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ABOUT SYMMETRIES AND CONSERVATION LAWS IN LAGRANGIAN AND HAMILTONIAN k-SYMPLECTIC FORMALISM

BY

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Dedicated to Academician Radu Miron at his 80th anniversary

Abstract. The aim of this paper is to extend the study of symmetries and conservation laws from Classical Mechanics to the first-order classical field theories, both for the Lagrangian and Hamiltonian k-symplectic formalisms. More exactly, we will obtain new kinds of conservation laws for k-symplectic Hamiltonian systems, without the help of a Noether type theorem, only using symmetries and pseudosymmetries.

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1. Introduction.

The k-symplectic formalism is the generalization to the field theories of the standard symplectic formalism in Mechanics, which is the geometric framework for describing autonomous dynamical systems. Like in the classical case, the k-symplectic formalism allow us to study toghether the Lagrangian and the Hamiltonian formalisms for field theories (using the Legendre transformation or the k-tangent structure ([18], [19], [20], [30], [35])). So, many results obtained by Crampin ([7], [8], [9]), Grifone ([14]), Klein ([17]), de Leon ([18], [19], [20]), Miron and Anastasiei ([22], [23], [24], [25], [26], [27]), Noether ([31]), for the symplectic formalism was already extend or can be extend to the k-symplectic case.

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More that, in the last ten years a new Lagrangian and Hamiltonian formalism was introduced by R. Miron. So, R. Miron defined the higher order Lagrangians and higher order Hamiltonians and made a very complete study of the geometries of the higher order Lagrange and higher order Hamilton spaces ([22], [23], [24], [25], [26], [27] and many other papers of MIRON). The geometric framework created by this high order geometries allows us to study the properties of some very interesting Hamiltonian dynamical systems and, particularly, allows us to obtain new kinds of conservation laws for this (see [10], [29]).

The aim of this paper is to extend the study of symmetries and conservation laws from Classical Mechanics to the first-order classical field theories, both for the Lagrangian and Hamiltonian formalisms, using Günther's k-symplectic description, and considering only the regular case. The study of symmetries and conservation laws for k-symplectic Hamiltonian systems is, like in the classical case, a topic of great interest and was developped recently by SALGADO, ROMAN-ROY and VILARINO in [35], by generalization of the Noether's type theorem for Hamiltonian and Lagrangian systems in k-symplectic field theories (see [32], [9]). In [35] the Noether's theorem, obtained for a k-symplectic Hamiltonian system, associates conservation laws to so-called Cartan symmetries. However, these kinds of symmetries do not exhaust the set of symmetries. As is known, in mechanics and physics there are symmetries which are not of Cartan type, and which generate conserved quantities, i.e. conservation laws (see [21], [33], [34] for some examples).

In the present paper we will obtain new kinds of conservation laws for k-symplectic Hamiltonian systems, without the help of a Noether type theorem and without the use of a variational principle, using only symmetries and pseudosymmetries associated to the k-vector fileds $\mathbf{X} = (X_1, \dots, X_k)$ which are solutions of the equation $\sum_{A=1}^{k} i_{X_A}\omega_A = dH$. The main result is a generalization from the classical (symplectic) case ($k = 1$) of a results of JONES ([16]) and CRÂŞMĂREANU ([10]). Applications for Lagrangian and Hamiltonian k-symplectic formalisms are also presented.

The structure of the paper is the following: after this introduction, in the second section we present the main notions and tools necessarly in the third section. Two very interesting examples of k-symplectic Lagrangian and Hamiltonian systems are presented. In the last section we enounce and prove the main result and, finally, we present some applications for k-symplectic Lagrangian and Hamiltonian systems.

All manifolds are real, paracompact, connected and C^∞. All maps are

C^∞. Sum over crossed repeated indices is understood. The Lagrangian and Hamiltonian functions are regular.

2. Geometric framework. In this section we present, shortly, the geometric elements necessarily for obtain and explain the results from the next section.

2.1. The tangent bundle of k^1-velocities of a manifold M. An almost tangent structure J on a $2n$-dimensional manifold M is tensor field of type $(1,1)$ of constant rank n such that $J^2 = 0$. The manifold M is then called an almost tangent manifold. Almost tangent structures were introduced by Clark and Bruckheimer [5] and Eliopoulos [13] around 1960 and have been studied by many authors (see [3], [6], [7], [8], [14], [19]).

The canonical model of these structures is the tangent bundle $\tau_M : TM \to M$ of an arbitrary manifold M. The canonical tangent structure J on TM is locally given by

$$J = \frac{\partial}{\partial v^i} \otimes dx^i \tag{1}$$

with respect the bundle coordinates on TM. This tensor J can be regarded as the vertical lift of the identity tensor on M to TM ([28]).

The almost k-tangent structures were introduced as generalization of the almost tangent structures [18, 19].

Definition 2.1. *An almost k-tangent structure J on a manifold M of dimension $n+kn$ is a family $(J^1,\dots,J^k)$ of tensor fields of type $(1,1)$ such that*

$$J^A \circ J^B = J^B \circ J^A = 0, \quad rank\, J^A = n, \quad Im\, J^A \cap (\oplus_{B\neq A} Im\, J^B) = 0, \tag{2}$$

for $1 \le A, B \le k$. In this case the manifold M is then called an almost k-tangent manifold.

The canonical model of these structures is the k-tangent vector bundle $T^1_k M = J^1_0(\mathbf{R}^k, M)$ of an arbitrary manifold M, that is the vector bundle with total space the manifold of 1-jets of maps with source at $0 \in \mathbf{R}^k$ and with projection map $\tau : T^1_k M \to M$, $\tau(j^1_0\sigma) = \sigma(0)$. This bundle is also known as the *tangent bundle of k^1-velocities* of M [28].

The manifold T^1_kM can be canonically identified with the Whitney sum of k copies of TM, that is

$$\begin{array}{rcl} T^1_kM & \equiv & TM\oplus\cdots\oplus TM, \\ j^1_0\sigma & \equiv & (j^1_0\sigma_1 = v_1,\ldots,j^1_0\sigma_k = v_k) \end{array}$$

where $\sigma_A{=}\sigma(0,\ldots,t,\ldots,0)$ with $t\in\mathbf{R}$ at position A and $v_A{=}(\sigma_A)_*(0)(\frac{d}{dt}{}_0)$.

If (x^i) are local coordinates on $U\subseteq M$ then the induced local coordinates (x^i,v^i_A), $1\le i\le n$, $1\le A\le k$, on $\tau^{-1}(U)\equiv T^1_kU$ are given by

$$x^i(j^1_0\sigma) = x^i(\sigma(0)), \qquad v^i_A(j^1_0\sigma) = \frac{d}{dt}(x^i\circ\sigma_A)|_{t=0} = v_A(x^i)\,.$$

Definition 2.2. *For a vector X_x at M we define its vertical A-lift $(X_x)^A$ as the vector on T^1_kM given by*

$$(X_x)^A(j^1_0\sigma) = \frac{d}{dt}((v_1)_x,\ldots,(v_{A-1})_x,(v_A)_x+tX_x,(v_{A+1})_x\ldots,(v_k)_x)|_{t=0} \in T_{j^1_0\sigma}(T^1_kM)$$

for all points $j^1_0\sigma\equiv((v_1)_x,\ldots,(v_k)_x))\in T^1_kM$.

In local coordinates we have $(X_x)^A = \sum_{i=1}^n a^i\frac{\partial}{\partial v^i_A}$, for a vector $X_x = a^i\,\partial/\partial x^i$. The *canonical vertical vector fields* C^A_B on T^1_kM are defined by $C^A_B(x,X_1,X_2,\ldots,X_k) = (X_B)^A$ and are locally given by $C^A_B = v^i_B\frac{\partial}{\partial v^i_A}$.

The *canonical k-tangent structure* $(J^1,\ldots,J^k)$ on T^1_kM is defined by $J^A(Z_{j^1_0\sigma}) = (\tau_*(Z_{j^1_0\sigma}))^A$, for all vectors $Z_{j^1_0\sigma}\in T_{j^1_0\sigma}(T^1_kM)$. In local coordinates we have

$$J^A = \frac{\partial}{\partial v^i_A}\otimes dx^i \tag{3}$$

The tensors J^A can be regarded as the $(0,\ldots,1_A,\ldots,0)$-lift of the identity tensor on M to T^1_kM defined in [28].

2.2. The cotangent bundle of k^1-covelocities of M and $(T^1_k)^*M$. Almost cotangent structures were introduced by Bruckheimer [4]. An *almost cotangent structure* on a $2m$-dimensional manifold M consists of a pair (ω,V) where ω is a symplectic form and V is a distribution such that

$$(i)\quad \omega\rfloor_{V\times V} = 0, \qquad (ii)\quad \ker\omega = \{0\}$$

The canonical model of this structure is the cotangent bundle $\tau_M^* : T^*M \to M$ of an arbitrary manifold M, where ω is the canonical symplectic form $\omega_0 = -d\theta_0$ on T^*M and V is the vertical distribution. The Liouville form θ_0 in T^*M is defined by $\theta_0(\alpha)(\tilde{X}_\alpha) = \alpha((\tau_M^*)_*(\alpha)(\tilde{X}_\alpha))$, for all vectors $\tilde{X}_\alpha \in T_\alpha(T^*M)$. In local coordinates (x^i, p_i) on T^*M

$$\theta_0 = p_i dx^i, \qquad \omega_0 = dx^i \wedge dp_i, \qquad V = \langle \frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial p_k} \rangle. \tag{4}$$

Definition 2.3. *[1, 2] A k-symplectic structure on a manifold M of dimension $N = n + kn$ is a family $(\omega_A, V; 1 \leq A \leq k)$, where each ω_A is a closed 2-form and V is an nk-dimensional distribution on M such that*

$$(i) \quad \omega_{A_{\rfloor V\times V}} = 0, \qquad (ii) \quad \cap_{A=1}^k \ker \omega_A = \{0\}.$$

In this case (M, ω_A, V) is called a k-symplectic manifold.

The canonical model of this structure is the k-cotangent bundle $(T_k^1)^*M = J^1(M, \mathbb{R}^k)_0$ of an arbitrary manifold M, that is the vector bundle with total space the manifold of 1-jets of maps with target at $0 \in \mathbf{R}^k$, and projection $\tau^*(j^1_{x,o}\sigma) = x$.

The manifold $(T_k^1)^*M$ can be canonically identified with the Whitney sum of k copies of T^*M, say

$$\begin{array}{rcl} (T_k^1)^*M & \equiv & T^*M \oplus \cdots \oplus T^*M, \\ j_{x,0}\sigma & \equiv & (j^1_{x,0}\sigma^1, \dots, j^k_{x,0}\sigma^k) \end{array}$$

where $\sigma^A = \pi_A \circ \sigma : M \longrightarrow \mathbb{R}$ is the A-th component of σ.

The canonical k-symplectic structure $(\omega_0)_A, V; 1 \leq A \leq k)$, on $(T_k^1)^*M$ is defined by

$$(\omega_0)_A = (\tau_A^*)^*(\omega_0)$$
$$V(j^1_{x,0}\sigma) = \ker(\tau^*)_*(j^1_{x,0}\sigma)$$

where $\tau_A^* = (T_k^1)^*M \to T^*M$ is the projection on the A^{th}-copy T^*M of $(T_k^1)^*M$, and ω_0 is the canonical symplectic structure of T^*M.

One can also define the 2-forms ω_A by $\omega_A = -d\theta_A$ where $(\theta_0)^A = (\tau_A^*)^*\theta_0$

If (x^i) are local coordinates on $U \subseteq M$ then the induced local coordinates (x^i, p_i^A), $1 \leq i \leq n$, $1 \leq A \leq k$ on $(T_k^1)^*U = (\tau^*)^{-1}(U)$ are given by

$$x^i(j^1_{x,0}\sigma) = x^i(x), \qquad p_i^A(j^1_{x,0}\sigma) = d_x\sigma^A \left(\left. \frac{\partial}{\partial x^i} \right|_x \right).$$

Then the canonical k-symplectic structure is locally given by

$$(\omega_0)^A = \sum_{i=1}^{n} dx^i \wedge dp_i^A, \qquad V = \langle \frac{\partial}{\partial p_i^1}, \dots, \frac{\partial}{\partial p_i^k} \rangle \quad 1 \leq A \leq k\,. \tag{5}$$

and $(\theta_0)^A = p_i^A \, dx^i$.

2.3. Second order partial differential equations on $T_k^1 M$. Let M be an arbitrary manifold and $\tau : T_k^1 M \longrightarrow M$ its tangent bundle of k^1-velocities.

Definition 2.4. *A section* $\mathbf{X} : M \longrightarrow T_k^1 M$ *of the projection* τ *will be called a* k***-vector field*** *on* M.

Since $T_k^1 M$ can be canonically identified with the Whitney sum $T_k^1 M \equiv TM \oplus \cdots \oplus TM$ of k copies of TM, we deduce that a k-vector field $\mathbf{X}$ defines a family of vector fields $X_1, \dots, X_k$ on M. Günther in [15] introduce the following definition.

Definition 2.5. *An* ***integral section*** *of the* k*-vector field* $\mathbf{X} = (X_1, \dots, X_k)$ *passing through a point* $x \in M$ *on* M *is a map* $\phi : U_0 \subset \mathbb{R}^k \to M$, *defined on some neighborhood* U_0 *of* $0 \in \mathbb{R}^k$, *such that*

$$\phi(0) = x, \quad \phi_*(t) \left(\frac{\partial}{\partial t^A} \right) = X_A(\phi(t)) \quad \forall t \in U, \quad 1 \leq A \leq k,$$

or equivalently, ϕ *satisfies*

$$X \circ \phi = \phi^{(1)}, \tag{6}$$

where $\phi^{(1)}$ *is the first prolongation of* ϕ *defined by*

$$\begin{array}{rccl} \phi^{(1)} : & U_0 \subset \mathbb{R}^k & \longrightarrow & T_k^1 M \\ & t & \longrightarrow & \phi^{(1)}(t) = j_0^1 \phi_t \end{array}$$

where $\phi_t(s) = \phi(s+t)$ *for all* $t, s \in \mathbb{R}^k$ *such that* $s + t \in U_0$.

In local coordinates,

$$\phi^{(1)}(t^1, \dots, t^k) = (\phi^i(t^1, \dots, t^k), \frac{\partial \phi^i}{\partial t^A}(t^1, \dots, t^k)), 1 \leq A \leq k\,, 1 \leq i \leq n\,. \tag{7}$$

Definition 2.6. *We say that a k-vector field $\mathbf{X} = (X_1, \dots, X_k)$ on M is* ***integrable*** *if there is an integral section passing through each point of M.*

Remark. If ϕ is an integral section of a k-vector field $(X_1, \dots, X_k)$ then each curve on M defined by $\phi_A = \phi \circ h_A$, where $h_A : \mathbb{R} \to \mathbb{R}^k$ is the natural inclusion $h_A(t) = (0, \dots, t, \dots, 0)$, is an integral curve of the vector field X_A on M, with $1 \leq A \leq k$. We refer to [11, 12] for a discussion on the existence of integral sections.

Definition 2.7. *A k-vector field on $T^1_k M$, that is, a section $\xi \equiv (\xi_1, \dots, \xi_k) : T^1_k M \to T^1_k(T^1_k M)$ of the projection $\tau_{T^1_k M} : T^1_k(T^1_k M) \to T^1_k M$, is a* ***Second Order Partial Differential Equation (SOPDE)*** *if and only if it is also a section of the vector bundle $T^1_k(\tau) : T^1_k(T^1_k M) \to T^1_k M$, where $T^1_k(\tau)$ is defined by $T^1_k(\tau)(j^1_0 \sigma) = j^1_0(\tau \circ \sigma)$.*

Let (x^i) be a coordinate system on M and (x^i, v^i_A) the induced coordinate system on $T^1_k M$. From the definition we deduce that the local expression of a SOPDE ξ is

$$\xi_A(x^i, v^i_A) = v^i_A \frac{\partial}{\partial x^i} + (\xi_A)^i_B \frac{\partial}{\partial v^i_B}, \quad 1 \leq A \leq k. \tag{8}$$

We recall that the first prolongation $\phi^{(1)}$ of $\phi : U \subset \mathbb{R}^k \to M$ is defined by

$$\begin{array}{rccl} \phi^{(1)}: & U \subset \mathbb{R}^k & \longrightarrow & T^1_k M) \\ & t & \longrightarrow & \phi^{(1)}(t) = j^1_0 \phi_t \end{array}$$

where $\phi_t(s) = \phi(s + t)$ for all $t, s \in \mathbb{R}$.

Proposition 2.8. *Let ξ an integrable k-vector field on $T^1_k M$. The necessary and sufficient condition for ξ to be a Second Order Partial Differential Equation (SOPDE) is that its integral sections are first prolongations $\phi^{(1)}$ of maps $\phi : \mathbb{R}^k \to M$. That is*

$$\xi_A(\phi^{(1)}(t)) = (\phi^{(1)})_*(t)(\frac{\partial}{\partial t_A})(t)$$

for all $A = 1, \dots, k$. These maps ϕ will be called solutions of the SOPDE ξ.

From (7) and (8) we have

Proposition 2.9. *$\phi : \mathbb{R}^k \to M$ is a solution of the SOPDE $\xi = (\xi_1, \dots, \xi_k)$, locally given by (8), if and only if*

$$\frac{\partial \phi^i}{\partial t^A}(t) = v^i_A(\phi^{(1)}(t)), \qquad \frac{\partial^2 \phi^i}{\partial t^A \partial t^B}(t) = (\xi_A)^i_B(\phi^{(1)}(t)).$$

If $\xi : T^1_k M \to T^1_k T^1_k M$ is an integrable SOPDE then for all integral sections $\sigma : U \subset \mathbb{R}^k \to T^1_k M$ we have $(\tau_M \circ \sigma)^{(1)} = \sigma$, where $\tau : T^1_k M \to M$ is the canonical projection.

Now we show how to characterize the SOPDEs using the canonical k-tangent structure of $T^1_k M$.

Definition 2.10. ***The Liouville (or canonical) vector field*** *C on $T^1_k M$ is the infinitesimal generator of the one parameter group*

$$\begin{array}{rcl} \mathbb{R} \times (T^1_k M) & \longrightarrow & T^1_k M \\ (s, (x^i, v^i_B)) & \longrightarrow & (x^i, e^s\, v^i_B)\,. \end{array}$$

Thus C is locally expressed as follows:

$$C = \sum_B C_B = \sum_{i,B} v^i_B \frac{\partial}{\partial v^i_B}, \tag{9}$$

where each C_B corresponds with the canonical vector field on the B-th copy of TM on $T^1_k M$.

From (3), (8) and (9) we deduce the following

Proposition 2.11. *A k-vector field $\xi = (\xi_1, \dots, \xi_k)$ on $T^1_k M$ is a SOPDE if and only if*

$$J^A(\xi_A) = C_A, \qquad \forall\, 1 \le A \le k,$$

where $(J^1, \dots, J^k)$ is the canonical k-tangent structure on $T^1_k M$.

2.4. Hamiltonian and Lagrangian formalisms. The role played by symplectic manifolds in classical mechanics is here played by the k-symplectic manifolds (see Günther, [15]). Let $(M, \omega_A, V; 1 \le A \le k)$ be a

k–symplectic manifold. Let us consider the vector bundle morphism defined by GÜNTHER ([15]):

$$
\begin{array}{rccl}
\Omega^\sharp : & T^1_k M & \longrightarrow & T^*M \\
& (X_1, \dots, X_k) & \longrightarrow & \Omega^\sharp(X_1, \dots, X_k) = \displaystyle\sum_{A=1}^{k} i_{X_A}\,\omega_A \,.
\end{array}
\tag{10}
$$

Definition 2.12. *Let $H : M \longrightarrow \mathbb{R}$ be a function on M. Any k-vector field $(X_1, \dots, X_k)$ on M such that*

$$\Omega^\sharp(X_1, \dots, X_k) = dH$$

will be called an evolution k-vector field on M associated with the Hamiltonian function H.

It should be noticed that in general the solution to the above equation is not unique. Nevertheless, it can be proved [20] that there always exists an *evolution k-vector field* associated with a Hamiltonian function H.

We denote by $\mathcal{X}^k_H(M)$ the set of k-vector fields $\mathbf{X} = (X_1, \dots, X_k)$ on M which are the solutions of the equation

$$\sum_{A=1}^{k} i_{X_A}\,\omega_A \;= dH. \tag{11}$$

Let (x^i, p^A_i) be a local coordinate system on M. Then we have

Proposition 2.13. *If $\mathbf{X} = (X_1, \dots, X_k)$ is an integrable evolution k-vector field associated to H, i.e. $\mathbf{X} \in \mathcal{X}^k_H(M)$, then its integral sections*

$$
\begin{array}{rccl}
\sigma : & \mathbb{R}^k & \longrightarrow & M \\
& (t^B) & \longrightarrow & (\sigma^i(t^B), \sigma^A_i(t^B)),
\end{array}
$$

are solutions of the classical local Hamilton equations associated with a regular multiple integral variational problem [36]:

$$\frac{\partial H}{\partial x^i} = -\sum_{A=1}^{k} \frac{\partial \sigma^A_i}{\partial t^A}, \quad \frac{\partial H}{\partial p^A_i} = \frac{\partial \sigma^i}{\partial t^A}, \quad 1 \leq i \leq n,\, 1 \leq \; A \leq k\,. \tag{12}$$

If we consider the canonical k-symplectic structure $((\omega_0)_A, V; 1 \leq A \leq k)$ on $(T^1_k)^*M$ and $H : (T^1_k)^*M \to \mathbb{R}$ be a Hamiltonian function on $(T^1_k)^*M$, then the family $\big((T^1_k)^*M, (\omega_0)_A, H\big)$ is called **a k-symplectic Hamiltonian system** and the equations (12) are called **the Hamilton-de Donder-Weyl equations** associated to this system.

Example 1. ([30]) We shall use the above formalism to obtain an intrinsic version for the electrostatic equations. Let us consider $\mathbf{R}^3$ with a metric g with components g_{ij}. Let $\sigma : \mathbf{R}^3 \to \mathbf{R}$ be the electric potential and $P = (P_1, P_2, P_3) : \mathbf{R}^3 \to \mathbf{R}^3$ the electric field. We denote by (t^1, t^2, t^3) the standard coordinates on $\mathbf{R}^3$ and we set $\sqrt{g} = \sqrt{\det g_{ij}}$. By $r(t)$ we denote the scalar function which gives the density of the electric charge on $\mathbf{R}^3$. In this example we suppose that $r(t)$ is constant, $r(t) = r$, that is the distribution of the electric charge is constant on $\mathbf{R}^3$ and, also, we suppose that the metric g on $\mathbf{R}^3$ is the Euclidian metric.

Let us consider on $M = (T^1_3)^*\mathbf{R}$ the canonical polysymplectic structure $((\omega_0)_1, (\omega_0)_2, (\omega_0)_3)$. We denote by (q, p^1, p^2, p^3) the local coordinates on $M = (T^1_3)^*\mathbf{R}$ induced by the standard coordinates (q) on $\mathbf{R}$, and we define a Hamiltonian function $H : (T^1_3)^*\mathbf{R} \to \mathbf{R}$ by $H(q, p^1, p^2, p^3) = 4\pi r q + \frac{1}{2}\sum_{A=1}^{3}(p^A)^2$.

Consider the equation

$$\Omega^\sharp(X_1, X_2, X_3) = \sum_{A=1}^{3} i_{X_A}(\omega_0)_A = dH, \tag{13}$$

where (X_1, X_2, X_3) is a 3-vector field on $(T^1_3)^*\mathbf{R}$.

Let $\phi : \mathbf{R}^3 \to (T^1_3)^*\mathbf{R}$, $\phi(t) = (\psi(t), \psi^1(t), \psi^2(t), \psi^3(t))$ be an integral section of an evolution 3-vector field which is a solution of (13). Then we obtain the Hamilton-de Donder-Weyl equations associated to this 3-symplectic Hamiltonian system:

$$4\pi r = -\left(\frac{\partial\psi^1}{\partial t^1} + \frac{\partial\psi^2}{\partial t^2} + \frac{\partial\psi^3}{\partial t^3}\right), \quad \psi^A = \frac{\partial\psi}{\partial t^A}, \; A = 1, 2, 3,$$

which are the electrostatic equations, and then the components $\psi(t)$ and $(\psi^1(t), \psi^2(t), \psi^3(t))$ of ϕ are the electric potential σ and the electric field $P = (P_1, P_2, P_3)$ on $\mathbf{R}^3$, respectively. So, the equation (13) is a geometric version of the electrostatic equations.

Next, if we consider a Lagrangian function $L : T^1_k M \to \mathbb{R}$, $L = L(x^i, v^i_A)$, then we obtain, by using a variational principle, the *generalized Euler-Lagrange equations* for L:

$$\sum_{A=1}^{k} \frac{d}{dt^A}\left(\frac{\partial L}{\partial v^i_A}\right) - \frac{\partial L}{\partial x^i} = 0, \qquad v^i_A = \frac{\partial x^i}{\partial t^A}. \tag{14}$$

Following the ideas of GÜNTHER [15], we will describe the above equations (14) in terms of the geometry of k-tangent structures. In classical mechanics the symplectic structure of Hamiltonian theory and the tangent structure of Lagrangian theory play complementary roles [7], [8], [14], [19]. Also, that the k-symplectic structures and the k-tangent structures play similarly complementary roles. So, we construct a k–symplectic structure on the manifold $T^1_k M$, using its canonical k–tangent structure for each $1 \leq A \leq k$.

Let us consider the 1–forms $(\beta_L)_A = dL \circ J^A$, $1 \leq A \leq k$. In a local coordinate system (x^i, v^i_A) we have

$$(\theta_L)_A = \frac{\partial L}{\partial v^i_A} dx^i, \ 1 \leq A \leq k. \tag{15}$$

Definition 2.14. *A Lagrangian L is called **regular** if*

$$det\left(\frac{\partial^2 L}{\partial v^i_A \partial v^j_B}\right) \neq 0, \qquad 1 \leq i, j, \leq n, \quad 1 \leq A, B \leq k\,. \tag{16}$$

By introducing the following 2–forms

$$(\omega_L)_A = -d(\theta_L)_A\,,\ 1 \leq A \leq k, \tag{17}$$

one can easily prove the following.

Proposition 2.15. *$L : T^1_k M \longrightarrow \mathbb{R}$ is a regular Lagrangian if and only if $((\omega_L)_1, \dots, (\omega_L)_k, V)$ is a k-symplectic structure on $T^1_k M$, where V denotes the vertical distribution of $\tau : T^1_k M \to M$.*

Let $L : T^1_k M \longrightarrow \mathbb{R}$ be a regular Lagrangian and let us consider the k–symplectic structure $((\omega_L)_1, \dots, (\omega_L)_k, V)$ on $T^1_k M$ defined by L. Let $\Omega^\sharp_L$ be the morphism defined by this k–symplectic structure

$$\Omega^\sharp_L : T^1_k(T^1_k M) \longrightarrow T^*(T^1_k M).$$

Thus, we can set the following equation:

$$\Omega_L^\sharp(X_1,\dots,X_k)=dE_L, \tag{18}$$

where $E_L=C(L)-L=\sum\limits_{i,B} v_B^i\dfrac{\partial L}{\partial v_A^i}-L.$

The family $\left(T_k^1M,(\omega_L)_A,E_L\right)$ is called **a k-symplectic Lagrangian system**. As in the Hamiltonian case, we will denote by $\mathcal{X}_L^k(T_k^1M)$ the set of k-vector fields $\xi=(\xi_1,\cdots,\xi_k)$ on T_k^1M which are the solutions of the equation (18).

Proposition 2.16. *Let L be a regular Lagrangian. If $\xi=(\xi_1,\cdots,\xi_k)$ is a solution of (18) then it is a SOPDE. In addition if ξ is integrable then the solutions of ξ are solutions of the generalized Euler-Lagrange equations (14).*

Proof. It is a direct computation in local coordinates using (8), (9) , (15) and (16). □

Remark. This k-symplectic structure, associated to a regular Lagrangian L, was also introduced by GÜNTHER ([15]) using the Legendre transformation, as follow.

The Legendre map $FL:T_k^1M\to(T_k^1)^*M$ was introduced by GÜNTHER ([15]) and was rewritten in [30] as follow: if $(v_{1x},\dots,v_{kx})\in(T_k^1)_xM$, then

$$\left[FL(v_{1x},\dots,v_{kx})\right]^A(u_x)=\frac{d}{ds}\,|_{s=0}\;L(v_{1x},\dots,v_{Ax}+su_x,\dots,v_{kx}), \tag{19}$$

for each $A=1,\dots,k$ and $u_x\in T_xM$. Locally, FL is given by

$$FL(x^i,v_A^i)=\left(x^i,\frac{\partial L}{\partial v_A^i}\right). \tag{20}$$

In fact, from (15), (17) and (20) we obtain the following lemma and proposition:

Lemma 2.17. *For every $A=1,\dots,k$, $(\omega_L)_A=(FL)^*(\omega_0)_A$, where $((\omega_0)_1,\dots,(\omega_0)_k)$ are the 2-forms of the canonical k-symplectic structure on $(T_k^1)^*M$.*

Proposition 2.18. *Let L be a Lagrangian. The following conditions are equivalent:*

(1) *L is regular.*
(2) *FL is a local diffeomorphism.*
(3) *$((\omega_L)_1, \ldots, (\omega_L)_k, V)$ is a k-symplectic structure on $T^1_k M$.*

Example 2. ([30]) In this example we consider the theory of a vibrating string. Coordinates (t^1, t^2) are interpreted as the time and the distance along the string, respectively. If $\phi(t^1, t^2)$ denotes the displacement of each point of the string as function of the time t^1 and the position t^2, the motion equations are

$$\sigma \frac{\partial^2 \phi}{\partial (t^1)^2} - \tau \frac{\partial^2 \phi}{\partial (t^2)^2} = 0, \tag{21}$$

where σ and τ are certain constants of the mechanical system.

We shall show that the equations (21) can be described as the generalized Euler-Lagrange equations associated to a Lagrangian L defined on the jet bundle $T^1_k M$ with $M = \mathbf{R}$ and $k = 2$. Let us denote by (x, v_1, v_2) the coordinates on $T^1_2 \mathbf{R}$ and consider the Lagrangian $L : T^1_2 \mathbf{R} \to \mathbf{R}$, $(x, v_1, v_2) \to \frac{1}{2}(\sigma v_1^2 - \tau v_2^2)$.

Since L is regular there exists a k-symplectic structure $((\omega_L)_1, (\omega_L)_2)$, associated to L, given in local coordinates by $(\omega_L)_1 = \sigma dv_1 \wedge dx$, $(\omega_L)_1 = -\tau dv_2 \wedge dx$. The energy $E_L = C(L) - L$ is locally given by $E_L = \frac{1}{2}(\sigma v_1^2 - \tau v_2^2)$ and $dE_L = \sigma v_1 dv_1 - \tau v_2 dv_2$.

Now, we consider the map $\Omega^\sharp_L : T^1_2(T^1_2 \mathbf{R}) \longrightarrow T^*(T^1_2 \mathbf{R})$ and let us suppose that there exists (ξ_1, ξ_2) a solution of the equation

$$\Omega^\sharp_L(\xi_1, \xi_2) = i_{X_1}(\omega_L)_1 + i_{X_2}(\omega_L)_2 = dE_L. \tag{22}$$

Then, from the Proposition (2.16), we know that (ξ_1, ξ_2) is a SOPDE. Let us suppose that $(\xi_1, \xi_2) \in T^1_2(T^1_2 \mathbf{R})$ are locally given by $\xi_A = v_A \frac{\partial}{\partial x} + (\xi_A)_1 \frac{\partial}{\partial v_1} + (\xi_A)_2 \frac{\partial}{\partial v_2}, \quad A = 1, 2$.

If (ξ_1, ξ_2) is a solution of $\Omega^\sharp_L(\xi_1, \xi_2) = dE_L$, then we have $\sigma(\xi_1)_1 - \tau(\xi_2)_2 = 0$. So, if we consider $\phi : \mathbf{R}^2 \to \mathbf{R}$, $\phi = \phi(t^1, t^2)$, a solution of $\xi = (\xi_1, \xi_2)$, then we obtain

$$0 = \sigma(\xi_1)_1 - \tau(\xi_2)_2 = \sigma \frac{\partial^2 \phi}{\partial (t^1)^2} - \tau \frac{\partial^2 \phi}{\partial (t^2)^2}.$$

Thus, the equation (22) is a geometric version for the equations (21).

3. Main result and applications to k-symplectic Hamiltonian and Lagrangian systems. In this section we will present a result which allow us to obtain new kinds of conservation laws for k-symplectic Hamiltonian systems, without the help of a Noether type theorem and without the use of a variational principle, using only symmetries and pseudosymmetries associated to the k-vector fileds $\mathbf{X} = (X_1, \ldots, X_k)$ which are solutions of the equation $\sum_{A=1}^{k} i_{X_A}\omega_A = dH$. This result is a generalization from the classical case ($k = 1$) of a results of Jones ([16]) and Crâşmăreanu ([10]). Applications for Lagrangian and Hamiltonian k-symplectic formalisms are also presented.

First, let us introduce the notions:

Definition 3.1. *The map $\mathbf{\Phi} = (\Phi_1, \ldots, \Phi_k) : M \longrightarrow \mathbf{R}^k$ is called* ***conservation law*** *for a k-vector field $\mathbf{X} = (X_1, \ldots, X_k)$ on M if*

$$\sum_{A=1}^{k} L_{X_A}\Phi_A = 0. \tag{23}$$

Definition 3.2. *A k-vector field $\mathbf{Y} = (Y_1, \ldots, Y_k)$ on M is called* ***symmetry*** *for $\mathbf{X} = (X_1, \ldots, X_k)$ if*

$$L_{X_A}Y_A = 0, \ \ for \ all \quad A = 1, \ldots, k. \tag{24}$$

Definition 3.3. *If we fixed a k-vector field $\mathbf{Y} = (Y_1, \ldots, Y_k)$ on M, then a k-vector field $\mathbf{Z} = (Z_1, \ldots, Z_k)$ is called* ***$\mathbf{Y}$-pseudosymmetry*** *for $\mathbf{X} = (X_1, \ldots, X_k)$ if, for all $A = 1, \ldots, k$, there is a function $f_A \in C^\infty(M)$ such that*

$$L_{X_A}Z_A = f_A\ Y_A. \tag{25}$$

A $\mathbf{X}$-pseudosymmetry for $\mathbf{X}$ is called **pseudosymmetry** for $\mathbf{X}$. It's clear that a $\mathbf{O}$-pseudosymmetry for $\mathbf{X} = (X_1, \ldots, X_k)$ is a symmetry for $\mathbf{X}$.

Theorem 3.4. *Let be $\mathbf{X} = (X_1, \ldots, X_k)$ a k-vector field on M and $(\omega_1, \ldots, \omega_k)$ be a family of p-forms on M, invariant for $\mathbf{X}$, i.e. $L_{X_A}\omega_A = 0$, for all $A = 1, \ldots, k$. If the k-vector field $\mathbf{Y} = (Y_1, \ldots, Y_k)$ on M is a symmetry for $\mathbf{X}$ and the $p-1$ k-vector fields $\mathbf{S}^1 = (S^1_A)_{A=\overline{1,k}}$, ..., $\mathbf{S}^{p-1} = (S^{p-1}_A)_{A=\overline{1,k}}$ are $\mathbf{Y}$-pseudosymmetries for $\mathbf{X}$, then*

$$\mathbf{\Phi} = (\Phi_1, \ldots, \Phi_k)\,, \tag{26}$$

is a conservation law for $\mathbf{X} = (X_1, \dots, X_k)$*, where* $\Phi_A = \omega_A(S^1_A, \dots, S^{p-1}_A, Y_A)$*, for all* $A = 1, \dots, k$*.*

Particularly, if $\mathbf{Y}$*,* $\mathbf{S}^1$*, ...,* $\mathbf{S}^{p-1}$ *are symmetries for* $\mathbf{X}$ *then* $\mathbf{\Phi}$ *given by* (26) *is a conservation law for* $\mathbf{X} = (X_1, \dots, X_k)$*.*

Proof. Applying the properties of the Lie derivative and taking into account that, locally, $\Phi_A = (S^1_A)^{i_1} \cdots (S^{p-1}_A)^{i_{p-1}} (Y_A)^{i_p} \omega_{i_1 \dots i_{p-1} i_p}$, for all $A = 1, \dots, k$, we obtain that

$$\begin{aligned}
\sum_{A=1}^{k} L_{X_A} \Phi_A &= \sum_{A=1}^{k} L_{X_A} \left(\left(S^1_A\right)^{i_1} \cdots \left(S^{p-1}_A\right)^{i_{p-1}} (Y_A)^{i_p} (\omega_A)_{i_1 \dots i_{p-1} i_p} \right) \\
&= \sum_{A=1}^{k} \left(L_{X_A} S^1_A\right)^{i_1} \left(S^2_A\right)^{i_2} \cdots \left(S^{p-1}_A\right)^{i_{p-1}} (Y_A)^{i_p} (\omega_A)_{i_1 \dots i_{p-1} i_p} \\
&+ \dots + \sum_{A=1}^{k} \left(S^1_A\right)^{i_1} \cdots \left(S^{p-2}_A\right)^{i_{p-2}} \left(L_{X_A} S^{p-1}_A\right)^{i_{p-1}} (Y_A)^{i_p} (\omega_A)_{i_1 \dots i_{p-1} i_p} \\
&+ \sum_{A=1}^{k} \left(S^1_A\right)^{i_1} \cdots \left(S^{p-1}_A\right)^{i_{p-1}} (L_{X_A} Y_A)^{i_p} (\omega_A)_{i_1 \dots i_{p-1} i_p} \\
&+ \sum_{A=1}^{k} \left(S^1_A\right)^{i_1} \cdots \left(S^{p-1}_A\right)^{i_{p-1}} (Y_A)^{i_p} (L_{X_A} \omega_A)_{i_1 \dots i_{p-1} i_p} \\
&= \sum_{A=1}^{k} f_A (Y_A)^{i_1} \left(S^2_A\right)^{i_2} \cdots \left(S^{p-1}_A\right)^{i_{p-1}} (Y_A)^{i_p} (\omega_A)_{i_1 \dots i_{p-1} i_p} \\
&+ \dots + \sum_{A=1}^{k} \left(S^1_A\right)^{i_1} \cdots \left(S^{p-2}_A\right)^{i_{p-2}} f_A (Y_A)^{i_{p-1}} (Y_A)^{i_p} (\omega_A)_{i_1 \dots i_{p-1} i_p} = 0,
\end{aligned}$$

because $L_{X_A} Y_A = 0$, $L_{X_A} \omega_A = 0$ and taking into account of the antisymmetry of p-forms ω_A, $A = 1, \dots, k$. $\square$

As an immediate consequence of the previous theorem, we have the result:

Theorem 3.5. *Let* $(M, \omega_A, V; 1 \leq A \leq k)$ *be a* k*–symplectic manifold and* $H : M \longrightarrow \mathbb{R}$ *be a function on* M*. Let* $\mathbf{X} = (X_1, \dots, X_k)$ *be an integrable evolution* k*-vector field associated to* H*, i.e.* $\mathbf{X} \in \mathcal{X}^k_H(M)$*. If we* *suppose that* $L_{X_A}\omega_A = 0$, *for all* $A = 1, \dots, k$, *then for any* k*-vector field* $\mathbf{Y} = (Y_1, \dots, Y_k)$ *on* M, *which is a symmetry for* $\mathbf{X}$ *and for any* k*-vector field* $\mathbf{S} = (S_1, \dots, S_k)$, *which is a* $\mathbf{Y}$*-pseudosymmetry for* $\mathbf{X}$, *we have that*

$$\mathbf{\Phi} = (\Phi_1, \dots, \Phi_k),$$

is a conservation law for $\mathbf{X} = (X_1, \dots, X_k)$, *where* $\Phi_A = \omega_A(S_A, Y_A)$, *for all* $A = 1, \dots, k$.

Particularly, if $\mathbf{Y}$, $\mathbf{S}$ *are symmetries for* $\mathbf{X}$ *then* $\mathbf{\Phi}$ *given above is a conservation law for* $\mathbf{X} = (X_1, \dots, X_k)$.

Remarks. a) Obviously, for any k-vector field $\mathbf{X} \in \mathcal{X}_H^k(M)$, using (11), we have $\sum_{A=1}^{k} L_{X_A}\omega_A = 0$. But, for our purpose we have need more that, that is we need that $L_{X_A}\omega_A = 0$, for all $A = 1, \dots, k$.

b) Obviously, the Hamiltonian function H is not a conservation law for an integrable evolution k-vector field $\mathbf{X} = (X_1, \dots, X_k) \in \mathcal{X}_H^k(M)$. Neither the map $\mathbf{H} = (H, \dots, H) : M \longrightarrow \mathbf{R}^k$ is not a conservation law for any integrable evolution k-vector field $\mathbf{X} \in \mathcal{X}_H^k(M)$, because $\sum_{A=1}^{k} L_{X_A}H \neq 0$.

Now, using this last result we can obtain new kinds of conservation laws for k-symplectic Lagrangian systems and k-symplectic Lagrangian systems.

Corollary 3.6. *Let* $\left((T_k^1)^*M, (\omega_0)_A, H\right)$ *be a* k*-symplectic Hamiltonian system and* $\mathbf{X} = (X_1, \dots, X_k)$ *be an integrable evolution* k*-vector field associated to* H, *i.e.* $\mathbf{X} \in \mathcal{X}_H^k((T_k^1)^*M)$. *If we suppose that* $L_{X_A}(\omega_0)_A = 0$, *for all* $A = 1, \dots, k$, *then for any* k*-vector field* $\mathbf{Y} = (Y_1, \dots, Y_k)$ *on* $(T_k^1)^*M$, *which is a symmetry for* $\mathbf{X}$ *and for any* k*-vector field* $\mathbf{S} = (S_1, \dots, S_k)$, *which is a* $\mathbf{Y}$*-pseudosymmetry for* $\mathbf{X}$, *we have that*

$$\mathbf{\Phi} = (\Phi_1, \dots, \Phi_k),$$

is a conservation law for $\mathbf{X} = (X_1, \dots, X_k)$, *where* $\Phi_A = (\omega_0)_A(S_A, Y_A)$, *for all* $A = 1, \dots, k$.

Particularly, if $\mathbf{Y}$, $\mathbf{S}$ *are symmetries for* $\mathbf{X}$ *then* $\mathbf{\Phi}$ *given above is a conservation law for* $\mathbf{X} = (X_1, \dots, X_k)$.

Corollary 3.7. *Let* $\left(T_k^1M, (\omega_L)_A, E_L\right)$ *be a* k*-symplectic Lagrangian system and* $\mathbf{X} = (X_1, \dots, X_k)$ *be an integrable evolution* k*-vector field associated to* H, *i.e.* $\mathbf{X} \in \mathcal{X}_L^k(T_k^1M)$. *If we suppose that* $L_{X_A}(\omega_L)_A = 0$, *for all* $A = 1, \dots, k$, *then for any* k*-vector field* $\mathbf{Y} = (Y_1, \dots, Y_k)$ *on* T_k^1M, *which* *is a symmetry for* $\mathbf{X}$ *and for any* k*-vector field* $\mathbf{S} = (S_1, \ldots, S_k)$*, which is a* $\mathbf{Y}$*-pseudosymmetry for* $\mathbf{X}$*, we have that*

$$\mathbf{\Phi} = (\Phi_1, \ldots, \Phi_k)\,,$$

is a conservation law for $\mathbf{X} = (X_1, \ldots, X_k)$*, where* $\Phi_A = (\omega_L)_A\,(S_A, Y_A)$*, for all* $A = 1, \ldots, k$*.*

Particularly, if $\mathbf{Y}$*,* $\mathbf{S}$ *are symmetries for* $\mathbf{X}$ *then* $\mathbf{\Phi}$ *given above is a conservation law for* $\mathbf{X} = (X_1, \ldots, X_k)$*.*

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