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V AND H COHOMOLOGY OF COMPLEX FINSLER BUNDLES

BY

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Dedicated to Academician Radu Miron at his 80th anniversary

Abstract. In the first section we recall briefly the geometry of the total space of holomorphic vector bundles. In the second section, we define the complex forms with values on complex Finsler bundles and, by analogy with [6], starting from a natural decomposition of the exterior differential, we prove Dolbeault type theorems and define new cohomology groups. In the last section we prove the invariance of these groups at complex gauge transformation.

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1. The geometry of total space of a holomorphic vector bundle. In this section we recall briefly, the notion of complex nonlinear connection on holomorphic vector bundle, see [1], [4]. Let $\pi : E \longrightarrow M$ be a holomorphic vector bundle over the complex manifold M, $dim_C M = n$, $rank E = m$.

Consider $\mathcal{U} = (U_\alpha)$ a covering set of M, $(z^k)_{k=\overline{1,n}}$ local coordinates in a chart (U, φ) and $s_U = (s_a)_{a=\overline{1,m}}$ a local frame for the sections of E over U . The covering $\{(U, s_U)_{U\in\mathcal{U}}\}$ induces the complex coordinate system $u = (z^k, \eta^a)$ on $\pi^{-1}(U)$, where $s = \eta^a s_a$ is a section on E_z. Denote by $g_{UV} : U \cap V \longrightarrow GL(m, C)$ the transition functions. In $z \in U \cap V$, $g_{UV}(z)$ has a local representation by the complex matrix $M^a_b(z)$, and if (z'^k, η'^a) are the complex coordinates in $\pi^{-1}(V)$ the transition laws of these coordinates are

$$z'^k = z'^k(z)\,;\eta'^a = M^a_b(z)\,\eta^b. \tag{1}$$

As we already know, the holomorphic vector bundle E has a structure of complex manifold because the transition functions $M_b^a(z)$ are holomorphic. Consider $T^{'}E$ the holomorphic tangent bundle and $T^{''}E = \overline{T^{'}E}$. Then the complexified tangent bundle is $T_C E = T^{'}E \oplus T^{''}E$.

The vertical holomorphic tangent bundle $V(E) = ker\pi_*$ is the relative tangent bundle of the holomorphic projection π. At any $z \in U$, a local frame field on $V_z(E)$ is $\{\frac{\partial}{\partial \eta^a}\}_{a=\overline{1,m}}$.

Moreover, we denote by $\pi^*(T^{'}M)$ the pull-back bundle of the holomorphic tangent bundle on E. Then, the general theory of vector bundles [2] gets the following exact sequence

$$0 \longrightarrow V(E) \xrightarrow{i} T^{'}E \xrightarrow{d\pi} \pi^*\left(T^{'}M\right) \longrightarrow 0. \tag{2}$$

If a splitting $C : T^{'}E \longrightarrow V(E)$ is given in this sequence we have a natural connection on the vertical bundle which determines a smooth distribution $H(E) \subset T^{'}E$ for which, the morphism $d\pi$ in the exact sequence (2), induces an isomorphism of $H(E) \approx \pi^*(T^{'}M)$ called *complex nonlinear connection*, shortly c.n.c., which determines the decomposition

$$T^{'}E = H(E) \oplus V(E). \tag{3}$$

By conjugation, we obtain a decomposition of the complexified tangent bundle

$$T_C E = H(E) \oplus V(E) \oplus \overline{H(E)} \oplus \overline{V(E)}. \tag{4}$$

Note that the splitting C determines, via the fiberwise isomorphism of $H(E)$ with $\pi^*\left(T^{'}M\right)$, a map $l^h : \pi^*\left(T^{'}M\right) \longrightarrow H(E)$ called the horizontal lift, which acts on the natural frame field as follows

$$l^h\left(\frac{\partial}{\partial z^k}\right) := \delta_k = \frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^a \frac{\partial}{\partial \eta^a}. \tag{5}$$

The functions $N_k^a(z, \eta)$are called the coefficients of the *c.n.c* on E and, obvioulsy, are homogeneous functions on η. Using the Jacobi matrix of the (1), we deduce the change laws of the natural vector fields on $T_u^{'}E$,

$$\frac{\partial}{\partial z^j} = \frac{\partial z'^k}{\partial z^j}\frac{\partial}{\partial z'^k} + \frac{\partial M_b^a}{\partial z^j}\eta^b \frac{\partial}{\partial \eta'^a}; \frac{\partial}{\partial \eta^b} = M_b^a \frac{\partial}{\partial \eta'^a}. \tag{6}$$

The transformation rule of $\frac{\delta}{\delta z^j}$ is

$$\frac{\delta}{\delta z^j} = \frac{\partial z'^k}{\partial z^j}\frac{\delta}{\delta z'^k} \tag{7}$$

which, in view of (6), leads to the transformation rule for N_k^a, namely

$$\frac{\partial z'^k}{\partial z^j}N_k'^a = M_b^a N_j^b - \frac{\partial M_b^a}{\partial z^j}\eta^b. \tag{8}$$

The frame field $\{\delta_k := \frac{\delta}{\delta z^k}, \dot{\partial}_a := \frac{\partial}{\partial \eta^a}\}$ will be called the adapted frame of the *c.n.c* N. The conjugate frame is denoted by $\{\delta_{\overline{k}}, \dot{\partial}_{\overline{a}}\}$. The dual adapted bases are denoted by

$$\{dz^k\},\ \left\{\delta\eta^a = d\eta^a + N_k^a dz^k\right\},\ \left\{d\overline{z}^k\right\},\ \left\{\delta\overline{\eta}^a = d\overline{\eta}^a + N_{\overline{k}}^{\overline{a}} d\overline{z}^k\right\}.$$

We denote by $\mathcal{V}(E)$ the module of sections $V(E)$, called *vector field of* v-type, by $\mathcal{H}(E)$ the module of sections $H(E)$, called *vector field of* h- type. Then elements of their conjugates are called *vector fields of* $\overline{v}$-type, and $\overline{h}$-type, respectively.

2. Complex valued forms on E. Let us consider, as in [6], the set $F(E)$ of the complex valued differential forms on E given by the direct sum,

$$F(E) = \oplus F^{p,q,r,s}(E), \tag{9}$$

where $p, r = \overline{1,n},\ q, s = \overline{1,m}$ and $F^{p,q,r,s}(E)$ [or $F^{p,q,r,s}(U)$ for the open set U of E, or simply $F^{p,q,r,s}$ when there is no confusion danger] is the set of $(p+q+r+s)-$ forms which can be non zero only when these act on p vector fields of h-type, on q vector fields of v-type, on r vector fields of $\overline{h}$-type, and on s vector fields of $\overline{v}$-type. The elements of $F^{p,q,r,s}(U)$ are called $(p,q,r,s)-forms$ on U.

In the adapted dual bases we have the following local expression of $(p,q,r,s)-$forms ω,

$$\omega = \sum \omega_{IAJB} dz^I \wedge \delta\eta^A \wedge d\overline{z}^J \wedge \delta\overline{\eta}^B, \tag{10}$$

where $I = i_1...i_p,\ A = a_1...a_q,\ J = j_1...j_r,\ B = b_1...b_s$ and the sum is the indices $i_1 \le ... \le i_p,\ a_1 \le ... \le a_q,\ j_1 \le ... \le j_r, b_1 \le ... \le b_s$.

Now, let us consider f a complex valued differential function defined on E. In [1],[4] the following operators are considered:

$$d'^h f = \frac{\delta f}{\delta z^i} dz^i = \left(\frac{\partial f}{\partial z^i} - N_i^a \frac{\partial f}{\partial \eta^a} \right) dz^i; \; d'^v f = \frac{\partial f}{\partial \eta^a} \delta \eta^a$$
$$d''^h f = \frac{\delta f}{\delta \overline{z}^i} d\overline{z}^i = \left(\frac{\partial f}{\partial \overline{z}^i} - N_{\bar{i}}^{\bar{a}} \frac{\partial f}{\partial \overline{\eta}^a} \right) d\overline{z}^i; \; d''^v f = \frac{\partial f}{\partial \overline{\eta}^a} \delta \overline{\eta}^a$$

and they give a natural decomposition of the exterior differential df of f.

We shall generalize these operators to any differential form. Now, assume that E is a complex Finsler bundle with Finsler structure L see [1], definition 3.1, and we consider that N is canonical connection $\overset{c\ a}{N_k} = h^{\bar{c}a} \frac{\partial^2 L}{\partial z^k \partial \overline{\eta}^c}$ [1], prop. 3.1 and it is satisfy [1], prop 3.2

$$[\delta_j, \delta_k] = R^a_{jk} \dot{\partial}_a = 0. \tag{11}$$

Proposition 1. *If* (E, L) *is a complex Finsler vector bundle endowed with the canonical connection we have:*

$$dF^{p,q,r,s} \subset F^{p+1,q,r,s} \oplus F^{p,q+1,r,s} \oplus F^{p,q,r+1,s} \oplus F^{p,q,r,s+1}$$
$$\oplus F^{p+1,q-1,r+1,s} \oplus F^{p+1,q-1,r,s+1} \oplus F^{p+1,q,r+1,s-1} \oplus F^{p,q+1,r+1,s-1}.$$

From the above decomposition it follows that we can define eight morphisms of complex vector spaces if we consider the different components

$$d'^h : F^{p,q,r,s} \longrightarrow F^{p+1,q,r,s}; \; d'^v : F^{p,q,r,s} \longrightarrow F^{p,q+1,r,s}$$
$$d''^h : F^{p,q,r,s} \longrightarrow F^{p,q,r+1,s}; \; d''^v : F^{p,q,r,s} \longrightarrow F^{p,q,r,s+1}$$
$$\partial_1 : F^{p,q,r,s} \longrightarrow F^{p+1,q-1,r+1,s}; \; \partial_2 : F^{p,q,r,s} \longrightarrow F^{p+1,q-1,r,s+1}$$
$$\partial_3 : F^{p,q,r,s} \longrightarrow F^{p+1,q,r+1,s-1}; \; \partial_4 : F^{p,q,r,s} \longrightarrow F^{p,q+1,r+1.s-1}.$$

We remark that these operators and the classical d', d'' that appear in the decomposition $d = d' + d''$ of the differential on a complex bundle, are related by the following relations:

$$d' = d'^h + d'^v + \partial_3 + \partial_4; \; d'' = d''^h + d''^v + \partial_1 + \partial_2. \tag{12}$$

Moreover, by equalizing the terms of the same type in the relation

$$\left(d'' \right)^2 = (d''^h + d''^v + \partial_1 + \partial_2)^2 = 0$$

we obtain:

$$\begin{aligned} (d^{''h})^2 &= (d^{''v})^2 = (\partial_1)^2 = (\partial_2)^2 = 0 \\ d^{''v}d^{''h} + d^{''h}d^{''v} &= d^{''h}\partial_1 + \partial_1 d^{''h} = d^{''v}\partial_2 + \partial_2 d^{''v} = 0 \\ \partial_1\partial_2 + \partial_2\partial_1 &= d^{''v}\partial_1 + \partial_1 d^{''v} + d^{''h}\partial_2 + \partial_2 d^{''h} = 0. \end{aligned}$$

By the same argument we have:

$$d^{''v}(\omega \wedge \theta) = d^{''v}\omega \wedge \theta + (-1)^{deg\omega}\omega \wedge d^{''v}\theta, \tag{13}$$

for any $\omega \in F^{p,q,r,s}$, $\theta \in F^{p^{'},q^{'},r^{'},s^{'}}$ and similar equalities for the other operators defined above.

From (13) and from the linearity of $d^{''v}$ we deduce that if $\omega \in F^{p,q,r,s}$, is given by (10) then

$$d^{''v}\omega = \sum \frac{\partial \omega_{IAJB}}{\partial \overline{\eta}^a} \delta\overline{\eta}^a \wedge dz^I \wedge \delta\eta^A \wedge d\overline{z}^J \wedge \delta\overline{\eta}.^B \tag{14}$$

We know that the local bases $\{\frac{\partial}{\partial \overline{\eta}^a}\}$; $\{\frac{\partial}{\partial \overline{\eta}^{'a}}\}$ of $\overline{\mathcal{V}(E)}$, are related by

$$\frac{\partial}{\partial \overline{\eta}^a} = \overline{M}_a^{b^{'}} \frac{\partial}{\partial \overline{\eta}^{'b}}. \tag{15}$$

The formulas (15) prove that if f is a complex differentiable function defined on E then the condition

$$\frac{\partial f}{\partial \overline{\eta}^a} = 0 \ ; \ a = 1, 2, ..., m \tag{16}$$

are independent with respect to this change. Moreover, the form $\omega \in F^{p,q,r,0}$ is $d^{''v}-$closed (i.e. $d^{''v}\omega = 0$) if only if its local components satisfy the conditions (16). We denote by $\Phi^{p,q,r}$ the sheaf of germs of these forms.

Another property of the operator $d^{''v}$ is a Grothendieck-Dolbeault type lemma, namely:

Theorem 1. *Let ω be a $d^{''v}$-closed (p, q, r, s)-form defined on a neighborhood U on E and $s \geq 1$. Then there exists a $(p, q, r, s-1)$-form θ defined on some neighborhood $U^{'} \subset U$ and such that $d^{''v}\theta = \omega$ on $U^{'}$.*

Proof. Using a similarly argument from [7], [8], the proof is similar with that from particular case when $E = T^{'}M$ from [6]. □

Let $\mathcal{F}^{p,q,r,s}$ be the sheaf of germs of $(p, q, r, s)-$forms and we denote by $i : \Phi^{p,q,r} \rightarrow \mathcal{F}^{p,q,r,0}$ the natural inclusion. The sheaves $\mathcal{F}^{p,q,r,s}$ are fine and taking into account the Theorem 1 it follows that the sequence of sheaves

$$0 \longrightarrow \Phi^{p,q,r} \xrightarrow{i} \mathcal{F}^{p,q,r,0} \xrightarrow{d^{''v}} \mathcal{F}^{p,q,r,1} \xrightarrow{d^{''v}} ... \xrightarrow{d^{''v}} \mathcal{F}^{p,q,r,s} \xrightarrow{d^{''v}} \mathcal{F}^{p,q,r,s+1} \xrightarrow{d^{''v}} ...$$

is a fine resolution of $\Phi^{p,q,r}$ and we denote by $H^s(E, \Phi^{p,q,r})$ the cohomology groups of E with coefficients in the sheaf $\Phi^{p,q,r}$, called *v-cohomology groups of E*. Then, we obtain a de Rham type theorem:

Theorem 2. *The v-cohomology groups of the complex Finsler bundle* (E, L) *are given by*

$$H^s(E, \Phi^{p,q,r}) \approx Z^{p,q,r,s}/d^{''v}F^{p,q,r,s-1}(E), \tag{17}$$

where $Z^{p,q,r,s}$ *is the space of* $d^{''v}-$*closed* $(p, q, r, s)-$*forms globally defined on* E.

Obtaining h-cohomology groups of complex Finsler vector bundle is not possible like above, because the horizontal distribution $H(E)$ is not integrable. In the sequel, we will present a particular situation when this fact is possible.

Consider $E = T^{'}M$ the holomorphic tangent bundle of M endowed with c.n.c Chern-Finsler. According to [6] if $\omega = \omega_k \delta\overline{\eta}^k \in F^{0,0,0,1}(T^{'}M)$ we obtain that:

$$d^{''h}\omega = [\frac{\delta\omega_k}{\delta\overline{z}^h} + \overline{T^i_{kh}}]d\overline{z}^h \wedge \delta\overline{\eta}^k, \tag{18}$$

where $T^i_{jk} = \dot{\partial}_j\ (N^i_k) - \dot{\partial}_k\ (N^i_j)$, and N is Chern-Finsler connection.

By Definition 4.2.1 in [5], the complex Finsler manifold (M, F) is said strongly Kahler if $T^i_{jk} = 0$. By induction, it results the following proposition.

Proposition 2. *If* M *is a complex Finsler manifold strongly Kahler and* ω *is locally given by* (10) *then:*

$$d^{''h}\omega = \delta_{\overline{k}}(\omega_{IJHK})d\overline{z}^k \wedge dz^I \wedge \delta\eta^J \wedge d\overline{z}^H \wedge \delta\overline{\eta}^K. \tag{19}$$

At changes of local complex coordinates $z^i, \eta^i \longrightarrow z'^i, \eta'^i$ on $T'M$ we have the following law transformation on $\overline{H(T'M)}$:

$$\frac{\delta}{\delta \overline{z}^i} = \frac{\partial \overline{z}'^j}{\partial \overline{z}^i} \frac{\delta}{\delta \overline{z}'^j}. \tag{20}$$

The formulas (20) prove that if f is a complex valued differentiable function defined on $T'M$ then, the condition

$$\frac{\delta f}{\delta \overline{z}^i} = 0\,,\; i = 1, ..., n \tag{21}$$

is independent with respect to this change.

Moreover, the form $\omega \in F^{p,q,0,s}(T'M)$ is d''^h−closed (i.e. $d''^h\omega = 0$) if and only if its local components satisfy the conditions (21). We denote by $\Phi^{p,q,s}$ the sheaf of germs of these forms.

Similarly with theorem 1, it results:

Theorem 3. *Let ω be a d''^h−closed (p, q, r, s)−forms defined on a neighborhood U on $T'M$ and $r \geq 1$. Then, there exists a $(p, q, r-1, s)$−form θ defined on some neighborhood $U' \subset U$ and such that $d''^h\theta = \omega$ on U'.*

Proof. One proceeds as in the proof of Theorem 1 from [6], but the induction process is made after a k index, defined by the condition that the form ω, locally defined by (10), does not contain the differentials $\delta\overline{\eta}^{k+1}, ..., \delta\overline{\eta}^n$. □

Let $\mathcal{F}^{p,q,r,s}$ be the sheaf of germs of (p, q, r, s)−forms and we denote by $i : \Phi^{p,q,s} \longrightarrow \mathcal{F}^{p,q,0,s}$ the natural inclusion. The sheaves $\mathcal{F}^{p,q,r,s}$ are fine and taking into account the Theorem 3 it follows that the sequence of sheaves

$$0 \longrightarrow \Phi^{p,q,s} \xrightarrow{i} \mathcal{F}^{p,q,0,s} \xrightarrow{d''^h} \mathcal{F}^{p,q,1,s} \xrightarrow{d''^h} ... \xrightarrow{d''^h} \mathcal{F}^{p,q,r,s} \xrightarrow{d''^h} \mathcal{F}^{p,q,r+1,s} \xrightarrow{d''^h} ...$$

is a fine resolution of $\Phi^{p,q,s}$ and we denote by $H^r(M, \Phi^{p,q,s})$ the cohomology groups of M with coefficients in the sheaf $\Phi^{p,q,s}$, called *h-cohomology groups of M*. Then we obtain a de Rham type theorem:

Theorem 4. *The h-cohomology groups of the complex Finsler strongly Kahler manifold (M, L) are given by*

$$H^r(M, \Phi^{p,q,s}) \approx Z^{p,q,r,s}/d''^h F^{p,q,r-1,s}(T'M), \tag{22}$$

where $Z^{p,q,r,s}$ is the space of d''^h−closed (p, q, r, s)−forms globally defined on $T'M$.

3. V-cohomology groups invariance at gauge complex transformations. According to [5], a gauge complex transformation on complex vector bundle E is a pair $\Psi = (F_0, F_1)$, where locally $F_0 : M \longrightarrow M$ and $F_1 : E \longrightarrow E$ is an F_0-holomorphic isomorphism which satisfies

$$\pi \circ F_1 = F_0 \circ \pi. \tag{23}$$

The local character concerns the open sets of local charts. Let us consider that $\Psi = (F_0, F_1)$ applies the point $u = (z^i, \eta^a) \in \pi^{-1}(U_\alpha)$ in the point $\widetilde{u} = (\widetilde{z}^i, \widetilde{\eta}^a) \in \pi^{-1}(U_\beta)$, where $(U_\alpha, (z^i, \eta^a))_{i=\overline{1,n};a=\overline{1,m}}$ and $(U_\beta, (\widetilde{z}^i, \widetilde{\eta}^a))_{i=\overline{1,n};a=\overline{1,m}}$ are the complex coordinates in two local charts on E.

Proposition 3. [5] *A gauge complex transformation $\Psi : u \longrightarrow \widetilde{u}$ is locally given by a system of analytic functions:*

$$\widetilde{z}^i = X^i(z) \ ; \widetilde{\eta}^a = Y^a(z, \eta). \tag{24}$$

with the regularity condition: $det(\frac{\partial X^i}{\partial z^j}) \cdot det(\frac{\partial Y^a}{\partial \eta^b}) \neq 0$.

We denote by $X^i_j := \frac{\partial X^i}{\partial z^j}$ and $Y^a_b := \frac{\partial Y^a}{\partial \eta^b}$ and by $X^{\overline{i}}_{\overline{j}}$, $Y^{\overline{a}}_{\overline{b}}$ their conjugates. Obviously, from the holomorphy requirements we have $X^i_{\overline{j}} = \frac{\partial X^i}{\partial \overline{z}^j} = 0$ and $Y^a_{\overline{j}} = \frac{\partial Y^a}{\partial \overline{z}^j} = 0$, $Y^a_{\overline{b}} = \frac{\partial Y^a}{\partial \overline{\eta}^b} = 0$.

Defintion 1. [5] *A c.n.c is said to be gauge, shortly g.c.n.c., if it transforms by Ψ_* the adapted frames after rules*

$$\delta_j = X^i_j \delta_{\widetilde{i}} ; \dot{\partial}_b = Y^a_b \, \dot{\partial}_{\widetilde{a}}, \tag{25}$$

where $\delta_{\widetilde{i}} = \frac{\delta}{\delta \widetilde{z}^i}$ and $\dot{\partial}_{\widetilde{b}} = \frac{\partial}{\partial \widetilde{\eta}^b}$.

Let $L(z, \eta)$ be a gauge invariant Lagrangian on E, i.e. $L(z, \eta) = L(\widetilde{z}, \widetilde{\eta})$. Then we have:

Proposition 4. [5] *The canonical c.n.c $\overset{c}{N}{}^a_k$ from (11) is g.c.n.c*

Proposition 5. *At local maps changes on E, we have:*

$$X'^i_h = \frac{\partial \widetilde{z}'^i}{\partial \widetilde{z}^j} X^j_h \ , \ Y'^a_c = M^a_b(\widetilde{z}) Y^b_c. \tag{26}$$

From section 2, we have the following superior semiexact sequences:

$$F(U_\alpha) : 0 \longrightarrow \Phi^{p,q,r}(U_\alpha) \xrightarrow{i} \mathcal{F}^{p,q,r,0}(U_\alpha) \xrightarrow{d''^v} ... \xrightarrow{d''^v} \mathcal{F}^{p,q,r,s}(U_\alpha) \xrightarrow{d''^v} ...$$

$$F(U_\beta) : 0 \longrightarrow \Phi^{p,q,r}(U_\beta) \xrightarrow{i} \mathcal{F}^{p,q,r,0}(U_\beta) \xrightarrow{d''^v} ... \xrightarrow{d''^v} \mathcal{F}^{p,q,r,s}(U_\beta) \xrightarrow{d''^v} ...$$

Using an argument from to [3] prop. 8.2, pag. 272 one obtain:

Proposition 6. *The gauge diffeomorphism* $\Psi : \pi^{-1}(U_\alpha) \longrightarrow \pi^{-1}(U_\beta)$ *defined by* (24)*, induces a superior semiexact sequence morphism defined by:* $\widetilde{\Psi} = (\widetilde{\Psi}^{p,q,r})^s_{s\geq 1} : F(U_\beta) \longrightarrow F(U_\alpha)$ *where* $(\widetilde{\Psi}^{p,q,r})^s : \mathcal{F}^{p,q,r,s}(U_\beta) \longrightarrow \mathcal{F}^{p,q,r,s}(U_\alpha)$ *is localy given by:*

$$((\widetilde{\Psi}^{p,q,r})^s \omega_\beta)(\delta_I, \dot{\partial}_A, \overline{\delta}_J, \overline{\dot{\partial}}_B) = \omega_\beta(X^H_I \delta_{\widetilde{H}}, Y^C_A \, \dot{\partial}_{\widetilde{C}}, \overline{X}^K_J \overline{\delta}_{\widetilde{K}}, Y^D_B \overline{\dot{\partial}}_{\widetilde{D}}), \tag{27}$$

which is globaly defined, according to (25)*.*

Proof. For proving this proposition, is sufficient to show that the next diagram is commutative:

$$\begin{array}{ccccccc}
... \xrightarrow{d''^v} & \mathcal{F}^{p,q,r,s}(U_\beta) & \xrightarrow{d''^v} & \mathcal{F}^{p,q,r,s+1}(U_\beta) & \xrightarrow{d''^v} ... \\
 & (\widetilde{\Psi}^{p,q,r})^s \downarrow & & \downarrow (\widetilde{\Psi}^{p,q,r})^{s+1} & \\
... \xrightarrow{d''^v} & \mathcal{F}^{p,q,r,s}(U_\alpha) & \xrightarrow{d''^v} & \mathcal{F}^{p,q,r,s+1}(U_\alpha) & \xrightarrow{d''^v} ...
\end{array}$$

(i.e)

$$(\widetilde{\Psi}^{p,q,r})^{s+1}(d''^v \omega_\beta) = d''^v (\widetilde{\Psi}^{p,q,r})^s \omega_\beta \tag{28}$$

for every $\omega_\beta \in \mathcal{F}^{p,q,r,s}(U_\beta)$.

A straightforward calculus in the local coordinates, in both members of (28), using (27), (25), (14) proves that

$$((\widetilde{\Psi}^{p,q,r})^{s+1} d''^v \omega_\beta)_{IAJB} = (d''^v (\widetilde{\Psi}^{p,q,r})^s \omega_\beta)_{IAJB}. \tag{29}$$

□

Now we take into account the v-cohomology groups $H^s(U_\beta, \Phi^{p,q,r})$ and $H^s(U_\alpha, \Phi^{p,q,r})$ of complex Finsler vector bundle (E, L) in local charts U_β respectively U_α.

The morphism $(\widetilde{\Psi}^{p,q,r})^s : \mathcal{F}^{p,q,r,s}(U_\beta) \longrightarrow \mathcal{F}^{p,q,r,s}(U_\alpha)$ induces canonically a groups homomorphism $((\widetilde{\Psi}^{p,q,r})^s)^* : H^s(U_\beta, \Phi^{p,q,r}) \longrightarrow H^s(U_\alpha, \Phi^{p,q,r})$.

Indeed accordingly to above proposition it results: $(\widetilde{\Psi}^{p,q,r})^s : Z^{p,q,r,s}(U_\beta) \longrightarrow Z^{p,q,r,s}(U_\alpha)$ and $(\widetilde{\Psi}^{p,q,r})^s : B^{p,q,r,s}(U_\beta) \to B^{p,q,r,s}(U_\alpha)$, where $B^{p,q,r,s}(U) = \{\omega \in F^{p,q,r,s}(U)/\theta \in F^{p,q,r,s-1}(U)\,,\, \omega = d''^v\theta\}$.

We define

$$((\widetilde{\Psi}^{p,q,r})^s)^*([\omega]) = [(\widetilde{\Psi}^{p,q,r})^s(\omega)] \tag{30}$$

which is well defined because for $\omega'\,\tilde{}\,\omega$ we have $\omega - \omega = d''^v\theta \Rightarrow (\widetilde{\Psi}^{p,q,r})^s(\omega) - (\widetilde{\Psi}^{p,q,r})^s(\omega) = (\widetilde{\Psi}^{p,q,r})^s(d''^v\theta) = d''^v(\widetilde{\Psi}^{p,q,r})^{s-1}\theta$ the last equality being true because of (2 9). Thus, we obtain that: $(\widetilde{\Psi}^{p,q,r})^s(\omega)\,\tilde{}\,(\widetilde{\Psi}^{p,q,r})^s(\omega)$. As Ψ is diffeomorphism it results:

Theorem 5.

$$H^s(U_\beta, \Phi^{p,q,r}) \approx H^s(U_\alpha, \Phi^{p,q,r}) \quad by \quad ((\widetilde{\Psi}^{p,q,r})^s)^*). \tag{31}$$

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