ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" DIN IAŞI (S.N.)
MATEMATICĂ, Tomul LIII, 2007, Supliment

GEOMETRICAL MODEL OF A FINSLERIAN NONHOLONOMIC MECHANICAL SYSTEM

BY

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Abstract. One considers a Finslerian nonholonomic mechanical system $\sum_F = (M, F(x,y), Q_\sigma(x,dx), F_e(x,y))$ and determines the evolution equations, canonical semispray, canonical nonlinear connection and metrical connection.

Mathematics Subject Classification 2000: 53B60.
Key words: Finsler space, semispray, Finslerian mechanical systems.

1. Introduction. In the recent paper [3], we studied the geometrical model of nonholonomic Lagrangian mechanical system determining its evolution equations, the canonical semispray and studying the corresponding geometrical model. In the last section we remarked a very important particular case, given by the nonholonomic Finslerian mechanical systems. But this case is a remarkable one. So, it must to be studied separately.

In the present paper a Finslerian mechanical system, scleronomic with the nonholonomic constraints is defined as a 4-uple:

$$\sum_F = (M, F(x,y), Q_\sigma(x,dx), F_e(x,y)) \tag{1.1}$$

where $F^n = (M, F(x,y))$ is a Finsler space [1], [2], [4], $Q_\sigma(x,dx) = a_{\sigma_i}(x)dx^i = 0$, $(\sigma = m+1, ..., n)$, $m < n$, $n = dimM$ are the nonholonomic constraints and $F_e = F^i(x,y)\frac{\partial}{\partial y^i}$ are the external forces. Since the d-vector $F^i(x,y)$ depends on velocities $y^i = \frac{dx^i}{dt}$, F_e are nonconservative forces.

Such kind of mechanical systems were considered for the first time by Joseph Klein [1], [3], who studied these systems only by means of their evolution equations.

Now, using the Finsler geometry of F^n and the geometry of semisprays on the phase space TM we determine all fundamental geometric objects of system $\sum_F$ and give their geometrical meaning. The present paper is a continuation of the joint paper [3]. This is the reason for which we take some results from the mentioned paper.

This paper has been presented by the second author to the International Conference dedicated to academician Radu Miron, at his 80^{th} anniversary, Sept. 3-8, 2007, "Al.I.Cuza" University.

2. Preliminaries: Finsler spaces. The real differentiable manifold M of dimension n is called the configuration space of system $\sum_F$. A generic point $x \in M$ has local coordinates $(x^i), (i = 1, ..., n)$. The tangent bundle (TM, π, M) has the total space TM- called the phase space of $\sum_F$ as a real, orientable 2n dimensional manifold, with local coordinates (x^i, y^i) of the points $(x, y), (x = \pi(x, y))$. A change of coordinates on TM is of the form

$$\begin{aligned} &\tilde{x}^i = \tilde{x}^i(x^1, ..., x^n),\, det\left(\frac{\partial \tilde{x}^i}{\partial x^j}\right) \neq 0 \\ &\tilde{y}^i = \frac{\partial \tilde{x}^i}{\partial x^j} y^j. \end{aligned} \tag{2.1}$$

The pair $F^n = (M, F(x, y))$ is a Finsler space-called associated to $\sum_F$. Then, $F(x, y)$ is the fundamental function of F^n and its fundamental (or metric) tensor field is

$$g_{ij}(x, y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j},\ \forall (x, y) \in \tilde{TM} = TM \setminus \{0\}. \tag{2.2}$$

The integral of action $I(c)$ of F^n is defined by

$$I(c) = \int_0^1 F^2(x, \dot{x}) dt. \tag{2.3}$$

The definition of $I(c)$ is correct because the Kynetic energy of F^n is

$$E(x, y) = g_{ij} y^i y^j = F^2(x, y),\ y^i = \frac{dx^i}{dt}. \tag{2.4}$$

The variational problem on $I(c)$ leads to the Euler-Lagrange equations, [1],[2],[3],[4]

$$\frac{d}{dt}\left(\frac{\partial F^2}{\partial y^i}\right) - \frac{\partial F^2}{\partial x^i} = 0,\ y^i = \frac{dx^i}{dt}. \tag{2.5}$$

This system of differential equations is equivalent to the SODE:

$$\frac{d^2x^i}{dt^2} + \gamma^i_{jk}(x, \frac{dx}{dt})\frac{dx^j}{dt}\frac{dx^k}{dt} = 0, \tag{2.6}$$

where $\gamma^i_{jk}(x,y)$ are the Christoffel symbols of the fundamental tensor $g_{ij}(x,y)$. In the case when t is arclength, (2.6) give the geodesics of the Finsler space F^n. The space F^n gives rise to a canonical semispray on the phase space TM:

$$\overset{o}{S}= y^i \frac{\partial}{\partial x^i} - 2\overset{o}{G}{}^i(x,y)\frac{\partial}{\partial y^i}, \tag{2.7}$$

where

$$2\overset{o}{G}{}^i = \gamma^i_{jk} y^j y^k. \tag{2.7'}$$

But $\overset{o}{S}$ is 1-homogeneous. This means that $\overset{o}{S}$ is a spray. The integral curves of the spray $\overset{o}{S}$ are given by (2.6). This is the reason why $\overset{o}{S}$ is called "canonical for F" .

3. Finslerian nonholonomic mechanical systems. The Finslerian nonholonomic mechanical system is defined by a qvadruple:

$$\sum_F = (M, F(x,y), Q_\sigma(x,dx), F_e(x,y)), \tag{3.1}$$

where $F(x,y)$ is the fundamental function of a Finsler space, Q_σ are the nonholonomic constrains

$$Q_\sigma(x,dx) \equiv a_{\sigma_i}(x)dx^i = 0, (\sigma = m+1, ..., n), \tag{3.2}$$

$Q_\sigma(x,dx) = 0$ being an nonintegrable Pfaff system,

$$F_e(x,y) = F^i(x,y)\frac{\partial}{\partial y^i} \tag{3.2}$$

are the external forces, $F^i(x,y)$ being a distinguished vector field on the phase space,[1], [2].

Since F^i depend on the velocities $y^i = \frac{dx^i}{dt}$, it follows that the mechanical system $\sum_F$ is nonconservative. The 1-forms $Q_\sigma(x,dx) = a_{\sigma_i}(x)dx^i$ determine n-m functions on TM:

$$Q_\sigma(x,y) = a_{\sigma_i}(x)y^i (\text{sum by } i = 1,...,n). \tag{3.3}$$

We can consider the following Lagrangian

$$\mathcal{L}(x,y) = F^2(x,y) + \sum_{\sigma=m+1}^{n} \lambda^\sigma(x)Q_\sigma(x,y) \tag{3.4}$$

the functions $\lambda^{m+1}(x),...,\lambda^n(x)$ being the Lagrange multiplier. The following properties hold:

1) $\mathcal{L}(x,y)$ is a Lagrangian on the phase space. This means $\mathcal{L}$ is a scalar field on TM.

2) The restriction of $\mathcal{L}(x,y)$ on the distribution $Q_\sigma(x,dx) = 0$ is $F^2(x,y)$, which is the energy of the Finsler space F^n.

3) The fundamental tensor of the Lagrangian $\mathcal{L}(x,y), g_{ij}(x,y) = \frac{1}{2}\frac{\partial^2\mathcal{L}}{\partial y^i \partial y^j}$ is coincides to the fundamental tensor of the Finsler space F^n:

$$g_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}. \tag{3.5}$$

4) $\mathcal{L}(x,y)$ is not homogeneous with respect to y^i.

5) The integral of action of $\mathcal{L}(x,y)$ is

$$I'(c) = \int_0^1 \mathcal{L}(x,\dot{x})dt = \int_0^1 F^2(x,\dot{x})dt + \sum_{\sigma=m+1}^{n} \int_0^1 \lambda^\sigma(x)a_{\sigma_i}(x)\dot{x}^j dt \tag{3.6}$$

and the Euler-Lagrange equations of $\mathcal{L}$ are given by:

$$\frac{d}{dt}\frac{\partial F^2}{\partial y^i} - \frac{\partial F^2}{\partial x^i} + \sum_{\sigma=m+1}^{n}\left[\frac{d}{dt}\frac{\partial}{\partial y^i}(\lambda^\sigma Q_\sigma) - \frac{\partial}{\partial x^i}(\lambda^\sigma Q_\sigma)\right] = 0,\ y^i = \frac{dx^i}{dt} \tag{3.7}$$

or equivalently

(3.7′)

$$\frac{d}{dt}\frac{\partial F^2}{\partial y^i} - \frac{\partial F^2}{\partial x^i} + \sum_{\sigma=m+1}^{n}\left[\frac{\partial\lambda^\sigma}{\partial x^j}a_{\sigma_i} - \frac{\partial\lambda^\sigma}{\partial x^i}a_{\sigma_j} + \lambda^\sigma\left(\frac{\partial a_{\sigma_i}}{\partial x^j} - \frac{\partial a_{\sigma_j}}{\partial x^i}\right)\right] = 0.$$

Now, as usually we ask for the 1-form

$$\sum_{\sigma=m+1}^{n} \lambda^\sigma(x) Q_\sigma(x, dx) \tag{3.8}$$

to be closed:

$$\sum_{\sigma=m+1}^{n} d[\lambda^\sigma(x) Q_\sigma(x, dx)] = 0. \tag{3.8$'$}$$

We obtain the equations for determining the multipliers $\lambda^\sigma(x)$

$$\sum_{\sigma=m+1}^{n} \left\{ \frac{\partial \lambda^\sigma}{\partial x^i} a_{\sigma_j} - \frac{\partial \lambda^\sigma}{\partial x^j} a_{\sigma_i} + \lambda^\sigma \left(\frac{\partial a_{\sigma_i}}{\partial x^j} - \frac{\partial a_{\sigma_j}}{\partial x^i} \right) \right\} = 0. \tag{3.9}$$

So we can conclude that the Lagrangians $\mathcal{L}$ and F^2 verify the same Euler Lagrange equations if and only if the multipliers $\lambda^\sigma(x), (\sigma = m+1, ..., n)$ satisfy the equations (3.8') or (3.9).

4. The evolution equations of the Finslerian nonholonomic mechanical systems. A Finslerian nonholonomic mechanical system $\sum_F$, given by (1.1) can be considered as a Lagrangian mechanical system

$$\sum_{\mathcal{L}} = (M, \mathcal{L}(x, y), F_e(x, y)) \tag{4.1}$$

for which

$$\mathcal{L}(x, y) = F^2(x, y) + \sum_{\sigma=m+1}^{n} \lambda^\sigma(x) Q_\sigma(x, y) \tag{4.1$'$}$$

is a regular Lagrangian,

$$Q_\sigma \equiv a_{\sigma_i}(x) dx^i = 0 \tag{4.1$''$}$$

are the Kynematic constraints,
(4.1$'''$)
F^2 is the Kynetic energy of a Finsler space $F^n = (M, F(x, y))$,

$$\lambda^\sigma(x) \tag{4.1^{iv}}$$

are the Lagrange multiplier which satisfy the exterior equations,

$$\sum_{\sigma=m+1}^{n} d[\lambda^{\sigma}(x)Q_{\sigma}(x,dx)] = 0, \tag{4.2}$$

$$F_e(x,y) = F^i(x,y)\frac{\partial}{\partial y^i} \tag{4.1^v}$$

are the given vertical vector fields called the external forces.

In the paper [3] we anounced the following Postulate, applied here in the case of Finslerian nonholonomic mechanical systems in the form:

Postulate. The evolution equations of the Finslerian nonholonomic mechanical system $\sum_F$ are:

$$\frac{d}{dt}\left(\frac{\partial F^2}{\partial y^i}\right) - \frac{\partial F^2}{\partial x^i} = F_i(x,y) + \sum_{\sigma=m+1}^{n} \lambda^{\sigma}(x)a_{\sigma_i}(x),\ y^i = \frac{dx^i}{dt}, \tag{4.3}$$

where the multipliers $\lambda^{\sigma}(x)$ satisfy the equations

$$\sum_{\sigma=m+1}^{n} d(\lambda^{\sigma}Q_{\sigma}(x,dx)) = 0 \tag{4.3$'$}$$

and where the covariant components of external forces are:

$$F_i(x,y) = g_{ij}(x,y)F^j(x,y). \tag{4.4}$$

It is not difficult to prove that the equations (4.3) have a geometrical meaning on the phase space TM.

These equations will be named **the Lagrange equations** of the Finslerian nonholonomic mechanical systems $\sum_F$.

They have some interesting properties:

1^o $F_i(x,y) = 0, a_{\sigma_i}(x) = 0$ imply the fact that the solution curves of (4.3) are the geodesics of the Finsler space F^n.

2^o Let $F^n = R^n$ be a Riemann space and $\frac{\partial F_i}{\partial y^j} = 0$. We obtain a classical Riemannian conservative nonholonomic mechanical system.

3^o $F^2 = g_{ij}(x)\frac{dx^i}{dt}\frac{dx^j}{dt}$ -Riemannian Kynetic energy and F^i depending of the velocity $\frac{dx^i}{dt} = y^i$ give us the Riemannian nonconservative nonholonomic mechanical systems.

4^o Let F^n be a localy Minkowski space, and $F^i(x,y) = cy^i$, $c = const. \neq 0$. $\sum_F$ is a Minkowskian nonconservative nonholonomic mechanical system, the external forces being of Liouville type [1].

Now, let us denote

$$a^i_\sigma(x,y) = g^{ij}(x,y)a_{\sigma_i}(x)\,(\sigma = m+1,...,n). \tag{4.5}$$

Then, the Lagrange system of equations (4.3), (4.3)' is equivalent to:
(4.6)

$$\frac{d^2x^i}{dt^2} + \gamma^i_{jk}\left(x,\frac{dx}{dt}\right)\frac{dx^j}{dt}\frac{dx^k}{dt} = \frac{1}{2}\left[F^i\left(x,\frac{dx}{dt}\right) + \sum_{\sigma=m+1}^{n}\lambda^\sigma(x)a^i_\sigma\left(x,\frac{dx}{dt}\right)\right]$$

$$\sum_{\sigma=m+1}^{n} d[\lambda^\sigma(x)Q_\sigma(x,dx)] = 0$$

where $\gamma^i_{jk}(x,y)$ are the Christoffel symbols of the fundamental tensor $g_{ij}(x,y)$ of the Finsler space F^n.

In the following section we use the Lagrange equations of the system $\sum_F$ to derive some important geometrical object fields on the phase space TM of $\sum_F$.

5. Canonical semispray of $\sum_F$. The equations (4.6) allow us to prove [1],[2],[3]:

Theorem 5.1 *The canonical semispray S of the Finslerian nonholonomic mechanical system $\sum_F$ is the following vector field on the phase space TM:*

$$S = y^i\frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}, \tag{5.1}$$

where

$$2G^i(x,y) = \gamma^i_{jk}(x,y)y^jy^k - \frac{1}{2}[F^i(x,y) + \sum_{\sigma=m+1}^{n}\lambda^\sigma(x)a^i_\sigma(x,y)]. \tag{5.2}$$

S is called canonical, since it depends on system $\sum_F$, only.

A first important result can be formulate:

Theorem 5.2 *The integral curves of the canonical semispray S are given by the evolution equations* (4.6).

Proof. Indeed, the integral curves of S are expressed by $\frac{dx^i}{dt} = y^i$, $\frac{dy^i}{dt} + 2G^i(x,y) = 0$, the functions $\lambda^\sigma(x)$ verifying the second equations (4.6).

But, these equations are exactly the first equations (4.6). q.e.d.

The vector field S on the phase space TM is a dynamical system determinated only by $\sum_F$. Consequently the global properties of the moving of the nonholonomic mechanical system $\sum_F$, the stability of integral curves, or equilibrum points, etc, can be studied by means of S.

We define the Lagrange-Finsler geometry of the Finslerian nonholonomic mechanical system $\sum_F$ as being the Lagrange geometry of the pair (TM, S), [3].

Remark. The canonical semispray S is not homogeneous with respect to velocities $y^i = \frac{dx^i}{dt}$. This is the reason the geometry of the pair (TM, S) is not purely Finslerian. But it can be constructed as a Lagrange geometry [1],[3].

6. Canonical nonlinear connection and metrical connection of $\sum_F$. The geometrical model of $\sum_F$ is based on the fundamental notions: the nonlinear connection N and the canonical metrical connection $C\Gamma(N)$, determined only by $\sum_F$.

The general theory, [2],[3] shows us that the canonical semispray S, with coefficients $G^i(x,y)$ determines a nonlinear connection N with the coefficients $N^i_j(x,y)$ expressed by $N^i_j = \frac{\partial G^i}{\partial y^j}$.

Evidently, N depends on $\sum_F$ only. So, it can be called canonical for $\sum_F$.

We have, by a direct calculation:

Theorem 6.1 *The canonical nonlinear connection N of $\sum_F$ has the coefficients:*

$$N^i_j = \overset{o}{N}{}^i_j - \frac{1}{4}\left(\frac{\partial F^i}{\partial y^j} + \sum_{\sigma=m+1}^{n} \lambda^\sigma \frac{\partial a^i_\sigma}{\partial y^j}\right) \tag{6.1}$$

where $\overset{o}{N}{}^i_j$ are the coefficients of the Cartan nonlinear connection of the Finsler space F^n.

From the previous theorem, we obtain:

Theorem 6.2. *The Berwald connection of the system* $\sum_F$ *has the coefficients*

$$B^i_{jk} = \overset{o}{B}{}^i_{jk} - \frac{1}{4}\left(\frac{\partial^2 F^i}{\partial y^j \partial y^k} - \sum_{\sigma=m+1}^{n} \lambda^\sigma \frac{\partial^2 a^i_\sigma}{\partial y^j \partial y^k}\right) \tag{6.2}$$

$\overset{o}{B}{}^i_{jk}$ *being the coefficients of the Berwald connection of the Finsler space* F^n.

But, the torsion of Berwald connection vanishes. Consequently,

Corollary 6.1. *The weak torsion of the canonical nonlinear connection* N *vanishes.*

The nonlinear connection N on TM determines a direct decomposition of tangent space $T_u(TM)$:

$$T_u(TM) = N(u) \oplus V(u),\ \forall u \in TM. \tag{6.3}$$

An adapted local basis to (6.3) is of the form $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)_u$ where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i(x,y)\frac{\partial}{\partial y^j}. \tag{6.4}$$

Thus, the integrability tensor R^i_{jk} of N is $R^i_{jk} = \frac{\delta N^i_j}{\delta x^k} - \frac{\delta N^i_k}{\delta x^j}$. We know that $R^i_{jk} = 0$ characterises the integrability of the canonical nonlinear connection N of $\sum_F$.

Let us consider the N-linear connection $C\Gamma(N) = (L^i_{jk}, C^i_{jk})$ and the h-covariant derivative "$^|$" and v-covariant derivative "$|$" defined by $C\Gamma(N)$. Then, we can prove:

Theorem 6.3. *For the Finslerian nonholonomic mechanical system* $\sum_F$*, there exists only one N-linear connection* $C\Gamma(N) = (L^i_{jk}, C^i_{jk})$ *having properties* $g_{ij|k} = 0$, $g_{ij}|_k = 0$, $T^i_{jk} = 0$, $S^i_{jk} = 0$. *The coefficients of this connections are given by the following generalized Christoffel symbols:*

$$\begin{cases} L^i_{jk} = \frac{1}{2} g^{ih}\left(\frac{\delta g_{jh}}{\delta x^k} + \frac{\delta g_{kh}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^h}\right) \\ C^i_{jk} = \frac{1}{2} g^{ih}\left(\frac{\partial g_{jh}}{\partial y^k} + \frac{\partial g_{kh}}{\partial y^j} - \frac{\partial g_{jk}}{\partial y^h}\right) \end{cases}. \tag{6.5}$$

A general remark. All geometric properties of $\sum_F$ can be expressed in terms of the following geometric object fields: $g_{ij}, N^i_j, (L^i_{jk}, C^i_{jk})$ and Q_σ. So, for the system $\sum_F$ we can determine the electromagnetic tensor field F_{ij} and its generalized Maxwell equations, as well as the Einstein equations of the gravitational tensor field g_{ij}, [3].

REFERENCES

1. Bucătaru, I.; Miron, R. – *Finsler-Lagrange Geometry,Applications to Dynamical Systems*, ED.Acad.Române, Bucureşti, 2007.
2. Miron, R.; Anastasiei, M. – *The geometry of Lagrange Spaces. Theory and Applications*, Kluwer Acad.Publ. FTPH 59. , 1994.
3. Miron, R.; Nimineţ, V. – *The Lagrangian Geometrical Model and the Associated Dynamical System of a Nonholonomic mechanical system,* Facta Universitatis, S.Mechanics, Automatic Control and Robotics, Nis, Serbia vol.5, No 1, pp.1-24, 2006.
4. Nimineţ, V. – *The exterior differential system. Applications in the Control theory,* Ph. D.Thesis Univ. "Al.I. Cuza", 2005.

Received: 15.X.2007

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