ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" DIN IAŞI (S.N.)
MATEMATICĂ, Tomul LIII, 2007, Supliment

ON THE GEOMETRIZATION OF THE NONHOLONOMIC MECHANICAL SYSTEM

BY

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Dedicated to Academician Radu Miron at his 80th anniversary

Abstract. In this paper we study the geometrical problem of general nonholonomic mechanical systems. We associate to the system a semispray on the phases space TM and we use the differential geometry of Lagrange spaces to study the nonholonomic mechanical systems. An example of nonholonomic mechanical systems is pointed out.

Mathematics Subject Classification 2000: 53C60, 58F05, 70H05.
Key words: Lagrangian systems, semispray, nonholonomic constraints.

1. Introduction. Holonomic mechanical systems where geometrically defined by the Riemannian space determined by kinetic energy and were studied by Levi-Civita in 1922. Nonholonomic systems were geometrically characterized by Vranceanu in 1926 who introduced the notion of nonholonomic Riemann space.

We study the geometrical problem of general nonholonomic mechanical system

$$\sum = (M, L(x,y), F_i(x,y), Q_\sigma(x,y)) \tag{1.1}$$

from the Lagrangian geometries view point. Here M is a real differentiable manifold of dimension n, $L^n = (M, L(x,y))$ is a Lagrange space, $F_i(x,y)$ are external forces given so that $F_i(x,y)$ are the components on TM of a d-covector field and Pfaff equations $Q_\sigma(x,dx) = a_{\sigma i}(x)dx^i = 0\,(\sigma = m+1,...,n)$ are the kinematic constrains of the system.

In his Ph.D. Thesis (1956) [2] Radu Miron resolved the general case in which for the system $Q_\sigma = 0$ exist a number $r < m$ of derived subsystems.

E. Cartan considered this case unreachable because of the calculating difficulties.

The evolution of the system $\sum$ is described by the Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y^i}\right) - \frac{\partial L}{\partial x^i} = \sum_{\sigma=m+1}^{n} \lambda^\sigma(x) a_{\sigma i}(x) + F_i(x,y), y^i = \frac{dx^i}{dt} \tag{1.2}$$

and by the Pfaff equations system

$$Q_\sigma(x, dx) = a_{\sigma i}(x) dx^i = 0. \tag{1.3}$$

In this paper we present the Lagrange geometry of the Lagrangian nonholonomic mechanical system $\sum$, associating to it a canonical semispray S^*, whose integral curves are given by the evolution equations of $\sum$ and in addition they satisfy the equations $Q_\sigma(x, \frac{dx}{dt}) = 0$. The Lagrange geometry of $(S^*, Q_\sigma = 0)$ represents the Lagrange geometry of the system $\sum$ on the phases space.

2. Lagrangian nonholonomic mechanical systems. Let $\sum$ (1.1) be a Lagrangian nonholonomic, mechanical system with the evolution equations (1.2), (1.3). $\sum$ determines in a canonic way a semispray S^* on the phases space TM, which we will study next.

We denote by $g_{ij}(x,y)$ the fundamental tensor of the Lagrange space $L^n = (M, L(x,y))$ and $g^{ij}(x,y)$ its contravariant part. As it is known [3], $g_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j}$, $rank\|g_{ij}\| = n$ on $\tilde{TM} := TM \setminus 0$ and g_{ij} has constant signature.

External forces $F_i(x,y)$ determine a d-covariant vector field and

$$F_{ij} = \frac{\partial F_j}{\partial y^i} - \frac{\partial F_i}{\partial y^j} \tag{2.1}$$

is a skew-symmetric d-tensor field, named elicoidal tensor of system $\sum$. The functions that determine the constrains of the system

$$Q_\sigma(x,y) = a_{\sigma i}(x) y^i, \ (\sigma = m+1, ...n) \tag{2.2}$$

are scalars with respect to the changes of the coordinates on TM. So, $a_{\sigma i}(x)$ are n-m covariant vector fields on M and

$$\sum_{\sigma=m+1}^{n} \lambda^{\sigma}(x) Q_{\sigma}(x,y) \tag{2.2'}$$

is also a scalar function on TM.

Let be the Lagrangian

$$L^{*}(x,y) = L(x,y) + \sum_{\sigma=m+1}^{n} \lambda^{\sigma}(x) Q_{\sigma}(x,y). \tag{2.3}$$

We have $L^{*}\left(x, \frac{dx}{dt}\right) = L\left(x, \frac{dx}{dt}\right)$ on the distribution $Q_{\sigma}(x,dx) = 0$. The Lagrangians L^{*} and L are equivalent if they satisfy simultaneously the Euler-Lagrange equations $\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial y^i} = 0$ and have the same fundamental tensor. The second condition is obviously satisfied.

Theorem 2.1 *The Lagrangians L^{*} and L are equivalent if and only if the functions $\lambda(x)$ satisfy the system of equations*

$$\frac{\partial \lambda^{\sigma}}{\partial x^{i}} a_{\sigma i} - \frac{\partial \lambda^{\sigma}}{\partial x^{j}} a_{\sigma i} + \lambda^{\sigma}\left(\frac{\partial a_{\sigma i}}{\partial x^{i}} - \frac{\partial a_{\sigma i}}{\partial x^{j}}\right) = 0. \tag{2.4}$$

We observe that the system (2.4) is equivalent with the fact that 1-form $\sum\limits_{\sigma}(\lambda^{\sigma}(x)Q_{\sigma}(x,dx))$ is closed. Indeed, the equations (2.4) are equivalent with exterior equations

$$\sum_{\sigma} d(\lambda^{\sigma}(x) Q_{\sigma}(x,dx)) = 0. \tag{2.4'}$$

So, we admit that: The evolution equations of the Lagrangian nonholonomic mechanical system $\sum$ are given by (1.2), where the multipliers $\lambda^{\sigma}(x)$ satisfy the equations (2.4').

The Lagrange equations (1.2) can be written in the following form

$$\frac{d^2 x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = \frac{1}{2}\left[F^i\left(x, \frac{dx}{dt}\right) + \sum_{\sigma} \lambda^{\sigma}(x) a^{i}_{\sigma}(x,dx)dt\right], \tag{2.5}$$

where

$$2G^i(x,y) = \frac{1}{2} g^{ij}(x,y)\left[\frac{\partial^2 L}{\partial y^j \partial x^k} y^k - \frac{\partial L}{\partial x^j}\right] \tag{2.6}$$

$$F^i(x,y) = g^{ij}(x,y)F_j(x,y); a_{\sigma i}(x,y) = g^{ij}(x,y)a_{\sigma j}(x). \tag{2.7}$$

As $G^i(x,y)$ are coefficients of a semispray S determined by the Lagrangian $L(x,y)$, and the second member from (2.5) is a d-vector field, it follows:

Theorem 2.2 *The system of functions:*

$$2G^{*i}(x,y) = 2G^i(x,y) - \frac{1}{2}\left[F^i(x,y) + \sum_\sigma \lambda^\sigma(x)a^i_\sigma(x,y)\right] \tag{2.8}$$

are the coefficients of a semispray S^* determined only by the Lagrangian nonholonomic mechanical systems $\sum$, the parameters $\lambda^\sigma(x)$ satisfying (2.4).

Obviously S^* is given by:

$$S^* = y^i\frac{\partial}{\partial x^i} - 2G^{*i}(x,y) = S + \frac{1}{2}[F^i + \sum_\sigma \lambda^\sigma a^i_\sigma(x,y)]\frac{\partial}{\partial y^i}. \tag{2.9}$$

The semispray S^* with the coefficients G^{*i} in (2.8) and λ^σ verifying (2.4) will be called the canonic semispray of the system $\sum$. If $S, F^i, \lambda^\sigma a^i_\sigma$ are vector fields defined on the phases space TM, then semispray S^* is globally defined on the manifold $\tilde{TM}$. The vector field S^* defines a dynamic system on TM that has the following important property:

Theorem 2.3 *The integral curves of the canonic semispray S^* are given by the evolution equations of the system $\sum$* (2.5), (2.4′).

Proof. The integral curves of S^* are given by $\frac{dx^i}{dt} = y^i, \frac{dy^i}{dt} + 2G^{*i}(x,y) = 0$ λ^σ verifying (2.4') and G^{*i} from (2.8). But these equations are equivalent to (2.5), (2.4'). □

We define Lagrange geometry of system $\sum$ as being Lagrange geometry on the phases space TM of the canonic semispray S^*.

3. An example of nonholonomic mechanical systems. Consider a "free particle" on $\mathbb{R}^2$ which moves along curve angular coefficient of which is proportional to time passed from the beginning of the motion.

We denote by (t) the coordinate on $X = \mathbb{R}$, by (t,x,y) the fiber coordinates on $Y = \mathbb{R} \times \mathbb{R}^2$ and associated coordinates $(t,x,y,\dot{x},\dot{y})$ on $J^1Y = \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2$. The Lagrangian of this problem has the simple form

$$\lambda = Ldt = \frac{1}{2}(\dot{x}^2 + \dot{y}^2)dt \tag{3.1}$$

and we consider a first order mechanical system $[\alpha]$ on the fibered manifold $\mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$, related with a dynamical form

$$E = (A_\sigma + B_{\sigma\rho}\tilde{q}^\rho)dq^\rho \wedge dt = -m\ddot{x}dx \wedge dt - m\ddot{y} \wedge dt, \tag{3.2}$$

where components $A_\sigma, B_{\sigma\rho}$ are in this case $A_\sigma = 0$, $B_{\sigma\rho} = -m\delta_{\sigma\rho}$, $\sigma, \rho = 1, 2$. Associated Lepage 2-form α is then expressed by

$$\alpha = -mw^1 \wedge d\dot{x} - mw^2 \wedge d\dot{y}, \tag{3.3}$$

where $w^1 = dx - \dot{x}dt, w^2 = dy - \dot{y}dt.$

The motion of the mechanical system $[\alpha]$ is for $t > 0$ subjected to one nonholonomic constraint Q given by equation

$$f(t, x, y, \dot{x}, \dot{y}) \equiv kt\dot{x} - \dot{y} = 0, \tag{3.4}$$

or equivalently in normal form $\dot{y} = g(t, x, y, \dot{x}) = ky\dot{x}$, where k is a positive constant. This equation defines constraint submanifold $Q \subset J^1Y$. The constraint 1-form (16) takes the form

$$\phi = ktdx - dy. \tag{3.5}$$

The constrained system $[\alpha_Q]$ related to the mechanical system $[\alpha]$ and the constraint Q is the equivalence class of the 2-forms

$$\alpha_Q = A'_1 w^1 \wedge dt + B'_{11} w^1 \wedge d\dot{x} + F + \phi_{(2)} \tag{3.6}$$

on Q, where $A'_1 = -mk^2t\dot{x}$, $B'_{11} = -m(1 + k^2t^2)$ and F is any 2-contact 2-form and $\phi_{(2)}$ is any constraint 2-form. The matrix (B'_{ls}) is only the number $(-m(1 + k^2t^2))$, which is always nonzero, it means, that the constrained system $[\alpha_Q]$ is always regular.

The equation of motion of the constrained system (reduced equation) is

$$[mk^2t\dot{x} + m(1 + k^2t^2)\ddot{x}] \circ J^2\tilde{\tau} = 0 \tag{3.7}$$

for the Q-admissible sections $\tilde{\tau} = (t, x(t), y(t))$ (i.e. for the sections satisfying the constraint equation $f \circ J^1\tau = 0$). Equation (3.7) is easy solvable: using substitution $u = \dot{x}$ and after separation of variables we obtain

$$u(t) = \dot{x}(t) = \frac{v_0}{\sqrt{1 + k^2t^2}}, \tag{3.8}$$

and substituting to the equation (3.4) of the constraint

$$\dot{y}(t) = \frac{kv_0 t}{\sqrt{1+k^2t^2}}. \tag{3.9}$$

Integrating (3.8) and (3.9) we get general solution of the considered problem in the parametric form: $x(t) = x_0 + \frac{v_0}{k} ln(kt + \sqrt{1+k^2t^2})$, $y(t) = y_0 + \frac{v_0}{k}(kt + \sqrt{1+k^2t^2})$, where v_0 is the initial velocity and x_0, y_0 are coordinates of the initial position of the particle.

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Received: 15.X.2007

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