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MIRON FRAMES ON A COMPLEX FINSLER MANIFOLD

BY

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Dedicated to Academician Radu Miron at his 80th anniversary

Abstract. In this note we introduce an orthonormal frame by adapting to the Hermitian structure of a complex Finsler space Miron's idea that generalizes the Berwald and Moór frames for a real $n-$dimensional Finsler space.

In the first section we briefly set the notions and terminology of the work.

The complex Miron frame construction follows the same idea introduced by MIRON ([Mi, Ma]), but the orthogonality is with respect to the Hermitian structure defined by the fundamental metric tensor of an $n-$dimensional complex Finsler space on the holomorphic tangent manifold $T'M$.

In the last section we apply the results to a two-dimensional complex Finsler manifold and the obtained frame is named complex Miron-Berwald frame. The expression of the complex Chern-Finsler connection with respect to this frame is obtain and finally we are able to compute the vertical holomorphic sectional curvature of the space with respect to complex Miron-Berwald frame.

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1. Introduction. In the geometry of two-dimensional Finsler spaces a great contribution is due to L. Berwald (see [Ma]). His theory is developed based on the choosing of an orthonormal frame consisting in the normalized Liouville field and an unit orthogonal field to it. Many other posterior remarkable results are known for two-dimensional real Finsler spaces ([Ma, As, B-C-S, B-F]).

Getting going the Berwald work, A. Moór introduces in study an orthonormal frame on a three-dimensional Finsler space. Here the main results belong to A. Moór and M. Matsumoto. The idea of finding an

orthonormal frame intrinsical related to the geometry of a $n-$dimensional Finsler space was solved by MIRON ([Mi]) and communicated to Matsumoto yet 1974.

The complex Finsler geometry was impulsed by the study of hyperbolic spaces with Kobayashi and Caratheodory metrics and long period its main purpose was to gave characterizations for these metrics on a complex Finsler space.

Investigations from the last decade ([A-P, Ai, Wo, Mu, Mu1, Al2]) proved that this geometry is likewise interesting as real geometry and furnish suitable models for physics theories. Therefore, one problem for this geometry was to extend the number of examples of complex Finsler spaces. We believe that our recent paper [A-M] comes with a contribution in this sense.

Part of general themes from real Finsler geometry can be broached in complex Finsler geometry, in the last case having the advantage of a powerful instrument, namely (cf. Abate and Patrizio) the Chern-Finsler connection, which is one Hermitian and of $(1,0)-$type and with special properties, but also the disadvantage of some uncomfortable calculus on account of extending the theory to the complexified holomorphic tangent bundle $T'M$.

In the present paper our aim is to introduce the Miron frames on a complex Finsler manifold, particularly analyzing the Berwald complex frame. Some of the ideas are natural extensions to the Hermitian geometry of that from the real case. For this reason we pass over from some technical details in our work. For comparison to real case you can look on [Ma] and on [B-F] for the Berwald frame case.

For the beginning will make a survey on complex Finsler geometry and will set the basic used notions and terminology. For more see [A-P, Mu].

Let M be a complex $n-$dimensional manifold, $(z^k)_{k=\overline{1,n}}$ complex coordinates in a local chart.

The complexified of the real tangent bundle T_CM splits into the sum of holomorphic tangent bundle $T'M$ and its conjugate $T''M$. The bundle $T'M$ is in its turn a complex manifold, the local coordinates in a chart will be denoted by (z^k, η^k).

A *complex Finsler space* is a pair (M, F), where $F : T'M \to \mathbb{R}^+$ is a continuous function satisfying the conditions:

i) $L := F^2$ is smooth on $\widetilde{T'M} := T'M\backslash\{0\}$;

ii) $F(z, \eta) \geq 0$, the equality holds if and only if $\eta = 0$;

iii) $F(z, \lambda\eta) = |\lambda|F(z, \eta)$ for $\forall\lambda \in \mathbb{C}$, the homogeneity condition;

iv) the Hermitian matrix $\left(g_{i\bar{j}}(z,\eta)\right)$, with $g_{i\bar{j}} = \frac{\partial^2 L}{\partial\eta^i \partial\bar{\eta}^j}$ the fundamental metric tensor, is positive defined. Equivalently, this means that the indicatrice is strongly pseudoconvex.

Consequently, from *iii*) we have:

$$\frac{\partial L}{\partial \eta^k}\eta^k = \frac{\partial L}{\partial \bar{\eta}^k}\bar{\eta}^k = L;\ \frac{\partial g_{i\bar{j}}}{\partial \eta^k}\eta^k = \frac{\partial g_{i\bar{j}}}{\partial \bar{\eta}^k}\bar{\eta}^k = 0 \tag{1.1}$$

and $L = g_{i\bar{j}}\eta^i\bar{\eta}^j$.

Roughly speaking, the geometry of a complex Finsler space consists in study of geometric objects of the complex manifold $T'M$ endowed with the Hermitian metric structure defined by $g_{i\bar{j}}$.

In this sense the first step is the study of sections of complexified tangent bundle of $T'M$ which is decomposed in the sum $T_C(T'M) = T'(T'M) \oplus T''(T'M)$. Let $V(T'M) \subset T'(T'M)$ be the vertical bundle, spanned locally by $\{\frac{\partial}{\partial\eta^k}\}$ and $VT''M$ its conjugate.

In this point the idea of complex nonlinear connection, briefly (*c.n.c.*), is fundamental in 'linearizing' of this geometry. A (*c.n.c.*) is a supplementary complex subbundle to $V(T'M)$ in $T'(T'M)$, i.e. $T'(T'M) = H(T'M) \oplus V(T'M)$. The horizontal distribution $H_u(T'M)$ is locally spanned by $\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j\frac{\partial}{\partial\eta^j}\}$, where $N_k^j(z,\eta)$ are the coefficients of the (*c.n.c.*), which obey a certain rule of change at the charts changes such that $\frac{\delta}{\delta z^k} = \frac{\partial z'^j}{\partial z^k}\frac{\delta}{\delta z'^j}$. Obvious also we have that $\frac{\partial}{\partial\eta^k} = \frac{\partial z'^j}{\partial z^k}\frac{\partial}{\partial\eta'^j}$. The pair $\{\dot{\partial}_k := \frac{\partial}{\partial\eta^k}, \delta_k := \frac{\delta}{\delta z^k}\}$ will be called the adapted frame of the (*c.n.c.*). By conjugation everywhere is obtained an adapted frame $\{\dot{\partial}_{\bar{k}}, \delta_{\bar{k}}\}$ on $T''_u(T'M)$. The dual adapted bases are $\{dz^k, \delta\eta^k\}$ and $\{d\bar{z}^k, \delta\bar{\eta}^k\}$.

Certainly, one main problem in this geometry is to determine a (*c.n.c.*) related only to the fundamental function of the complex Finsler space (M, F).

The next step is the action of a derivative law D on the sections of $T_C(T'M)$. First let us consider the Sasaky type lift of the metric tensor $g_{i\bar{j}}$,

$$G = g_{i\bar{j}}dz^i \otimes d\bar{z}^j + g_{i\bar{j}}\delta\eta^i \otimes \delta\bar{\eta}^j. \tag{1.2}$$

An Hermitian connection D, of $(1,0)-$type, which satisfies in addition $D_{JX}Y = JD_XY$, for all X horizontal vectors and J the natural complex structure of the manifold, is the so called (improperly in [A-P]) Chern-Finsler connection, in brief $C-F$, and locally it is given by the following

coefficients see notations in [Mu]:

(1.3)
$$N_j^k = g^{\bar{m}k}\frac{\partial g_{l\bar{m}}}{\partial z^j}\eta^l;\ L^i_{jk} = g^{\bar{l}i}\delta_k(g_{j\bar{l}});\ C^i_{jk} = g^{\bar{l}i}\dot{\partial}_k(g_{j\bar{l}});\ L^{\bar{i}}_{\bar{j}k} = 0;\ C^{\bar{i}}_{\bar{j}k} = 0,$$

where $D_{\delta_k}\delta_j = L^i_{jk}\delta_i$, $D_{\dot{\partial}_k}\dot{\partial}_j = C^i_{jk}\dot{\partial}_i$, etc..

The $C-F$ connection is the main tool in this study, but there exists another interesting Hermitian connection in this geometry (see [Mu]). The nonzero curvatures of the Chern-Finsler connection are denoted by

$$R(\delta_h,\delta_{\bar{k}})\delta_j = R^i_{j\bar{k}h}\delta_i;\ R(\dot{\partial}_h,\delta_{\bar{k}})\delta_j = P^i_{j\bar{k}h}\delta_i;\ R(\delta_h,\dot{\partial}_{\bar{k}})\delta_j = Q^i_{j\bar{k}h}\delta_i$$
$$R(\delta_h,\delta_{\bar{k}})\dot{\partial}_j = R^i_{j\bar{k}h}\dot{\partial}_i;\ R(\dot{\partial}_h,\delta_{\bar{k}})\dot{\partial}_j = P^i_{j\bar{k}h}\dot{\partial}_i;\ R(\delta_h,\dot{\partial}_{\bar{k}})\dot{\partial}_j = Q^i_{j\bar{k}h}\dot{\partial}_i$$
$$R(\dot{\partial}_h,\dot{\partial}_{\bar{k}})\delta_j = S^i_{j\bar{k}h}\delta_i;\quad R(\dot{\partial}_h,\dot{\partial}_{\bar{k}})\dot{\partial}_j = S^i_{j\bar{k}h}\dot{\partial}_i$$

where

(1.4)
$$R^i_{j\bar{k}h} = -\delta_{\bar{k}}L^i_{jh} - \delta_{\bar{k}}(N^l_h)C^i_{jl};\ P^i_{j\bar{k}h} = -\delta_{\bar{k}}C^i_{jh}$$
$$Q^i_{j\bar{k}h} = -\dot{\partial}_{\bar{k}}L^i_{jh} - \dot{\partial}_{\bar{k}}(N^l_h)C^i_{jl};\ S^i_{j\bar{k}h} = -\dot{\partial}_{\bar{k}}C^i_{jh}.$$

The $h\bar{h}$ Riemann tensor is $R_{i\bar{j}k\bar{h}} = g_{l\bar{j}}R^l_{i\bar{h}k}$ and the corresponding Ricci tensor is $R_{\bar{j}k} = R_{i\bar{j}k\bar{h}}\eta^i\bar{\eta}^h$.

For the vertical section $\mathcal{L} = \eta^k\dot{\partial}_k$, called the Liouville complex field (or the vertical radial vector field in [A-P]), we consider its horizontal lift $\chi := \eta^k\delta_k$.

According to [A-P], p. 108, [Mu], p. 81, the holomorphic sectional curvature of the complex Finsler space (M,F) in η direction,

(1.5)
$$K_F(z,\eta) = \frac{2}{L^2}G(\mathbf{R}(\chi,\bar{\chi})\chi,\bar{\chi}),$$

has locally the following expression (see [Al1]):

(1.6)
$$K_F(z,\eta) = \frac{2}{F^4}R_{\bar{j}k}\bar{\eta}^j\eta^k,\ \text{where } R_{\bar{j}k} = -g_{l\bar{j}}\delta_{\bar{h}}(N^l_k)\bar{\eta}^h.$$

2. Complex Miron frame. Let (M,F) be a $n-$dimensional complex Finsler space. In the following will keep similar notations used by

Matsumoto for the real case, [Ma], but adapted to our framework. In our previous notations, we consider the Hermitian tensors:

$$\begin{aligned}
L_i &= \dot\partial_i L = g_{i\bar j}\bar\eta^j =: \eta_i \\
L_{i\bar j} &= \partial^2 L \,/\, \partial\eta^i\partial\bar\eta^j =: g_{i\bar j} \\
L_{i\bar jk} &= \partial^3 L \,/\, \partial\eta^i\partial\bar\eta^j\partial\eta^k =: C_{i\bar jk} \text{ (the mixed Cartan Tensor)} \\
L_{i\bar jkh} &= \partial^4 L \,/\, \partial\eta^i\partial\bar\eta^j\partial\eta^k\partial\bar\eta^h \\
&\qquad \text{.....}
\end{aligned}$$

By help of these let us construct the covectors:

$$\begin{aligned}
L_{1)i} &= L_i \\
L_{2)i} &= L_{i\bar jk}g^{\bar jk} = C^k_{ik} \\
L_{3)i} &= L_{i\bar jk\bar hl}g^{\bar jk}g^{\bar hl} \\
&\qquad \text{.....} \\
L_{\alpha)i} &= L_{i\bar j_1k_1\ldots\ldots\bar j_{\alpha-1}k_{\alpha-1}}g^{\bar j_1k_1}\ldots\ldots g^{\bar j_{\alpha-1}k_{\alpha-1}}
\end{aligned} \tag{2.1}$$

$\alpha = 1, 2, ..., n$.

By conjugation everywhere the tensors $L_{\alpha)\bar\imath}$ are obtained and raising the indices with the metric tensor is obtain the vectors $L^i_{\alpha)} = g^{\bar mi}L_{\alpha)\bar m}$ are obtained. These could be looked as vertical vectors $\overset{v}{L}_{\alpha)} = L^i_{\alpha)}\dot\partial_i$ as well, by means of the horizontal lift χ, as horizontal vectors $\overset{h}{L}_{\alpha)} = L^i_{\alpha)}\delta_i$ relative to a chosen (*c.n.c.*). Subsequently we consider only the (1.3) $C-F$ (*c.n.c.*). Generally, the vectors $L_{\alpha)}$ are not linear independent and when they are, according with Matsumoto terminology, we say that the space is strongly non-Hermitian and all subsequent considerations will be in this circumstances. The pure Hermitian case, that is $g_{i\bar j}$ do not depends on η, is excluded in the further discussions.

Next, let us begin the construction of an orthonormal frame on the complex Finsler space (M, F).

First we consider the unit Liouville vector $\mathcal{L}$ and using it we define

$$e^i_{1)} = \frac{1}{F}L^i_{1)}. \tag{2.2}$$

Obviously we have $L^i_{1)} = \eta^i$ and $g_{i\bar j}e^i_{1)}e^{\bar j}_{1)} = \frac{1}{F^2}g_{i\bar j}\eta^i\bar\eta^j = 1$. Hence $e^i_{1)}$ is an unit vector.

Further, let us consider $M_{1)i\bar{j}} = g_{i\bar{j}} - e_{1)i}e_{1)\bar{j}} = g_{i\bar{j}} - \frac{1}{F^2}\eta_i\bar{\eta}_j =: h_{i\bar{j}}$ the angular metric. The $h_{i\bar{j}}$ metric is not invertible, it has $rank\ \ h_{i\bar{j}} = n-1$.

Raising the indices one obtains

$$M_{1)i}^{\ \ k} = g^{\bar{j}k}M_{1)i\bar{j}} = \delta_i^k - \frac{1}{F^2}\eta^i\eta^k$$

and its easy to check that $M_{1)i}^{\ \ k}e_{1)}^i = 0$.

Also we observe from a direct computation that $L_{2)}^i = g^{\bar{m}i}g^{\bar{k}j}\frac{\partial g_{j\bar{m}}}{\partial\bar{\eta}^k}$ obeys the conditions

$$M_{1)i}^{\ \ k}L_{2)}^i = L_{2)}^k \ \text{ and } g_{i\bar{j}}L_{2)}^i e_{1)}^{\bar{j}} = 0$$

and therefore $L_{2)}^i$ lies is the orthogonal subspace to $e_{1)}$. Thus the second vector of the orthonormal frame is

$$e_{2)}^i = \frac{1}{\parallel L_{2)} \parallel}L_{2)}^i. \tag{2.3}$$

In the third step, analogously, let us set $M_{2)i\bar{j}} = M_{1)i\bar{j}} - e_{2)i}e_{2)\bar{j}}$ and $M_{2)i}^{\ \ k} = g^{\bar{j}k}M_{2)i\bar{j}}$.

Since $e_{1)}^i e_{2)i} = 0$, it follows $M_{2)i}^{\ \ k}e_{1)}^i = 0$ and $M_{2)i}^{\ \ k}e_{2)}^i = 0$.

A direct calculus prove that if X is orthogonal to the subspace $\{e_{1)}, e_{2)}\}$ then $M_{2)i}^{\ \ k}X^i = M_{1)i}^{\ \ k}X^i - e_{2)i}e_{2)}^k X^i = X^k$. On the other hand, it is easy to check that $L_{3)}^i = g^{\bar{m}i}g^{\bar{k}j}g^{\bar{l}h}L_{\bar{m}j\bar{k}h\bar{l}}$ performs such condition $M_{2)i}^{\ \ k}L_{3)}^i = L_{3)}^k$ and thus $L_{3)}^i$ is orthogonal to $\{e_{1)}, e_{2)}\}$. Taking its norm one obtain the third vector of the orthonormal frame, $e_{3)}^i = \frac{1}{\|L_{3)}\|}L_{3)}^i$.

An inductive procedure finally get the proposed orthonormal frame $\{e_{1)}, e_{2)},\ ...., e_{n)}\}$, where $M_{r+1)i\bar{j}} = M_{r)i\bar{j}} - e_{r+1)i}e_{r+1)\bar{j}}$ and $M_{r+1)i}^{\ \ k} = g^{\bar{j}k}M_{r+1)i\bar{j}}$ and as above is proved that $M_{r)i}^{\ \ k}L_{r+1)}^i = L_{r+1)}^k$, $r = 1, 2, .., n-1$, and $e_{\alpha)}^i = \frac{1}{\|L_{\alpha)}\|}L_{\alpha)}^i$.

Furthermore, by this construction we observe that $e_{\alpha)}^i$ are $0-$homogeneous $d-$tensors (see this notion in [Mu]) because $L_{\alpha)}$ are homogeneous with respect to η.

Definition 2.1. *The obtained orthonormal frame $\{e_{1)}, e_{2)}, ..., e_{n)}\}$ will be called the complex Miron frame.*

With respect to $C-F$ connection we may consider the horizontal and vertical derivatives of the frame vectors:

$$e^{\ i}_{\alpha)\ |\ j}=\delta_j e^i_{\alpha)}+L^i_{jk}e^k_{\alpha)};\ \ e^{\ i}_{\alpha)\ |\ j}=\dot\partial_j e^i_{\alpha)}+C^i_{jk}e^k_{\alpha)} \tag{2.4}$$

and respectively, $e^{\ i}_{\alpha)\ |\ \bar j}=\delta_{\bar j}e^i_{\alpha)}\ ,\ \ e^{\ i}_{\alpha)\ |\ \bar j}=\dot\partial_{\bar j}e^i_{\alpha)}$.

These rules of derivative generate $d-$tensors which can be decomposed with respect to complex Miron frame. Likewise for the real case, [Ma], will have the decomposition:

$$\begin{aligned}&e^{\ i}_{\alpha)\ |\ j}=H_{\alpha)\beta\gamma}\ e^i_{\beta)}e_{\gamma)j};\ \ F\ e^{\ i}_{\alpha)\ |\ j}=V_{\alpha)\beta\gamma}\ e^i_{\beta)}e_{\gamma)j}\\&e^{\ i}_{\alpha)\ |\ \bar j}=H_{\alpha)\beta\bar\gamma}\ e^i_{\beta)}e_{\gamma)\bar j};\ \ F\ e^{\ i}_{\alpha)\ |\ \bar j}=V_{\alpha)\beta\bar\gamma}\ e^i_{\beta)}e_{\gamma)\bar j}\end{aligned}$$

By conjugation one obtains the other derivations, for instance $e^{\ \bar i}_{\alpha)\ |\ j}$ or $e^{\ \bar i}_{\alpha)\ |\ j}$, etc.

By deriving the orthogonality condition $g_{i\bar j}e^i_{\alpha)}e^{\bar j}_{\beta)}=0,\ \alpha\neq\beta$, taking into account that $C-F$ connection is metrical $g_{i\bar j\ |\ k}=g_{i\bar j\ |\ k}=0$, some direct calculus get $H_{\alpha)\beta\gamma}=-H_{\beta)\gamma\alpha};\ V_{\alpha)\beta\gamma}=-V_{\beta)\gamma\alpha}$.

We end here the considerations about these derivatives (curvature, etc.) because subsequently will be studied only the particular case $n=2$.

3. Complex Miron-Berwald type frame. Let (M,F) be a complex two-dimensional Finsler space, (z^k,η^k) complex coordinates on $T'M$ and $VT'M$ the vertical distribution spanned by $\{\dot\partial_k=\frac{\partial}{\partial\eta^k}\}$. Everywhere in this section the indices $i,j,k,...$run over $\{1,2\}$.

Let $g_{i\bar j}$ be the fundamental metric tensor of the space and G the (1.2) Hermitian metric structure defined on $T_CT'M$ with respect to adapted frames of Chern-Finsler (*c.n.c.*).

The vertical distribution $VT'M$ is two-dimensional in any point and an orthonormal frame one obtains by particularizing the construction from the above section. Thus, for simplicity, in the next we denote $e_{1)}:=l$ and in view of (2.2) we have $l=l^i\dot\partial_i$ and its dual form is $\omega=l_i\delta\eta^i$, where

$$l^i=\frac{1}{F}\eta^i\ \text{ and }\ l_i=\frac{1}{F}g_{i\bar j}\bar\eta^j=g_{i\bar j}l^{\bar j}. \tag{3.1}$$

Then $VT'M$ decomposes in $VT'M=\{l\}\oplus\{l\}^\perp$, where $\{l\}^\perp$ is spanned by the unit vector $m:=e_{2)}=\frac{1}{\|L_{2)}\|}L^i_{2)}\dot\partial_i$ with $L^i_{2)}=g^{\bar mi}g^{\bar kj}\frac{\partial g_{j\bar m}}{\partial\bar\eta^k}$.

The calculus of $\| L_{2)} \|$ seems uncomfortable, this is one of the motive that we act directly for an acceptable expression of m vector by requiring the orthogonality conditions $G(l, \bar{m}) = 0$ and $G(m, \bar{m}) = 1$. The second motive is that in the previous section we work in the strongly non-Hermitian circumstances which exclude the purely Hermitian case. Using $m_i := g_{i\bar{j}} m^{\bar{j}}$, the above two conditions get the following linear system

$\begin{cases} l_1 m^1 + l_2 m^2 = 0 \\ m_1 m^1 + m_2 m^2 = 1 \end{cases}$ with the solutions $m^1 = \frac{-l_2}{\Delta}$ and $m^2 = \frac{l_1}{\Delta}$, where $\Delta = l_1 m_2 - l_2 m_1$. A straight computation prove that $\Delta = \bar{\Delta}$ is real and if we replace these solutions in the second equation of the system, by expanding the calculus one obtains that $\Delta^2 = g = \det(g_{i\bar{j}})$. Thus, we have

$$m = \frac{1}{\sqrt{g}} \left(-l_2 \frac{\partial}{\partial \eta^1} + l_1 \frac{\partial}{\partial \eta^2} \right). \tag{3.2}$$

We recall that $l_i l^i = 1$, $l_i m^i = l^i m_i = 0$, $m_i m^i = 1$ and, from the definition (1.2) of the metric structure G, the $(1,0)$ vectors are orthogonal to $(0,1)$ vectors, thus $l_i \bar{l}^i = 0$, etc.

The frame $\{l, m, \bar{l}, \bar{m}\}$ will be called the *complex Miron-Berwald frame* on $V_C T'M$. By the natural isomorphism of $VT'M$ and $T'M$ bundles, composed with the horizontal lift to $HT'M$, one obtains the following orthonormal frame on $H_C T'M$, $\{\chi(l) = l^i \delta_i, \chi(m) = m^i \delta_i, \chi(\bar{l}) = l^{\bar{i}} \delta_{\bar{i}}, \chi(\bar{m}) = m^{\bar{i}} \delta_{\bar{i}}\}$.

In the preceding section we introduced a derivative algorithm for the Miron frame. Particularly, let us know its action on the Miron-Berwald frame. For this, let P the projector on $\{l^{\perp}\}$ in $VT'M$. Indeed $P(l) = 0$, $P^2 = I$ and for $X \in \Gamma(VT'M)$ we have $X = \omega(X) l + P(X)$, since $\omega(X) = \frac{1}{F} g_{i\bar{j}} X^i \bar{\eta}^j = G(X, \bar{l})$. It follows that locally P has the expression $P^i_j = \delta^i_j - l_j l^i$. Particularly, $\frac{\partial}{\partial \eta^i}$ is decomposed as follow

$$\frac{\partial}{\partial \eta^i} = l_i l + m_i m \text{ and hence } g_{i\bar{j}} = l_i l_{\bar{j}} + m_i m_{\bar{j}}. \tag{3.3}$$

Let D be the $C - F$ connection on (M, L). An immediate computation get $D_{\delta_k} \mathcal{L} = 0$, $D_{\delta_{\bar{k}}} \mathcal{L} = 0$ and $D_{\dot{\partial}_k} \mathcal{L} = \delta^j_k \dot{\partial}_j$, $D_{\dot{\partial}_{\bar{k}}} \mathcal{L} = 0$ for $\mathcal{L} = \eta^i \dot{\partial}_i$ the Liouville vector field. Thus, it result that $D_X \mathcal{L} = vX$, for any $X \in \Gamma T_C(T'M)$, where v is the projector on $VT'M$. Consequently, in account of $\delta_k F = \delta_{\bar{k}} F = 0$, we have $D_{\delta_k} l = D_{\delta_{\bar{k}}} l = 0$ and direct calculus get

$$D_{\dot{\partial}_k} l = \frac{1}{2F} (\dot{\partial}_k + m_k m); \;\; D_{\dot{\partial}_{\bar{k}}} l = \frac{-1}{2F} l_{\bar{k}} l. \tag{3.4}$$

Further let us give an explicit expression for $C-F$ connection with respect to Miron-Berwald local frame. Using the (3.3) expression of $\dot\partial_k$ and of its conjugate in (3.4), by contraction in turn with m^k and l^k, respective with $\bar m^k$ and $\bar l^k$, one obtains:

Proposition 3.1. *We have:*

$$D_l l=\frac{1}{2F}\,l;\ D_{\bar l}l=\frac{-1}{2F}l;\ D_m l=\frac{1}{F}m;\ D_{\bar m}l=0 \tag{3.5}$$

and their conjugates.

A part of the $D_X m$ components are obtained a bit more complicated and involve some preliminary calculus which could seem tediously. The first one is the calculus of $m(g)=2\sqrt{g}\{m(l_1)m_2+l_1m(m_2)-m(l_2)m_1-l_2m(m_1)\dot\}$. In view of $0-$homogeneity of the metric tensor, by using that $m(l_1)m_2=m^k\dot\partial_k(g_{1\bar h})l^{\bar h}m_2$, and similar for the others, finally we get $m(g)=gm^kC^i_{ik}$, where the sum for all $i,k\in\{1,2\}$ is assumed. Now we can obtain easy the expression of $m(m_i)=m(g_{i\bar h}\bar m^h)=m^k\dot\partial_k(g_{i\bar h})\bar m^h-g_{i\bar 1}m(\frac{l_{\bar 2}}{\sqrt{g}})+g_{i\bar 2}m(\frac{l_{\bar 1}}{\sqrt{g}})$ and replacing the expressions of $l_{\bar 1}$ and $l_{\bar 2}$ we get that $m(m_i)=m^k\bar m^h\dot\partial_k(g_{i\bar h})-\frac{1}{2g}m_im(g)-\frac{1}{F}l_i$. Thus the final form of $m(m_i)$ obtained by substituting the above result for $m(g)$. Similar calculus can be made for the action of l and m vectors under the components l_i or m_i. Some of these calculations are immediate, others follow the same steps like in the above formula for $m(m_i)$.

Below we collect all these results.

Proposition 3.2. *We have:*

$$l(l_i)=\frac{-1}{2F}l_i;\ \bar l(l_i)=\frac{1}{2F}l_i;\ \ l(m_i)=\frac{1}{2F}m_i;\ \bar l(m_i)=\frac{-1}{2F}m_i \tag{3.6}$$

$$m(l_i)=m^jm^kl_hC^h_{kj}m_i;\ \ \bar m(l_i)=\frac{1}{F}m_i; \tag{3.7}$$

$$m(m_i)=\{m_hm^km^jC^h_{jk}\ -\frac{1}{2}m^kC^j_{jk}\}m_i-\frac{1}{F}l_i;\ \bar m(m_i)=\frac{1}{2}\bar m^kC^{\bar j}_{\bar j\bar k}m_i \tag{3.8}$$

By conjugation everywhere one obtains $l(l_{\bar i})$ and the others.

Also of interest is the actions of l and m vectors under the components l^i or m^i. For instance, we have $l(m^i)=g^{\bar ji}l(\bar m_j)-g^{\bar ji}l^k\dot\partial_k(g_{h\bar j})m^h$ and in account of $0-$homogeneity condition of the metric tensor it follow

$l^k\dot\partial_k(g_{h\bar j})=0$, thus $l(m^i)=\frac{-1}{2F}g^{\bar j i}\bar m_j=\frac{-1}{2F}m^i$ and analogously is deduced that $\bar l(m^i)=\frac{1}{2F}m^i$.

For $m(m^i)$ we have $m(m^i)=g^{\bar j i}m(\bar m_j)-m^km^h\dot\partial_k(g_{h\bar j})g^{\bar j i}$, and now using the second (3.8) formula one obtains $m(m^i)=\frac{1}{2}m^kC^j_{jk}m^i-m^km^jC^i_{jk}$. Similar calculus get the others.

Further, with the formulas from the above Proposition is a straight computation to obtain the other components of the vertical Chern-Finsler connection. For instance:

$D_lm=D_l(m^i\dot\partial_i)=l(m^i)\dot\partial_i-l^km^iC^j_{ik}\dot\partial_j=l(m^i)\dot\partial_i$,

this again in account of $0-$homogeneity condition of the metric tensor.

The calculus of D_mm is similar, but we use the obtained formula for $m(m^i)$.

Proposition 3.3. *Finally we have:*

$$\begin{aligned}(3.9)\qquad D_lm&=\frac{-1}{2F}m;\quad D_{\bar l}m=\frac{1}{2F}m;\\ D_mm&=\frac{1}{2}m^kC^j_{jk}m;\quad D_{\bar m}m=\frac{-1}{F}l\ -\frac{1}{2}\bar m^kC^{\bar j}_{\bar j\bar k},\end{aligned}$$

and their conjugates.

The (3.5), (3.9) formulas furnish the action of the vertical Chern-Finsler connection with respect to Miron-Berwald frame. Certainly, a complete image about Chern-Finsler connection could be obtained if is known also its action under the horizontal Miron-Berwald frame. Part of these are easy to deduce if we take innto account the fact that it is one normal connection (see [Mu]), but the others involve too much calculus which will be avoided here.

Now let us propose calculate the curvature tensors of vertical $C-F$ connection with respect to the Miron-Berwald frame. First, from (1.4) we have $S^i_{j\bar kh}=-\dot\partial_{\bar k}C^i_{jh}$. On the other hand the vertical curvature decomposes l and m directions, that is $S^i_{j\bar kh}\dot\partial_i=S^1_{j\bar kh}l+S^2_{j\bar kh}m$. By using (3.3) it follows

$$\begin{aligned}S^i_{j\bar kh}\dot\partial_i&=-\dot\partial_{\bar k}C^i_{jh}=-(l_{\bar k}\bar l+m_{\bar k}\bar m)\left(C^i_{jh}\right)(l_il+m_im)\\&=-m_{\bar k}\bar m\left(C^i_{jh}\right)l_il-m_{\bar k}\bar m\left(C^i_{jh}\right)m_im.\end{aligned}$$

Hence, we conclude

$$S^1_{j\bar kh}=-m_{\bar k}\bar m\left(C^i_{jh}\right)l_i\ \text{ and }S^2_{j\bar kh}=-m_{\bar k}\bar m\left(C^i_{jh}\right)m_i.$$

Next, the corresponding Riemann tensors are:

$$\begin{aligned}\mathbf{R}(l,\bar{l},l,\bar{l}) &= G(R(l,\bar{l})l,\bar{l}) = l^j l^{\bar{k}} l^h G(S^1_{j\bar{k}h} l + S^2_{j\bar{k}h} m, \bar{l}) \\ &= -l^j l^{\bar{k}} l^h m_{\bar{k}} \bar{m} \left(C^i_{jh}\right) l_i = 0.\end{aligned}$$

Similar we prove that $\mathbf{R}(l,\bar{l},m,\bar{m}) = \mathbf{R}(l,\bar{m},l,\bar{m}) = \mathbf{R}(m,\bar{l},m,\bar{l}) = 0$. The only nonzero Riemann tensor is

$$\mathbf{R}(m,\bar{m},m,\bar{m}) = -m^j m^h m_i \bar{m} \left(C^i_{jh}\right). \tag{3.10}$$

Following a similar terminology with real case, [Ma], the quantity $\mathbf{I} = -m^j m^h m_i \bar{m}(C^i_{jh})$ will call the *main scalar curvature*.

Since the Miron-Berwald frame is orthonormal, we may conclude:

Theorem 3.1. *The vertical holomorphic sectional curvature $K^v_{F,l}(z,\eta)$ in l direction of a two-dimensional complex Finsler space is vanishing and in m direction is*

$$K^v_{F,m}(z,\eta) = 2\mathbf{I}.$$

Indeed, the calculus of the main scalar curvature could be expanded yet. Also we can continue with the study of the bisectional curvature in l and m directions (see [Al1]).

But one problem of interest here is to apply the obtained results to some relevant classes of two-dimensional complex Finsler spaces. The other important problem concern the action $C-F$ connection on whole complexified tangent bundle and after then to apply the (1.6) definition of holomorphic sectional curvature in η direction. These issues will be completed in a coming study.

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