ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" DIN IAŞI (S.N.)
MATEMATICĂ, Tomul LIII, 2007, Supliment

FRACTIONAL DYNAMICAL SYSTEMS ON FRACTIONAL LEIBNIZ ALGEBROIDS

BY

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Dedicated to Academician Radu Miron at his 80th anniversary

Abstract. The main purpose of this paper is to describe fractional Leibniz structures on algebroids.

Mathematics Subject Classification 2000: 37J99, 17B60, 53C15.
Key words: Fractional tangent bundle, fractional Leibniz algebroid.

1. Introduction. Derivatives of fractional order have found many applications in recent studies in mechanics, physics, economics, medicine. Classes of fractional differentiable systems have studied in [10], [4], [7].

The fractional tangent bundle is defined, using the method of Radu Miron's from [8]. In this paper the fractional dynamical systems on fractional Leibniz algebroids are presented. The associated geometrical objects have an geometric character. Also, some illustrative examples are given.

2. Fractional tangent bundle on a manifold. Let $f : [a, b] \to \mathbf{R}$ and $\alpha \in \mathbf{R}, \alpha > 0$. The *Riemann-Liouville fractional derivative at to left of* a , respectively *at to right of* b is the function $f \to_a D_t^\alpha f$ resp. $f \to_t D_b^\alpha f$, where:

$$(1)\quad \begin{cases} {}_aD_t^\alpha f(t) &= \dfrac{1}{\Gamma(m-\alpha)}\left(\dfrac{d}{dt}\right)^m \displaystyle\int_a^t (t-s)^{m-\alpha-1}(f(s)-f(a))ds \\ {}_tD_b^\alpha f(t) &= \dfrac{1}{\Gamma(m-\alpha)}(-\dfrac{d}{dt})^m \displaystyle\int_t^b (t-s)^{m-\alpha-1}(f(s)-f(b))ds, \end{cases}$$

where $m \in \mathbf{N}^*$ such that $m-1 \leq \alpha \leq m, \Gamma$ is the Euler gamma function and $(\frac{d}{dt})^m = \frac{d}{dt} \circ \frac{d}{dt} \circ ... \circ \frac{d}{dt}$.

We will denote $D_t^\alpha =_a D_t^\alpha$. Clearly, if $\alpha \to 1$ then $D_t^\alpha f(t) = \frac{df}{dt}$.

Proposition 2.1 ([3]) (i) *If* $\lim_{n\to\infty} \alpha_n = p \in \mathbf{N}^*$, *then:*

$$\lim_{n\to\infty} ({}_aD_t^{\alpha_n} f(t)) = D_t^p f(t), \qquad \lim_{n\to\infty} ({}_tD_b^{\alpha_n} f(t)) = D_t^p f(t). \tag{2}$$

(ii) *If* $f(t) = c, (\forall) t \in [a,b]$, $D_t^\alpha f(t) = 0$.

(iii) *If* $f_1(t) = t^\gamma, (\forall) t \in [a,b]$, *then* $D_t^\alpha f_1(t) = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} t^{\gamma-\alpha}$.

(iv) *If* f_1, f_2 *are analytical functions on* (a,b), *then:*

$$D_t^\alpha (f_1 f_2)(t) = \sum_{k=0}^{\infty} \left(\begin{array}{c} \alpha \\ k \end{array} \right) D_t^{\alpha-k} f_1(t) (\frac{d}{dt})^k f_2(t). \tag{3}$$

(v) *If* $f : [a,b] \to \boldsymbol{R}$ *is an analytical function on* (a,b) *and* $0 \in (a,b)$ *then:*

$$f(t) = \sum_{h=0}^{\infty} E_{\alpha,h}(t) D_t^{\alpha h} f(t)|_{t=0}, \text{ where } E_{\alpha,h}(t) = \sum_{h=0}^{\infty} \frac{1}{\Gamma(1+\alpha h)} t^{\alpha h} \tag{4}$$

□

Let $\alpha \in \mathbf{R}, \alpha > 0$ and M be a manifold and U a local chart. We say that the curves $c_1, c_2 : I \to M, 0 \in I, c_1(0) = c_2(0) \in M$ have *fractional contact* α *in* x_0, if for all $f \in C^\infty(U), x_0 \in U$, the following relation holds:

$$D_t^\alpha (f \circ c_1)|_{t=0} = D_t^\alpha (f \circ c_2)|_{t=0}. \tag{5}$$

The set of equivalences classes $([c]_{x_0}^\alpha)$ is called the *fractional tangent space in* x_0 and it is denoted by $T_{x_0}^\alpha(U)$. Let $T^\alpha(M) = \bigcup_{x_0 \in M} T_{x_0}^\alpha(U)$ and the projection $\pi^\alpha : T^\alpha(M) \to M$ given by $\pi^\alpha([c]_{x_0}^\alpha) = x_0$.

On $T^\alpha(M)$ there exists a differentiable structure and we can prove that $(T^\alpha(M), \pi^\alpha, M)$ is a differentiable bundle.

In a system of local coordinates on M, if $x_0 \in U$ and $c : I \to M$ is a curve given by $x^i = x^i(t), (\forall) t \in I$, the class $([c]_{x_0}^\alpha)$ is given by:

$$x^i(t) = x^i(0) + \frac{1}{\Gamma(1+\alpha)} t^\alpha D_t^\alpha x^i(t)|_{t=0}, \quad t \in (-\varepsilon, \varepsilon). \tag{6}$$

On $(\pi^\alpha)^{-1}(U) \in T^\alpha(M)$, the coordinates of $([c]^\alpha_{x_0})$ are $(x^i, y^{i(\alpha)})$, where:

$$x^i = x^i(0), \qquad y^{i(\alpha)} = \frac{1}{\Gamma(1+\alpha)} D^\alpha_t x^i(t), i = \overline{1,n}. \tag{7}$$

Proposition 2.2 ([1], [2]). *Let $U, \overline{U}$ be two local charts on M such that $U \cap \overline{U} \neq \emptyset$ and the coordinate transformations:*

$$\overline{x}^i = \overline{x}^i(x^1, x^2, ..., x^n), \quad det\left(\frac{\partial \overline{x}^i}{\partial x^j}\right) \neq 0, \quad i = \overline{1,n} \tag{8}$$

The coordinate transformations on $(\pi^\alpha)^{-1}(U \cap \overline{U})$ are given by:

$$\overline{x}^i = \overline{x}^i(x^1, ..., x^n), \ \overline{y}^{i(\alpha)} = \frac{1}{\Gamma(1+\alpha)} D^\alpha_{\overline{x}^j}(x^i)^\alpha y^{j(\alpha)}, \tag{9}$$

and $D^\alpha_{x^i}$ is defined by:

$$D^\alpha_{x^i} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^{x^i} \frac{\partial f(x^1, ..., x^{i-1}, s, x^{i+1}, ..., x^n)}{\partial x^i} \frac{1}{(x^i - s)^\alpha} ds. \tag{10}$$

Let $\mathcal{D}^\alpha(U)$ the module of 1-forms on U. Using the fractional exterior derivative $d^\alpha : C^\infty(U) \to \mathcal{D}^\alpha(U), \ f \to d^\alpha(f)$ (see [2]), where $d^\alpha(f)$ is:

$$d^\alpha(f) = d(x^i)^\alpha D^\alpha_{x^i}(f) \tag{11}$$

it follows:

$$d(x^i)^\alpha = \overset{\alpha}{J}{}^i_j(x, \overline{x}) d(\overline{x}^j)^\alpha, \ D^\alpha_{x^i} = \overset{\alpha}{J}{}^j_i(\overline{x}, x) D^\alpha_{\overline{x}^j}, \ \overset{\alpha}{J}{}^i_k(x, \overline{x}) \overset{\alpha}{J}{}^k_j(\overline{x}, x) = \delta^i_j. \tag{12}$$

We denote by $\mathcal{X}^\alpha(U)$ the module of fractional vector fields generated by $\{D^\alpha_{x^i}, \ i = \overline{1,n}\}$. A fractional vector field $\overset{\alpha}{X} \in \mathcal{X}^\alpha(U)$ has the form:

$$\overset{\alpha}{X} = \overset{\alpha}{X}{}^i D^\alpha_{x^i}, \qquad \overset{\alpha}{X}{}^i \in C^\infty(U), i = \overline{1,n}, \tag{13}$$

which for a change of local charts, the correspondent components satisfies:

$$\overset{\alpha}{\overline{X}}{}^i = \overset{\alpha}{J}{}^i_j(x, \overline{x}) \overset{\alpha}{X}{}^j, \quad i, j = \overline{1,n}. \tag{14}$$

The fractional differentiable equations associated to $\overset{\alpha}{X}$ is:

$$D^\alpha_t x^i(t) = \overset{\alpha}{X}{}^i(x(t)), \quad \text{or} \quad \Gamma(1+\alpha) y^{i(\alpha)}(t) = \overset{\alpha}{X}{}^i(x(t)). \tag{15}$$

The equations (15) with initial conditions have solutions, see [3]. Examples of fractional differentiable equations on $\mathbf{R}$ can be find in [4].

3. Fractional Leibniz dynamical systems. Let the module $\mathcal{X}^\alpha(U)$ of fractional vector fields generated by the operators $\{D^\alpha_{x^i},\ i=\overline{1,n}\}$ and the module $\mathcal{D}^\alpha(U)$ generated by the 1-forms $\{d(x^i)^\alpha,\ i=\overline{1,n}\}$. Applying the Proposition 2.2 it follows:

$$(d(x^i)^\alpha)(D^\alpha_{x^j}) = D^\alpha_{x^j}(x^i)^\alpha = \Gamma(1+\alpha)\delta^i_j. \tag{16}$$

If $\overset{\alpha}{X} \in \mathcal{X}^\alpha(U)$ and $\overset{\alpha}{\omega} \in \mathcal{D}^\alpha(U)$ such that $\overset{\alpha}{\omega} = \overset{\alpha}{\omega}_i d(x^i)^\alpha$, then $\overset{\alpha}{\omega}(\overset{\alpha}{X}) = \Gamma(1+\alpha)\overset{\alpha}{X}{}^i\overset{\alpha}{\omega}_i$.

Let be a fractional 2− tensor field $\overset{\alpha}{B} \in \mathcal{X}^\alpha(U)\times\mathcal{X}^\alpha(U)$ and $d^\alpha f, d^\alpha g \in \mathcal{D}^\alpha(U)$. The bilinear map $[\cdot,\cdot]^\alpha : C^\infty(M)\times C^\infty(M) \to C^\infty(M)$ defined by:

$$[f,h]^\alpha = \overset{\alpha}{B}(d^\alpha f, d^\alpha h), \quad (\forall) f,h \in C^\infty(M), \tag{17}$$

is called the *fractional Leibniz bracket.*

If $\overset{\alpha}{B} = \overset{\alpha}{B}{}^{ij} D^\alpha_{x^i}\otimes D^\alpha_{x^j}$, from (17) it follows:

$$[f,h]^\alpha = \overset{\alpha}{B}{}^{ij} \cdot D^\alpha_{x^i} f \cdot D^\alpha_{x^j} h. \tag{18}$$

Since

$$D^\alpha_{x^i}(fh)(x) = \sum_{k=0}^{\infty} \begin{pmatrix} \alpha \\ k \end{pmatrix} (D^{\alpha-k}_{x^i} f(x)) \left(\frac{\partial}{\partial x^i}\right)^k h(x), \tag{19}$$

it follows

$$[fh,g]^\alpha = \sum_{k=0}^{\infty} \begin{pmatrix} \alpha \\ k \end{pmatrix} \cdot \overset{\alpha}{B}{}^{ij} (D^{\alpha-k}_{x^i} f) \cdot \left(\frac{\partial}{\partial x^i}\right)^k h \cdot D^\alpha_{x^j} g, \tag{20}$$

$$[f,gh]^\alpha = \sum_{k=0}^{\infty} \begin{pmatrix} \alpha \\ k \end{pmatrix} \cdot \overset{\alpha}{B}{}^{ij} (D^\alpha_{x^i} f) \cdot D^{\alpha-k}_{x^j}(g) \cdot \left(\frac{\partial}{\partial x^i}\right)^k h. \tag{21}$$

The pair $(M,[\cdot,\cdot]^\alpha)$ is called *fractional Leibniz manifold.* If the bracket $[\cdot,\cdot]^\alpha$ is skew-symmetric,we say that $(M,[\cdot,\cdot]^\alpha)$ is a *fractional almost Poisson manifold.* If $\alpha \to 1$, then one obtain the concept from [6], [9].

For $h \in C^\infty(M)$, the fractional vector field $\overset{\alpha}{X}_h$ defined by

$$\overset{\alpha}{X}_h(f) = [f,h]^\alpha,\ (\forall) f \in C^\infty(M), \tag{22}$$

is called the *fractional Leibniz vector field associated to* h and its associated dynamical system is the *fractional Leibniz dynamical system.*

Locally, the fractional Leibniz dynamical system is given by

$$D_t^\alpha x^i(t) = [x^i(t), h(t)]^\alpha, \quad \text{where} \quad [x^i,h]^\alpha = \overset{\alpha}{B}{}^{ij} \cdot D^\alpha_{x^j} h. \tag{23}$$

Example 3.1. Let the constant $2-$ tensor $\overset{\alpha}{g} = (\overset{\alpha}{g}{}^{ij})$ defined on $\mathbf{R}^3$ by

$$\overset{\alpha}{g} = \begin{pmatrix} s_1\gamma_1 & 0 & 0 \\ 0 & s_2\gamma_2 & 0 \\ 0 & 0 & s_3\gamma_3 \end{pmatrix},\ s_1,s_2,s_3 \in \{-1,1\}, \gamma_1+\gamma_2+\gamma_3 = 0. \tag{24}$$

For $h = x^1x^2x^3$, the associated fractional Leibniz dynamical system is

$$\left\{ \begin{array}{rcccl} D_t^\alpha x^1 &=& s_1\gamma_1 D^\alpha_{x^1}(h) &=& \dfrac{\Gamma(2)}{\Gamma(2-\alpha)} s_1\gamma_1 x^2x^3(x^1)^{1-\alpha} \\ D_t^\alpha x^2 &=& s_2\gamma_2 D^\alpha_{x^2}(h) &=& \dfrac{\Gamma(2)}{\Gamma(2-\alpha)} s_2\gamma_2 x^1x^3(x^2)^{1-\alpha} \\ D_t^\alpha x^3 &=& s_3\gamma_3 D^\alpha_{x^3}(h) &=& \dfrac{\Gamma(2)}{\Gamma(2-\alpha)} s_3\gamma_3 x^1x^2(x^3)^{1-\alpha}. \end{array} \right. \tag{25}$$

If $\alpha \to 1$, it follows the system (7) in [10].

For $\overset{\alpha}{h} = (x^1)^\alpha (x^2)^\alpha (x^3)^\alpha$ and $u = \Gamma(1+\alpha)$ the fractional system is

$D_t^\alpha x^1 = us_1\gamma_1 x^2x^3$, $D_t^\alpha x^2 = us_2\gamma_2 x^1x^3$, $D_t^\alpha x^3 = us_3\gamma_3 x^1x^2$. □

Let $\overset{\alpha}{P}$ resp. $\overset{\alpha}{g}$ be a skew-symmetric resp. symmetric fractional $2-$ tensor field on M. We define the bracket $[\cdot,\cdot]^\alpha : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ by

$$[f,h]^\alpha = \overset{\alpha}{P}(d^\alpha f, d^\alpha h) + \overset{\alpha}{g}(d^\alpha f, d^\alpha h), \quad (\forall) f,h \in C^\infty(M). \tag{26}$$

The 4-tuple $(M, \overset{\alpha}{P}, \overset{\alpha}{g}, [\cdot,\cdot]^\alpha)$ is called *fractional almost metriplectic manifold.* The fractional dynamical system associated to $h \in C^\infty(M)$ is

$$D_t^\alpha x^i(t) = [x^i(t), h(t)]^\alpha, \quad \text{where} \quad [x^i,h]^\alpha = \overset{\alpha}{P}{}^{ij} D^\alpha_{x^j} h + \overset{\alpha}{g}{}^{ij} D^\alpha_{x^j} h. \tag{27}$$

If we define the bracket $[\cdot,(\cdot,\cdot)]:C^\infty(M)\times C^\infty(M)\times C^\infty(M)\to C^\infty(M)$ by:

$$[f,h]^\alpha=\overset{\alpha}{P}(d^\alpha f,d^\alpha h_1)+\overset{\alpha}{g}(d^\alpha f,d^\alpha h_2),\quad (\forall)f,h_1,h_2\in C^\infty(M). \tag{28}$$

The fractional vector field $\overset{\alpha}{X}_{h_1h_2}$ defined by

$$\overset{\alpha}{X}_{h_1h_2}=[f,(h_1,h_2)],\quad (\forall)f\in C^\infty(M). \tag{29}$$

is called the *fractional almost Leibniz vector field associated to* $h_1,h_2\in C^\infty(M)$ and its associated dynamical system is the *fractional almost Leibniz dynamical system.* Locally, this system is given by:

$$D_t^\alpha x^i(t)=\overset{\alpha}{P}{}^{ij}D^\alpha_{x^j}h_1+\overset{\alpha}{g}{}^{ij}D^\alpha_{x^j}h_2. \tag{30}$$

Example 3.2. Let be the fractional 2- tensors fields $\overset{\alpha}{P}=(\overset{\alpha}{P}{}^{ij}),\overset{\alpha}{g}=(\overset{\alpha}{g}{}^{ij})$ on $\mathbf{R}^3$ and the functions $h_1,h_2\in C^\infty(\mathbf{R}^3)$ given by:

$$\overset{\alpha}{P}=\begin{pmatrix}0&1&0\\-1&0&x^1\\0&-x^1&0\end{pmatrix},\quad \overset{\alpha}{g}=\begin{pmatrix}-(x^2)^2-(x^3)^2&0&0\\0&-(x^3)^2&x^2x^3\\0&x^2x^3&-(x^2)^2\end{pmatrix},$$

$$\overset{\alpha}{h}_1=\frac{1}{2}(x^2)^{1+\alpha}+\frac{1}{2}(x^3)^{1+\alpha},\quad \overset{\alpha}{h}_2=\frac{1}{2}(x^1)^{1+\alpha}+(x^3)^\alpha.$$

Since $D^\alpha_{x^1}\overset{\alpha}{h}_1=0$, $D^\alpha_{x^2}\overset{\alpha}{h}_1=\frac{1}{2}\Gamma(2+\alpha)x^2$, $D^\alpha_{x^3}\overset{\alpha}{h}_1=\frac{1}{2}\Gamma(2+\alpha)x^3$, $D^\alpha_{x^1}\overset{\alpha}{h}_2=\frac{1}{2}\Gamma(2+\alpha)x^1$, $D^\alpha_{x^2}\overset{\alpha}{h}_2=0$, $D^\alpha_{x^3}\overset{\alpha}{h}_2=\Gamma(1+\alpha)$, the system (31) becomes:

$$\begin{pmatrix}D_t^\alpha x^1\\D_t^\alpha x^2\\D_t^\alpha x^3\end{pmatrix}=\begin{pmatrix}0&1&0\\-1&0&x^1\\0&-x^1&0\end{pmatrix}\begin{pmatrix}0\\\frac{1}{2}\Gamma(2+\alpha)x^2\\\frac{1}{2}\Gamma(2+\alpha)x^3\end{pmatrix}$$
$$+\begin{pmatrix}-(x^2)^2-(x^3)^2&0&0\\0&-(x^3)^2&x^2x^3\\0&x^2x^3&-(x^2)^2\end{pmatrix}\begin{pmatrix}\frac{1}{2}\Gamma(2+\alpha)x^1\\0\\\Gamma(1+\alpha)\end{pmatrix}$$ or equivalently

$$(31)\qquad \begin{cases} D_t^\alpha x^1 = \dfrac{1}{2}\Gamma(1+\alpha)[x^2 - x^1(x^2)^2 - x^1(x^3)^2] \\ D_t^\alpha x^2 = \dfrac{1}{2}\Gamma(2+\alpha)x^1x^3 + \Gamma(1+\alpha)x^2x^3 \\ D_t^\alpha x^3 = -\dfrac{1}{2}\Gamma(2+\alpha)x^1x^2 - \Gamma(1+\alpha)(x^2)^2 \end{cases}.$$

The system (31) is called the *revised fractional Maxwell- Bloch equations.*

If $\alpha \to 1$, the system (31) reduces to the revised Maxwell-Bloch equations.

4. Fractional Leibniz algebroids. Let M be a smooth manifold of dimension n , let $\pi : E \to M$ be a vector bundle and $\pi^* : E^* \to M$ the dual vector bundle.

A *fractional Leibniz algebroid structure* on a vector bundle $\pi : E \to M$ is given by a bracket $[\cdot,\cdot]^\alpha$ on the space of sections $Sec(\pi)$ and two vector bundle morphisms $\overset{\alpha}{\rho}_1, \overset{\alpha}{\rho}_2 : E \to T^\alpha M$ (called the *left* resp. *right fractional anchor*) such that for all $\sigma_1, \sigma_2 \in Sec(\pi)$ and $f, g \in C^\infty(M)$ we have:

$$(32)\qquad \begin{cases} [e_a, e_b]^\alpha = C_{ab}^c e_c \\ [f\sigma_1, g\sigma_2]^\alpha = f\overset{\alpha}{\rho}_1(\sigma_1)(g)\sigma_2 - g\overset{\alpha}{\rho}_2(\sigma_2)(f)\sigma_1 + fg[\sigma_1,\sigma_2]^\alpha. \end{cases}$$

A vector bundle $\pi : E \to M$ endowed with a fractional Leibniz algebroid structure on E , is called *fractional Leibniz algebroid* over M and denoted by $(E, [\cdot,\cdot]^\alpha, \overset{\alpha}{\rho}_1, \overset{\alpha}{\rho}_2)$.

A fractional Leibniz algebroid with an antisymmetric bracket $[\cdot,\cdot]^\alpha$ (in this case we have $\overset{\alpha}{\rho}_1 = -\overset{\alpha}{\rho}_2$) is called *fractional pre-Lie algebroid.*

Locally, if $\overset{\alpha}{\rho}_1(e_a) = \overset{\alpha i}{\rho}_{1a} D^\alpha_{x^i}$, $\overset{\alpha}{\rho}_2(e_b) = \overset{\alpha i}{\rho}_{2b} D^\alpha_{x^i}$ then:

$$(33)\qquad [\sigma_1^a e_a, \sigma_2^b e_b]^\alpha = \sigma_1^a \overset{\alpha i}{\rho}_{1a}(D^\alpha_{x^i}\sigma_2^b)e_b - \sigma_2^a \overset{\alpha i}{\rho}_{2a}(D^\alpha_{x^i}\sigma_1^b)e_b + \sigma_1^a\sigma_2^b C_{ab}^c e_c.$$

Let us we establish a correspondence between the fractional Leibniz algebroid structures on the vector bundle $\pi : E \to M$ and the fractional 2-tensor fields on manifold E^* of the dual vector bundle $\pi^* : E^* \to M$.

For $\sigma \in Sec(\pi)$, we define the function $i_{E^*}\sigma$ on E^* by the relation:

$$(34)\qquad i_{E^*}\sigma(a) = < \sigma(\pi^*(a)), a >, \quad \text{for} \quad a \in E^*,$$

where $< \cdot, \cdot >$ is the canonical pairing between E and E^*. If $\sigma = \sigma^a e_a$ and $a \in E^*$ has the coordinates (x^i, ξ_a), then $i_{E^*}\sigma(a) = \sigma^a \xi_a$.

Let $\overset{\alpha}{\Lambda}$ be a fractional 2-tensor field on E^* and $[\cdot,\cdot]_{\overset{\alpha}{\Lambda}}$ defined by:

$$[f,g]_{\overset{\alpha\beta}{\Lambda}} = \overset{\alpha\beta}{\Lambda}(d^{\alpha\beta}f, d^{\alpha\beta}g), \ (\forall)\ f,g \in C^\infty(E^*), \tag{35}$$

where

$$d^{\alpha\beta}f = d(x^i)^\alpha D^\alpha_{x^i} f + d(\xi_a)^\beta D^\beta_{\xi_a} f = d^\alpha(f) + d^\beta(f). \tag{36}$$

In the basis $\{D^\alpha_{x^i}, D^\beta_{\xi_a}\}, i = \overline{1,n}, a = \overline{1,m}$ of the module $\mathcal{X}^{\alpha\beta}(\pi^{*-1}(U))$, the components $\Lambda^{\alpha\beta}$ are given by:

$$\Lambda^{\alpha\beta} = A_{ab} D^\beta_{\xi_a} \otimes D^\beta_{\xi_b} + A^i_{1a} D^\beta_{\xi_a} \otimes D^\alpha_{x^i} + A^i_{2a} D^\alpha_{x^i} \otimes D^\beta_{\xi_a}. \tag{37}$$

For a given fractional 2-tensor field $\overset{\alpha\beta}{\Lambda}$ on E^*, we say that $\overset{\alpha\beta}{\Lambda}$ is *linear*, if for each pair (μ_1, μ_2) of sections of π^* , the function $\overset{\alpha\beta}{\Lambda}(d(i_{E^*}\mu_1)^\beta, d(i_{E^*}\mu_2)^\beta)$ defined on E^* is linear with respect the coordinates ξ_a.

If $\mu_1 = \mu_1^a(x)e_a, \mu_2 = \mu_2^a(x)e_a$, then $d_{E^*}\mu_1 = \mu_1^a(x)\xi_a, d_{E^*}\mu_2 = \mu_2^a(x)\xi_a$ and $\overset{\alpha\beta}{\Lambda}(d(i_{E^*}\mu_1)^\beta, d(i_{E^*}\mu_2)^\beta) = A_{ab}(x,\xi)(\mu_1^c(x))^\alpha(\mu_2^a(x))^\alpha) D^\beta_{\xi_a}(\xi_c) D^\beta_{\xi_b}(\xi_a)^\beta$ $= \frac{1}{\Gamma(1+\alpha)^2}(\mu_1^a(x))^\beta(\mu_2^b(x))^\beta A_{ab}(x,\xi)$. Then $\overset{\alpha\beta}{\Lambda}$ is linear $\iff$ $A_{ab}(x,\xi) = C^c_{ab}(x)\xi_c$.

The fractional formulation of the Grabowski and Urbanski's Theorem from [5], is the following.

Theorem 4.1. *For every fractional Leibniz algebroid* $(E, [\cdot,\cdot]^\alpha, \overset{\alpha}{\rho}_1, \overset{\alpha}{\rho}_2)$, *there exists an unique fractional* 2*-tensor* $\overset{\alpha}{\Lambda}$ *on* E^* *such that for all* $\sigma, \sigma_1, \sigma_2 \in Sec(\pi)$ *and* $f \in C^\infty(M)$ *the following relations hold:*

$$\begin{cases} i_{E^*}[\sigma_1,\sigma_2] = [(i_{E^*}\sigma_1)^\beta, (i_{E^*}\sigma_2)^\beta]_{\overset{\alpha\beta}{\Lambda}} \\ \pi^*(\overset{\alpha}{\rho}_1(\sigma)(f)) = [(i_{E^*}\sigma)^\beta, \pi^* f]_{\overset{\alpha\beta}{\Lambda}}, \quad \pi^*(\overset{\alpha}{\rho}_2(\sigma)(f)) = [\pi^* f, (i_{E^*}\sigma)^\beta]_{\overset{\alpha\beta}{\Lambda}}. \end{cases} \tag{38}$$

Conversely, every fractional 2*-linear tensor field* $\overset{\alpha\beta}{\Lambda}$ *on* E^* *defines a fractional Leibniz algebroid on* E *if the relations* (38) *hold.* □

Let $(x^i), i = \overline{1,n}$ be local coordinates on $U \subseteq M$ and let $\{e_1, \ldots, e_m\}$ be a basis of local sections of $E|_U$. We denote by $\{e^1, \ldots, e^m\}$ the dual basis of local sections of $E^*|_U$ and (x^i, y^a) (resp., (x^i, ξ_a)) the corresponding coordinates on E (resp., E^*).

Using (38), every linear fractional 2-tensor field $\overset{\alpha\,\beta}{\Lambda}$ on E^* has the form:

$$\overset{\alpha\,\beta}{\Lambda} = C_{ab}^d \xi_d D_{\xi_a}^{\beta} \otimes D_{\xi_b}^{\beta} + \overset{\alpha i}{\rho}_{1a} D_{\xi_a}^{\beta} \otimes D_{x^i}^{\alpha} - \overset{\alpha i}{\rho}_{2a} D_{x^i}^{\alpha} \otimes D_{\xi_a}^{\beta}. \tag{39}$$

The correspondence between $\overset{\alpha\,\beta}{\Lambda}$ and a fractional Leibniz algebroid structure is given by the following relations:

$$[e_a, e_b]^{\alpha} = C_{ab}^d e_d \ , \ \overset{\alpha}{\rho}_1(e_a) = \overset{\alpha i}{\rho}_{1a} D_{x^i}^{\alpha} \ , \ \overset{\alpha}{\rho}_2(e_a) = \overset{\alpha i}{\rho}_{2a} D_{x^i}^{\alpha}. \tag{40}$$

We call a *fractional dynamical system on* $(E, [\cdot,\cdot]^{\alpha}, \overset{\alpha}{\rho}_1, \overset{\alpha}{\rho}_2)$, the fractional system associated to vector field $\overset{\alpha\,\beta}{X}_h$ with $h \in C^{\infty}(E^*)$ given by:

$$\overset{\alpha\,\beta}{X}_h(f) = \overset{\alpha\,\beta}{\Lambda}(d^{\alpha\beta} f, d^{\alpha\beta} h), \quad \text{for all} \quad f \in C^{\infty}(E^*). \tag{41}$$

In the local coordinates (x^i, ξ_a) on E^*, the dynamical system (41) reads:

$$D_t^{\alpha} \xi_a = [\xi_a, h]_{\overset{\alpha\,\beta}{\Lambda}}, \quad D_t^{\alpha} x^i = [x^i, h]_{\overset{\alpha\,\beta}{\Lambda}}, \quad \textit{where} \tag{42}$$

$$[\xi_a, h]_{\overset{\alpha\,\beta}{\Lambda}} = C_{ab}^d \xi_d D_{\xi_b}^{\beta} h + \overset{\alpha i}{\rho}_{1a} D_{x^i}^{\alpha} h, \quad [x^i, h]_{\overset{\alpha\,\beta}{\Lambda}} = -\overset{\alpha i}{\rho}_{2a} D_{\xi_a}^{\beta} h. \tag{43}$$

If $\alpha \to 1, \beta \to 1$, the dynamical system (42) was studied in [6].

If $\alpha \to 1$ the dynamical system (42) has the form:

$$\dot{x}^i = -\overset{\alpha i}{\rho}_{2a} D_{\xi_a}^{\beta} h, \quad D_t^{\beta} \xi_a = C_{ab}^d \xi_d D_{\xi_b}^{\beta} h + \overset{\alpha i}{\rho}_{1a} \frac{\partial h}{\partial x^i}. \tag{44}$$

If $\beta \to 1$ the dynamical system (42) has the form:

$$D_t^{\alpha} x^i = -\overset{\alpha i}{\rho}_{2a} \frac{\partial h}{\partial \xi_a}, \quad \dot{\xi}_a = C_{ab}^d \xi_d \frac{\partial h}{\partial \xi_b} + \overset{\alpha i}{\rho}_{1a} D_{x^i}^{\alpha} h. \tag{45}$$

If the fractional Leibniz algebroid is a fractional pre-Lie algebroid (that is, $C_{ab}^d = -C_{ba}^d$), then the fractional dynamical system (42) is given by:

$$D_t^{\beta} \xi_a = C_{ab}^d \xi_d D_{\xi_a}^{\beta} h + \overset{\alpha i}{\rho}_{1a} D_{x^i}^{\alpha} h, \quad D_t^{\alpha} x^i = -\overset{\alpha i}{\rho}_{1a} D_{\xi_a}^{\beta} h. \tag{46}$$

If the fractional Leibniz algebroid is a fractional symmetric algebroid (that is, $C_{ab}^d = C_{ba}^d$), then the fractional dynamical system (42) is given by:

$$D_t^\beta \xi_a = C_{ab}^d \xi_d D_{\xi_a}^\beta h + \overset{\alpha}{\rho}{}_{1a}^{i} D_{x^i}^\alpha h, \quad D_t^\alpha x^i = \overset{\alpha}{\rho}{}_{1a}^{i} D_{\xi_a}^\beta h. \tag{47}$$

Example 4.1. Let the dual $\pi^* : E^* = \mathbf{R}^3 \times (\mathbf{R}^3)^* \to \mathbf{R}^3$ of the vector bundle $\pi : E = \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}^3$ and $\alpha > 0, \beta > 0$. We consider on E^* the fractional 2-contravariant linear tensor field $\overset{\alpha}{\Lambda}$ defined by the matrix P^β, the fractional anchors $\overset{\alpha}{\rho}_1, \overset{\alpha}{\rho}_2$ and the function h given by:

$$P^\beta = \begin{pmatrix} 0 & -\xi_3 x^3 & \xi_2 x^2 \\ \xi_3 x^3 & 0 & -\xi_1 x^1 \\ -\xi_2 x^2 & \xi_1 x^1 & 0 \end{pmatrix}, \quad \overset{\alpha}{\rho}_1 = \begin{pmatrix} 0 & -x^3 & x^2 \\ x^3 & 0 & 0 \\ -x^2 & 0 & 0 \end{pmatrix},$$

$$\overset{\alpha}{\rho}_2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -x^1 \\ 0 & x^1 & 0 \end{pmatrix} \text{ and } h(x,\xi) = (x^2)^\alpha (\xi_2)^\beta + (x^3)^\alpha (\xi_3)^\beta.$$

Since $D_{\xi_a}^\beta (\xi_b)^\gamma = \delta_b^a \xi_a^{\gamma-\beta} \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\beta)}$, $D_{x^i}^\alpha (x^j)^\gamma = \delta_i^j (x^i)^{\gamma-\alpha} \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)}$ follows:

$$\begin{cases} D_{\xi_1}^\beta h = 0, & D_{\xi_2}^\beta h = \Gamma(1+\beta)(x^2)^\alpha, & D_{\xi_3}^\beta h = \Gamma(1+\beta)(x^3)^\alpha \\ D_{x^1}^\alpha h = 0, & D_{x^2}^\alpha h = \Gamma(1+\alpha)(\xi_2)^\beta, & D_{x^3}^\alpha h = \Gamma(1+\alpha)(x^3)^\beta. \end{cases}$$

The fractional system (42) has the matrix form:

$$\begin{pmatrix} D_t^\beta \xi_1 \\ D_t^\beta \xi_2 \\ D_t^\beta \xi_3 \end{pmatrix} = \Gamma(1+\beta) \begin{pmatrix} 0 & -\xi_3 x^3 & \xi_2 x^2 \\ \xi_3 x^3 & 0 & -\xi_1 x^1 \\ -\xi_2 x^2 & \xi_1 x^1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ (x^2)^\alpha \\ (x^3)^\alpha \end{pmatrix}$$

$$+ \Gamma(1+\alpha) \begin{pmatrix} 0 & -x^3 & x^2 \\ x^3 & 0 & 0 \\ -x^2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ (\xi_2)^\beta \\ (\xi_3)^\beta \end{pmatrix},$$

$$\begin{pmatrix} D_t^\alpha x^1 \\ D_t^\alpha x^2 \\ D_t^\alpha x^3 \end{pmatrix} = -\Gamma(1+\beta) \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -x^1 \\ 0 & x^1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ (x^2)^\alpha \\ (x^3)^\alpha \end{pmatrix}.$$

From the above matrix equations follows the fractional dynamical sys-

tem:

$$
(48)\qquad \begin{cases} D_t^\beta \xi_1 = \Gamma(1+\beta)(-\xi_3 (x^2)^\alpha x^3 + \xi_2 x^2 (x^3)^\alpha) + \\ \qquad\qquad + \Gamma(1+\alpha)(-x^3(\xi_2)^\beta + x^2(\xi_3)^\beta) \\ D_t^\beta \xi_2 = -\Gamma(1+\beta) x^1 (x^3)^\alpha \xi_1, \\ D_t^\beta \xi_3 = \Gamma(1+\beta) x^1 (x^2)^\alpha \xi_1 \\ D_t^\alpha x^1 = \Gamma(1+\beta)(x^2)^\alpha \\ D_t^\alpha x^2 = \Gamma(1+\beta) x^1 (x^3)^\alpha \\ D_t^\alpha x^3 = -\Gamma(1+\beta) x^1 (x^2)^\alpha \end{cases} .
$$

The fractional dynamical system (48) is the $(\alpha, \beta)-$ *fractional dynamical system* associated to fractional Maxwell-Bloch equations. □

Conclusion. The numerical integration of the fractional systems presented in this paper will be discussed in future papers.

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Received: 15.X.2007

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