$ by Theorem 3.4. □

4. Length of indicatrix. We are concerning with the tangent plane F_x^2 at a fixed point x of a two-dimensional Finsler space F^2 with a fundamental function $F(x, y)$ and the fundamental tensor $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$.

Thus we are concerning with a two-dimensional Minkowski space M^2, a short notation for (F_x^2, F) with a norm function $F(y)$.

Let (l_i, m_i) be the Berwald frame field of M^2 ([1]), where $l_i = \frac{\partial F}{\partial y^i}$ and m_i is the unit vector orthogonal to l_i, that is, $(m_1, m_2) = \left(-l^2, l^1\right)\sqrt{g}$ where $g = \det(g_{ij})$. The contravariant components $\left(l^i, m^i\right)$ are given by $l^i = \frac{y^i}{F(y)}$ and $\left(m^1, m^2\right) = (-l_2, l_1) \diagup \sqrt{g}$. Then Cartan's tensor $C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}$ is written as

$$FC_{ijk} = I m_i m_j m_k,$$

where I is called the main scalar.

The Riemannian covariant differentiation in M^2 is nothing but the $v-$covariant one in F^2 with respect to the Cartan connection. Thus the Riemannian covariant derivative $X^i_j\ |_k$ of a tensor field $X^i_j(y)$ in M^2 is given by

$$X^i_j\ |_k = \frac{\partial X^i_j}{\partial y^k} + X^r_j C_r\ {}^i_k - X^i_r C_j\ {}^r_k.$$

Differentiating $Fl^i = y^i$ by y^j, we get $l_j l^i + F l^i\ |_j = \delta^i_j$ and then $g_{ik}g^{kj} = g^{kj}(l_i l_k + m_i m_k) = l^i l_j + m^i m_j$ leads to $Fl^i\ |_j = m^i m_j$. Also, differentiating $l_i m^i = 0$ and $g_{ij} = m^i m^j = 1$ by y^j, we obtain $Fm^i\ |_j = -(l^i + I m^i)m_j$. Now we get

$$\begin{aligned} \frac{\partial l^i}{\partial y^j} &= \frac{m^i m_j}{F}, \\ \frac{\partial m^i}{\partial y^j} &= -\frac{\left(l^i + I\ m^i\right) m_j}{F}. \end{aligned} \tag{4.24}$$

Since

$$\frac{\partial}{\partial y^j}\left(\frac{m_i}{F}\right) = \frac{\partial}{\partial y^i}\left(\frac{m_j}{F}\right)$$

holds the system of differential equations

$$\frac{\partial \theta}{\partial y^i} = \frac{m_i}{F} \tag{4.25}$$

is completely integrable. Its solution, the scalar θ is called the Landsberg angle. Then the pair (F, θ) may be regarded as a local coordinate system of F^2_x; we shall call it a polar coordinates system in F^2_x. The Jacobian matrix $\frac{\partial(F,\theta)}{\partial(y^1,y^2)}$ is $\left[l_i, \frac{m_i}{F}\right]$ and the inverse matrix $\frac{\partial(y^1,y^2)}{\partial(F,\theta)}$ is $\left[\frac{\partial y^i}{\partial F} = l^i, \frac{\partial y^i}{\partial \theta} = F m^i\right]$. It follows that the fundamental form $ds^2 = g_{ij}(x,y)dy^i dy^j$ of F^2_x becomes

$$ds^2 = dF^2 + F^2 d\theta^2. \tag{4.26}$$

In the polar coordinates system (F, θ) the movement of the Berwald frame $\left(l^i, m^i\right)$ is written

$$\begin{aligned} a)\quad & \frac{\partial l^i}{\partial F} = 0, \qquad \frac{\partial m^i}{\partial F} = 0, \\ b)\quad & \frac{\partial l^i}{\partial \theta} = m^i, \qquad \frac{\partial m^i}{\partial \theta} = -(l^i + I\ m^i). \end{aligned} \tag{4.27}$$

The equations ((4.27)a) show that the Berwald frame is constant along every ray issuing from the origin ($y = 0$).

We stress that the Landsberg angle θ is only locally defined as a solution of (4.25) and that the domain of θ is not necessarily equal to $[0, 2\pi]$.

We are now concerning with a curve c of M^2. Let $y^i = y^i(s)$ be its equations, where s is the arc-length of c with respect to the Riemannian metric ds^2 under consideration, and $(t^i = \frac{dy^i}{ds}, n^i)$ the unit tangent and normal vectors of c, respectively. Then the first Frenet formula is

$$Kn^i = \frac{dt^i}{ds} + C_j{}^i_k{}(y(s))\, t^j t^k, \tag{4.28}$$

where $K \geq 0$ is the curvature of c. The orientation of c with respect to s is assumed to be taken such that there exists an angle α satisfying

$$t^i = l^i \cos\alpha + m^i \sin\alpha, \qquad n^i = -l^i \sin\alpha + m^i \cos\alpha. \tag{4.29}$$

From (4.24) and (4.29) we get

$$\frac{dl^i}{ds} = \frac{\sin\alpha}{F} m^i, \qquad \frac{dm^i}{ds} = -\frac{\sin\alpha}{F}(l^i + I\ m^i). \tag{4.30}$$

We next consider the curve c in the polar coordinate system (F, θ). Let $(F(s), \theta(s))$ be the equation of c. From (1.2) and (4.29) we get

$$\frac{dF}{ds} = F\ |_i\ t^i = \cos\alpha, \qquad \frac{d\theta}{ds} = \theta\ |_i\ t^i = \frac{\sin\alpha}{F}. \tag{4.31}$$

Hence we have $\frac{dt^i}{ds} = -(\frac{\sin^2\alpha}{F})l^i + \frac{\sin\alpha}{F}(\cos\alpha - I\sin\alpha)m^i + (\frac{d\alpha}{ds})n^i$.

The equation (4.28) is now written in the following simple form:

$$\frac{d\alpha}{ds} + \frac{\sin\alpha}{F} = K. \tag{4.32}$$

It is observed that (4.31) and (4.32) constitute a complete system of differential equations for (F, θ, α), provided a non-negative function $K(s)$ be given. Therefore we have

Theorem 4.1. *Suppose a non-negative function $K(s)$ be given. Then, in a polar coordinate system (F, θ) of M^2, a curve having the curvature $K(s)$ is given by the equation $(F(s), \theta(s))$, where $F(s)$, $\theta(s)$ together with $\alpha(s)$ are the unique solutions of differential equations* (4.31), (4.32) *with a set of initial values $(F(0), \theta(0), \alpha(0))$.*

Next we are concerned with the equation $F = F(\theta)$ of a curve c. From (4.31) we get

$$\frac{dF}{d\theta} = F\cot\alpha. \tag{4.33}$$

Further (4.31) and (4.32) yield

$$\frac{d\alpha}{d\theta} = \frac{FK}{\sin\alpha} - 1. \tag{4.34}$$

The equations (4.33) and (4.34) uniquely determine a curve $F = F(\theta)$ having a given curvature $K(\theta)$, provided a set of initial values $(F(0), \theta(0))$ be given.

We differentiate (4.33) by θ. From the equation as thus obtained together with (4.33) and (4.34) we eliminate trigonometrical functions and we get

$$K(\theta) = \frac{2(L')^2 + L^2 - LL''}{\{L^2 + (L')^2\}^{\frac{3}{2}}}, \tag{4.35}$$

where $L' = \frac{dF}{d\theta}$ and $L'' = \frac{d^2F}{d\theta^2}$.

In particular, for the equation $F(\theta) = a\theta + b$ we have

$$K(\theta) = \frac{a}{(1 + ab + a^2\theta)^{\frac{3}{2}}} + \frac{a}{\sqrt{1 + ab + a^2\theta}},$$

where a, b are any constants.

On the other hand, for the curve c given by the equation $\theta = \theta(F)$, from (4.31) and (4.32) we get

$$\frac{d\theta}{dF} = \frac{\tan\alpha}{F}, \qquad \frac{d\alpha}{dF} = \frac{K}{\cos\alpha} - \frac{\tan\alpha}{F}.$$

These equations uniquely determine the curve $\theta = \theta(F)$ having the curvature

$$K(F) = \frac{\theta''F + \theta'(2 + \theta'F)}{(1 + \theta'F)^{\frac{3}{2}}}, \quad \text{where} \quad \theta' = \frac{d\theta}{dF}, \quad \theta'' = \frac{d^2\theta}{dF^2},$$

provided a set of initial values $(\theta(0), F(0))$ be given.

We apply the equations (4.31) and (4.32) to the indicatrix. As it is given by $F(y) = 1$, those equations become $\cos\alpha = 0$, $\frac{d\theta}{ds} = \sin\alpha$ and $\frac{d\alpha}{ds} + \sin\alpha = K$. As the curvature K should be non-negative, we get $\alpha = \frac{\pi}{2}$, $K = 1$ and $\frac{d\theta}{ds} = 1$; the last equation can be also derived from (4.26). Consequently we have

Corollary 4.2. *The indicatrix has the curvature $K = 1$ and the arc-length of the indicatrix is equal to the Landsberg angle θ within an additional constant.*

Remark. Therefore the Landsberg angle may be said to have a character of the radian, because the indicatrix is a generalization of the unit circle of a euclidean plane.

Also, we can prove that a curve c of M^2 with the curvature $K = 1$ and its arc-length is proportional with θ (i.e. $sc = \theta$, c=constant) is the indicatrix.

Indeed, on substituting the followings

$$\frac{d\alpha}{ds} = \frac{d\alpha}{d\theta}\frac{d\theta}{ds}, \qquad \frac{dF}{ds} = \frac{dF}{d\theta}\frac{d\theta}{ds}, \qquad \frac{d\theta}{ds} = c$$

in the equations (4.31) and (4.32), we obtain

$$\begin{aligned} c\frac{dF}{d\theta} &= \cos\alpha \\ c &= \frac{\sin\alpha}{F} \\ \frac{d\alpha}{d\theta}c &+ \frac{\sin\alpha}{F} = 1 \end{aligned}$$

From the last equation we find $\alpha = (\frac{1}{c} - 1)\theta + \alpha_0$ and using this in the first equation, we obtain $\frac{dF}{d\theta} = \frac{1}{c}\cos\big((\frac{1}{c} - 1)\theta + \alpha_0\big)$. If we take $c = 1$ and $\alpha_0 = \dfrac{\pi}{2}$ we get $F = 1$.

We now deal with the arc-length s of the indicatrix. From a linear coordinate system $\big(y^1, y^2\big)$ we define a pair (ρ, ϕ) of F_x^2 by

$$y^1 = \rho\cos\phi, \qquad y^2 = \rho\sin\phi, \qquad (\rho > 0).$$

The pair (ρ, ϕ) may be regarded as a coordinate system which is globally applicable to M^2 with the origin removed. Along the indicatrix we have

$F(\rho\cos\phi, \rho\sin\phi) = \rho F(\cos\phi, \sin\phi) = 1$, and so the indicatrix is given by

$$y^1 = \frac{\cos\phi}{F(\cos\phi, \sin\phi)}, \qquad y^2 = \frac{\sin\phi}{F(\cos\phi, \sin\phi)}. \tag{4.36}$$

The whole indicatrix is drawn by $\left(y^1, y^2\right)$ of (4.36) with the domain $0 \leq \phi < 2\pi$. As $F\left(y^1, y^2\right)$ is (1) $p-$ homogeneous, we have

$$F_1(\cos\phi, \sin\phi)\cos\phi + F_2(\cos\phi, \sin\phi)\sin\phi = F(\cos\phi, \sin\phi),$$

where $F_i\left(y^1, y^2\right) = \frac{\partial F\left(y^1, y^2\right)}{\partial y^i}$, $i = 1, 2$.

Along the indicatrix given by (4.36) it is observed that

$$\begin{aligned} dy^1 &= \left[\frac{-\sin\phi}{F} - \cos\phi\frac{F_1(-\sin\phi) + F_2\cos\phi}{F^2}\right] d\phi \\ &= \left[\frac{-\sin\phi}{F} + \sin\phi\frac{F - F_2\sin\phi}{F^2} - \frac{F_2\cos^2\phi}{F^2}\right] d\phi, \end{aligned}$$

which is equal to $-\left(\frac{F_2}{F^2}\right) d\phi$. Similarly we have

$$dy^1 = -\left(\frac{F_2}{F^2}\right) d\phi, \qquad dy^2 = \left(\frac{F_1}{F^2}\right) d\phi. \tag{4.37}$$

As we are concerned with the Riemannian length $ds = \left\{g_{ij}(y)dy^i dy^j\right\}^{\frac{1}{2}}$, the infinitesimal arc-length of the indicatrix is given by

$$ds = \left\{g_{ij}(y(\phi))\frac{dy^i}{d\phi}\frac{dy^j}{d\phi}\right\}^{\frac{1}{2}} d\phi.$$

From (4.37) this is rewritten as

$$ds = \left\{g_{11}(F_2)^2 - 2g_{12}F_1F_2 + g_{22}(F_1)^2\right\}^{\frac{1}{2}} F^{-2}d\phi.$$

Paying attention to $F_i = l_i$ and $(g_{11}, -g_{12}, g_{22}) = g(g^{22}, g^{12}, g^{11})$ $(g = \det(g_{ij}))$, we conclude

$$ds = \left(\frac{\sqrt{g}}{F^2}\right) d\phi. \tag{4.38}$$

Theorem 4.3 ([2]). *The infinitesimal arc-length ds of the indicatrix of a two-dimensional Minkowski space with respect to the Riemannian metric $g_{ij}(y)dy^i dy^j$ is given by* (4.38) *and the total length of the indicatrix is equal to the value of the integral*

$$F_I = \int_0^{2\pi} \left(\frac{\sqrt{g}}{F^2} \right) d\phi,$$

where $\left(y^1, y^2\right)$ *are taken as* $(\cos\phi, \sin\phi)$.

According to Proposition1, it may be said that the domain of the Landsberg angle θ is given by $0 \leq \theta < F_I$. The value of F_I is not necessarily equal to 2π, as the following example show, so that the name "angle" may be not suitable.

Example. We consider a Randers space F^n, a Finsler space with a Randers metric $F(x,y) = \alpha(x,y) + \beta(x,y)$, where $\alpha^2(x,dx) = a_{ij}(x)dx^i dx^j$ and $\beta(x,dx) = b_i(x)dx^i$ ([1] §16,30). In the following we suppose that $\alpha^2(x,dx)$ is a positive-definite Riemannian metric. With respect to this α^2 we denote by Y, B and ψ the length of y^i and of $B^i = a^{ij}b_j$ and the angle between y^i and B^i. Then we have $F(x,y) = Y(1 + B\cos\psi)$. Therefore $F(x,y)$ is positive for any non-zero y if and only if B is less than one. On this condition the Riemannian metric $ds^2 = g_{ij}(x,y)dy^i dy^j$ of F^n is positive-definite. Since g_{ij} is given by

$$g_{ij} = \frac{F}{\alpha}a_{ij} - \frac{F}{\alpha^3}Y_i Y_j + \left(\frac{Y_i}{\alpha} + b_i \right)\left(\frac{Y_j}{\alpha} + b_j \right),$$

where $Y_i = a_{ij}y^j$, we get

$$g_{ij}u^i u^j = \frac{F}{\alpha^3}\left\{ \left(a_{ij}y^i y^j\right)\left(a_{ij}u^i u^j\right) - \left(a_{ij}y^i u^j\right)^2 \right\} + \left(\frac{Y_i u^i}{\alpha} + b_i u^i \right)^2$$

for any u^i.

We now consider a two-dimensional Randers space satisfying $0 < B < 1$. In this case we may refer to an isothermal coordinate system $\left(x^i\right)$, in which we have $\alpha(x,y) = a(x)\left\{(y^1)^2 + (y^2)^2\right\}^{\frac{1}{2}}$. It is easily seen that if this $\left(y^i\right)$ is regarded as an orthonormal coordinate system in an euclidean plane, the

indicatrix is an ellipse such that the origin ($y = 0$) is on its inside. Theorem 4.3 shows that the total length of this indicatrix is given by

$$\begin{aligned} F_I &= \int_0^{2\pi} \left\{1 + \frac{b_1}{a}\cos\phi + \frac{b_2}{a}\sin\phi\right\}^{-\frac{1}{2}} d\phi = 2\int_0^{\pi} \left(1 + \overline{B}\cos\phi\right)^{-\frac{1}{2}} d\phi \\ &= 2\int_0^{\frac{\pi}{2}} \left\{\left(1 + \overline{B}\cos\phi\right)^{-\frac{1}{2}} + \left(1 - \overline{B}\cos\phi\right)^{-\frac{1}{2}}\right\} d\phi, \end{aligned}$$

where $\overline{B} = \dfrac{B}{a^2}$.

This is an elliptic integral and the value may be obtained by expansion in power series and termwise integration:

$$F_I = 2\pi\left[1 + \sum_{n=1}^{\infty} \frac{(4n)!\,(2n-1)!!}{\left(2^{2n}\,(2n)!\right)^2\,(2n)!!}\overline{B}^{2n}\right].$$

Thus F_I, that is, the domain of the Landsberg angle θ, certainly exceeds 2π. The inequality $F_I > 2\pi$ is easily shown by an inequality $(1+x)^{-\frac{1}{2}} + (1-x)^{-\frac{1}{2}} > 2$ for $0 < x < 1$.

Finally we consider the area A_I of the indicatrix. A_I is given by the usual integral $\iint \sqrt{g}dy^1dy^2$ in Riemannian geometry. Referring to the polar coordinate system (F, θ), from (4.26) we get

$$A_I = \iint F dF d\theta = \int_0^{F_I} \left[\frac{F^2}{2}\right]_0^1 d\theta.$$

Therefore we obtain

$$A_I = \frac{F_I}{2}.$$

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