ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" DIN IAŞI (S.N.)
MATEMATICĂ, Tomul LIII, 2007, Supliment

ASPECTS OF LIE ALGEBROID GEOMETRY AND HAMILTONIAN FORMALISM

BY

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Dedicated to Academician Radu Miron at his 80th anniversary

Abstract. In this paper we study some geometrical structures on the prolongation of Lie algebrioid over the vector bundle projection of a dual bundle. We continue the investigation of nonlinear connection and its properties starting in [4]. Some results known in the case of cotangent bundle are extended to Lie algebroid.

Mathematics Subject Classification 2000: 17B66, 53C05.
Key words: Lie algebroid, connection, Hamiltonian formalism.

1. Introduction. The cotangent bundle plays a very important role in symplectic geometry and its applications, since this carries a canonical symplectic structure induced by the Liouville form. The notion of prolongation of Lie algebroid over the vector bundle projection of the dual bundle generalize the concept of cotangent bundle. WEINSTEIN [12] gives a generalized theory of Lagrangian on Lie algebroids and obtains the equations of motion, using the Poisson structure of the dual and Legendre transformation associated with the regular Lagrangian. The Hamilton equations were later obtained by MARTINEZ [6] using the symplectic formalism and the notion of prolongation of Lie algebra over a smooth mapping introduced by HIGGINS and MACKENZIE [3]. In [5] the Hamiltonian formalism on Lie algebroids is presented.

The purpose of the present paper is to study diverse aspects of Lie algebroid geometry for Hamiltonian dynamics. The paper is organized as

*Partially supported by Grant 26C/ 2007

follows. In the section 2 the known results on Lie algebroid and its prolongation over the vector bundle projection of the dual bundle are presented. In sections 3 we continue the study of connections from [4] and introduce the notions of almost tangent structure and almost complex structure, using a regular Hamiltonian and characterize the integrability conditions of these structures in terms of torsion and curvature. We show that every regular section (in particular, any regular Hamiltonian on E^*) determines a canonical nonlinear connection (see [1], [2], [10] for particular case of tangent and cotangent bundles). In the particular case of standard Lie algebroid ($E = TM$, $\sigma = id$, $L^{\alpha}_{\beta\gamma} = 0$) the canonical nonlinear connection of MIRON [9] is obtained.

2. The notion of Lie algebroid. Let M be a differentiable, n-dimensional manifold and (TM, π_M, M) its tangent bundle. Let (E, π, M) be a vector bundle with m-dimensional fibres. A Lie algebroid over the manifold M is the triple $(E, [\cdot,\cdot], \sigma)$ with a Lie algebra structure $[\cdot,\cdot]$ on its space of sections $\Gamma(E)$ and a map $\sigma : E \to TM$ (called the anchor) which induces a Lie algebra homomorphism (also denoted σ) from sections of E to vector fields on M, satisfying the compatibility conditions

$$[s_1, fs_2] = f[s_1, s_2] + (\sigma(s_1)f)s_2,$$

where $f \in C^{\infty}(M)$ and s_1, s_2 are sections of E. Therefore, we also have the relations

$$[\sigma(s_1), \sigma(s_2)] = \sigma[s_1, s_2], \quad [s_1, [s_2, s_3]] + [s_2, [s_3, s_1]] + [s_3, [s_1, s_2]] = 0.$$

If ω is an element of $\bigwedge^p(E)$ with $p > 0$, then the exterior derivative $d\omega \in \bigwedge^{p+1}(E)$ is given by the formula

$$\begin{aligned} d\omega(s_1, ..., s_{p+1}) &= \sum_{i=1}^{p+1} (-1)^{i+1} \sigma(s_i)\omega(s_1, ..., \hat{s_i}, ..., s_{p+1}) \\ &\quad + \sum_{1 \le i < j \le p+1} (-1)^{i+j} \omega([s_i, s_j], s_1, ..., \hat{s_i}, ..., \hat{s_j}, ...s_{p+1}), \end{aligned}$$

If ξ is a section on E then the *Lie derivative* on Lie algebroid $\mathcal{L}_\xi$ is given by $\mathcal{L}_\xi = i_\xi \circ d + d \circ i_\xi$, where i_ξ is the contraction with ξ. If we take the local coordinates (x^i) on an open $U \subset M$ and a local basis $\{s_\alpha\}$ of sections of

the bundle $\pi^{-1}(U) \to U$, then we have the local coordinates (x^i, y^α) on E. These coordinates determine the local functions $\sigma^i_\alpha(x)$, $L^\gamma_{\alpha\beta}(x)$ on M given by

$$\sigma(s_\alpha) = \sigma^i_\alpha \frac{\partial}{\partial x^i}, \quad [s_\alpha, s_\beta] = L^\gamma_{\alpha\beta} s_\gamma, \quad i = \overline{1,n}, \quad \alpha, \beta, \gamma = \overline{1,m},$$

and satisfying the relations

$$(1) \qquad \sigma^j_\alpha \frac{\partial \sigma^i_\beta}{\partial x^j} - \sigma^j_\beta \frac{\partial \sigma^i_\alpha}{\partial x^j} = \sigma^i_\gamma L^\gamma_{\alpha\beta}, \qquad \sum_{(\alpha,\beta,\gamma)} \left(\sigma^i_\alpha \frac{\partial L^\delta_{\beta\gamma}}{\partial x^i} + L^\delta_{\alpha\eta} L^\eta_{\beta\gamma} \right) = 0,$$

which are called the *structure equations* of Lie algebroid.
Locally, if $f \in C^\infty(M)$ then $df = \frac{\partial f}{\partial x^i} \sigma^i_\alpha s^\alpha$ and if $\theta \in \Gamma(E^*)$, $\theta = \theta_\alpha s^\alpha$ then

$$d\theta = (\sigma^i_\alpha \frac{\partial \theta_\beta}{\partial x^i} - \frac{1}{2} \theta_\gamma L^\gamma_{\alpha\beta}) s^\alpha \wedge s^\beta,$$

where $\{s^\alpha\}$ is the dual basis of $\{s_\alpha\}$. Particularly, $dx^i = \sigma^i_\alpha s^\alpha$ and $ds^\alpha = -\frac{1}{2} L^\alpha_{\beta\gamma} s^\beta \wedge s^\gamma$.

2.1. The prolongation of a Lie algebroids over the vector bundle projection of the dual bundle. Let $\tau : E^* \to M$ be the dual of $\pi : E \to M$ and $(E, [\cdot,\cdot], \sigma)$ a Lie algebroid structure over M. Using [3],[6],[5] we get that the dual of Lie algebroid induces a vector bundle $\tau_1 : \mathcal{T}E^* \to E^*$ with a structure of Lie algebroid. The manifold $\mathcal{T}E^*$ is

$$\mathcal{T}E^* = \{(u, w) \in E \times_{TM} TE^* \mid \ w \in T_u E^*, \ \sigma(u) = T_\tau(w)\}$$

where $T_\tau : TE^* \to TM$ is the tangent application and the projection $\tau_1 : \mathcal{T}E^* \to E^*$ is given by $\tau_1(u, w) = \pi_{E^*}(w)$, where $\pi_{E^*} : TE^* \to E^*$ is the tangent projection. We have also the canonical projection $\tau_2 : \mathcal{T}E^* \to E$ given by $\tau_2(u, w) = u$. The projection onto the second factor $\sigma^1 : \mathcal{T}E^* \to TE^*$, $\sigma^1(u, w) = w$ will be the anchor of a new Lie algebroid over manifold E^*. An element of $\mathcal{T}E^*$ is said to be vertical if it belongs to the kernel of the projection τ_2. We will denote $Ker\tau_2 = V\mathcal{T}E^*$ and $(V\mathcal{T}E^*, \tau_{1|_{V\mathcal{T}E^*}}, E^*)$ is the vertical subbundle of $(\mathcal{T}E^*, \tau_1, E^*)$. A basis of sections of $\mathcal{T}E^*$ is given by

$$\mathcal{X}_\alpha(u^*) = \left(s_\alpha(\tau(u^*)), \sigma^i_\alpha \frac{\partial}{\partial x^i}\Big|_{u^*} \right), \quad \mathcal{P}^\alpha(u^*) = \left(0, \frac{\partial}{\partial \mu_\alpha}\Big|_{u^*} \right),$$

for $u^* \in \tau^{-1}(U)$, where $(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial \mu_\alpha})$ is the local basis on TE^*, and the associated coordinates will be denoted $(x^i, \mu_\alpha, z^\alpha, \omega_\alpha)$. The dual basis is $(\mathcal{X}^\alpha, \mathcal{P}_\alpha)$ and the structure functions of $\mathcal{T}E^*$ are given by the following formulas

$$\sigma^1(\mathcal{X}_\alpha) = \sigma^i_\alpha \frac{\partial}{\partial x^i},\ \sigma^1(\mathcal{P}^\alpha) = \frac{\partial}{\partial \mu_\alpha}, \tag{2}$$

$$[\mathcal{X}_\alpha, \mathcal{X}_\beta] = L^\gamma_{\alpha\beta}\mathcal{X}_\gamma, \quad [\mathcal{X}_\alpha, \mathcal{P}^\alpha] = 0, \quad [\mathcal{P}^\alpha, \mathcal{P}^\beta] = 0. \tag{3}$$

In the Lie algebroid $\mathcal{T}E^*$ there exist the *canonical symplectic structure* given by $\omega_E = -d\theta$ where $\theta = \mu_\alpha \mathcal{X}^\alpha$ is the *Liouville section*. In local coordinates we get

$$\omega_E = \mathcal{X}^\alpha \wedge \mathcal{P}_\alpha + \frac{1}{2}\mu_\alpha L^\alpha_{\beta\gamma}\mathcal{X}^\beta \wedge \mathcal{X}^\gamma. \tag{4}$$

We remark that $V\mathcal{T}E^*$ is Lagrangian for ω_E, i.e. $\omega_E(\rho_1, \rho_2) = 0$, for every $\rho_1, \rho_2 \in \Gamma(V\mathcal{T}E^*)$. The following section on $\mathcal{T}E^*$ given by

$$\mathcal{C} = \mu_\alpha \mathcal{P}^\alpha$$

is called *Liouville-Hamilton section*.
The Liouville-Hamilton section on $\mathcal{T}E^*$ measures the homogeneity of functions and sections. A function $f \in C^\infty(E^*)$ is said to be homogeneous of degree $r \in \mathbb{Z}$ if $\mathcal{L}_\mathcal{C} f = rf$ where $\mathcal{L}_\mathcal{C}$ is the Lie derivation with respect to Liouville-Hamilton section on Lie algebroid. A section ρ of $\mathcal{T}E^*$ is said to be homogeneous of degree $r \in \mathbb{Z}$ if $\mathcal{L}_\mathcal{C}\rho = r\rho$.

3. Connections on Lie algebroid $\mathcal{T}E^*$

Definition 1. *A nonlinear (Ehresmann) connection on $\mathcal{T}E^*$ is an almost product structure $\mathcal{N}$ on $\tau_1 : \mathcal{T}E^* \to E^*$ (i.e. a bundle morphism $\mathcal{N} : \mathcal{T}E^* \to \mathcal{T}E^*$, such that $\mathcal{N}^2 = id$) smooth on $\mathcal{T}E^*\backslash\{0\}$ such that $V\mathcal{T}E^* = \ker(id + \mathcal{N})$.*

If $\mathcal{N}$ is a connection on $\mathcal{T}E^*$ then $H\mathcal{T}E^* = \ker(id - \mathcal{N})$ is the horizontal subbundle associated to $\mathcal{N}$ and $\mathcal{T}E^* = V\mathcal{T}E^* \oplus H\mathcal{T}E^*$. Each $\rho \in \Gamma(\mathcal{T}E^*)$ can be written as $\rho = \rho^h + \rho^v$ where ρ^h, $\rho^v$are sections in the horizontal and respective, vertical subbundles. A connection $\mathcal{N}$ on $\mathcal{T}E^*$ induces two

projectors $h, v : \mathcal{T}E^* \to \mathcal{T}E^*$ such that $h(\rho) = \rho^h$ and $v(\rho) = \rho^v$ for every $\rho \in \Gamma(\mathcal{T}E^*)$. We have

$$h = \frac{1}{2}(id + \mathcal{N}), \quad v = \frac{1}{2}(id - \mathcal{N}), \tag{5}$$

$$\ker h = imv = V\mathcal{T}E^*, \quad imh = \ker v = H\mathcal{T}E^*.$$

Locally, a connection is given by $\mathcal{N}(\mathcal{X}_\alpha) = \mathcal{X}_\alpha + 2\mathcal{N}_{\alpha\beta}\mathcal{P}^\beta$ and $\mathcal{N}(\mathcal{P}^\alpha) = -\mathcal{P}^\alpha$,where $\mathcal{N}_{\alpha\beta} = \mathcal{N}_{\alpha\beta}(x, \mu)$ are the local coefficients of $\mathcal{N}$. The sections

$$\delta^*_\alpha = h(\mathcal{X}_\alpha) = \mathcal{X}_\alpha + \mathcal{N}_{\alpha\beta}\mathcal{P}^\beta, \tag{6}$$

generate a basis of $H\mathcal{T}E^*$. The frame $\{\delta^*_\alpha, \mathcal{P}^\alpha\}$ is a local basis of $\mathcal{T}E^*$ called *adapted*. The dual adapted basis is $\{\mathcal{X}^\alpha, \delta\mathcal{P}_\alpha\}$ where

$$\delta\mathcal{P}_\alpha = \mathcal{P}_\alpha - \mathcal{N}_{\alpha\beta}\mathcal{X}^\beta \tag{7}$$

We remark that the following formula holds

$$\sigma^1\left[\delta^*_\alpha, \delta^*_\beta\right] = \left[\sigma^1(\delta^*_\alpha), \sigma^1(\delta^*_\beta)\right].$$

A connection $\mathcal{N}$ is called *symmetric* if $H\mathcal{T}E^*$ is Lagrangian for ω_E. By a straightforward computation, using (4) and (6) we get

$$\omega_E(\delta^*_\alpha, \delta^*_\beta) = \mathcal{N}_{\alpha\beta} - \mathcal{N}_{\beta\alpha} - \mu_\gamma L^\gamma_{\alpha\beta},$$

and it results that $\mathcal{N}$ is symmetric if and only if

$$\mathcal{N}_{\alpha\beta} - \mathcal{N}_{\beta\alpha} = \mu_\gamma L^\gamma_{\alpha\beta}. \tag{8}$$

Proposition 1. *With respect to a symmetric nonlinear connection, the canonical symplectic structure ω_E can be written in the following form*

$$\omega_E = \mathcal{X}^\alpha \wedge \delta\mathcal{P}_\alpha + \mu_\alpha L^\alpha_{\beta\gamma}\mathcal{X}^\beta \wedge \mathcal{X}^\gamma \tag{9}$$

Proof. Using (4) and (7) we get

$$\omega_E = \mathcal{X}^\alpha \wedge \delta\mathcal{P}_\alpha + \frac{1}{2}(\mathcal{N}_{\alpha\beta} - \mathcal{N}_{\beta\alpha})\mathcal{X}^\alpha \wedge \mathcal{X}^\beta + \frac{1}{2}\mu_\alpha L^\alpha_{\beta\gamma}\mathcal{X}^\beta \wedge \mathcal{X}^\gamma$$

and from (8) we obtain (9). □

Proposition 2. *The Lie brackets of the adapted basis* $\{\delta^*_\alpha, \mathcal{P}^\alpha\}$ *are given by* [4]

$$[\delta^*_\alpha, \delta^*_\beta] = L^\gamma_{\alpha\beta}\delta^*_\gamma + \mathcal{R}_{\alpha\beta\gamma}\mathcal{P}^\gamma,\ [\delta^*_\alpha, \mathcal{P}^\beta] = -\frac{\partial \mathcal{N}_{\alpha\gamma}}{\partial \mu_\beta}\mathcal{P}^\gamma,\ [\mathcal{P}^\alpha, \mathcal{P}^\beta] = 0, \tag{10}$$

where

$$\mathcal{R}_{\alpha\beta\gamma} = \sigma^i_\alpha \frac{\partial \mathcal{N}_{\beta\gamma}}{\partial x^i} - \sigma^i_\beta \frac{\partial \mathcal{N}_{\alpha\gamma}}{\partial x^i} + \mathcal{N}_{\alpha\delta}\frac{\partial \mathcal{N}_{\beta\gamma}}{\partial \mu_\delta} - \mathcal{N}_{\beta\delta}\frac{\partial \mathcal{N}_{\alpha\gamma}}{\partial \mu_\delta} - L^\varepsilon_{\alpha\beta}\mathcal{N}_{\varepsilon\gamma}. \tag{11}$$

Proof. Using (2) and (3) we obtain

$$[\delta^*_\alpha, \delta^*_\beta] = \left(\sigma^i_\alpha \frac{\partial \mathcal{N}_{\beta\gamma}}{\partial x^i} - \sigma^i_\beta \frac{\partial \mathcal{N}_{\alpha\gamma}}{\partial x^i} + \mathcal{N}_{\alpha\delta}\frac{\partial \mathcal{N}_{\beta\gamma}}{\partial \mu_\delta} - \mathcal{N}_{\beta\delta}\frac{\partial \mathcal{N}_{\alpha\gamma}}{\partial \mu_\delta}\right)\mathcal{P}^\gamma + L^\varepsilon_{\alpha\beta}\mathcal{X}_\varepsilon$$

and considering $\mathcal{X}_\varepsilon = \delta^*_\varepsilon - \mathcal{N}_{\varepsilon\gamma}\mathcal{P}^\gamma$ we get $[\delta^*_\alpha, \delta^*_\beta] = L^\gamma_{\alpha\beta}\delta^*_\gamma + \mathcal{R}_{\alpha\beta\gamma}\mathcal{P}^\gamma$. □
The curvature of a connection $\mathcal{N}$ on $\mathcal{T}E^*$is given by $\Omega = -\mathrm{N}_h$ where h is horizontal projector and N_h is the Nijenhuis tensor of h.

Corollary 1. *In the local coordinates we get*

$$\Omega = -\frac{1}{2}\mathcal{R}_{\alpha\beta\gamma}\mathcal{X}^\alpha \wedge \mathcal{X}^\beta \otimes \mathcal{P}^\gamma$$

where $\mathcal{R}_{\alpha\beta\gamma}$ *is given by* (11) *and is called the curvature tensor of* $\mathcal{N}$.

Proof. Since $h^2 = h$ we obtain that $\Omega(h\rho_1, h\rho_2) = -v[h\rho_1, h\rho_2]$, $\Omega(h\rho_1, v\rho_2) = \Omega(v\rho_1, v\rho_2) = 0$ and in local coordinates we get $\Omega(\delta^*_\alpha, \delta^*_\beta) = -v[\delta^*_\alpha, \delta^*_\beta] = -\mathcal{R}_{\alpha\beta\gamma}\mathcal{P}^\gamma$ which concludes the proof. □
The curvature is an obstruction to the integrability of $H\mathcal{T}E^*$, understanding that a vanishing curvature entails that horizontal sections are closed under the Lie algebroid brackets of $\mathcal{T}E^*$.
Let Φ be a morphism of vector bundle E and $\widetilde{E}$. We recall that the connections N on E and $\widetilde{N}$ on $\widetilde{E}$ are Φ-related if $\Phi \circ N = \widetilde{N} \circ \Phi$. We consider the connections $\mathcal{N}$ on $\mathcal{T}E^*$ and N on TE^* which are σ^1-related and the adapted basis $(\delta_i, \frac{\partial}{\partial \mu_\alpha})$ of N is given by $\delta_i = \frac{\partial}{\partial x^i} + N_{i\alpha}\frac{\partial}{\partial \mu_\alpha}$, where $N_{i\beta}$ are the coefficients of N. From [4] we have

Theorem 1. *The following relations hold*

$$\sigma^1(\delta^*_\alpha) = \sigma^i_\alpha \delta_i, \quad \mathcal{N}_{\alpha\beta} = \sigma^i_\alpha N_{i\beta}, \quad \mathcal{R}_{\alpha\beta\gamma} = \sigma^i_\alpha \sigma^j_\beta R_{ij\gamma},$$

where $R_{ij\gamma} = \delta_i(N_{j\gamma}) - \delta_j(N_{i\gamma})$ *is the curvature of connection* N *on* TE^*.

Proof. Since $\mathcal{N}(\sigma^1(\delta^*_\alpha)) = \sigma^1(\delta^*_\alpha)$ the relation $N \circ \sigma^1 = \sigma^1 \circ \mathcal{N}$ leads to $N(\sigma^1(\delta^*_\alpha)) = \sigma^1(\delta^*_\alpha)$. But $N(\sigma^1(\delta^*_\alpha)) = 2\sigma^i_\alpha\delta_i - \sigma^1(\delta^*_\alpha)$ which concludes that $\sigma^1(\delta^*_\alpha) = \sigma^i_\alpha\delta_i$ from which we easily obtain $\mathcal{N}_{\alpha\beta} = \sigma^i_\alpha N_{i\beta}$. By straightforward computation we get

$$\mathcal{R}_{\alpha\beta\gamma} = \sigma^i_\alpha\sigma^j_\beta\left(\delta_i(N_{j\gamma}) - \delta_j(N_{i\gamma})\right) + N_{j\gamma}\left(\sigma^i_\beta\frac{\partial\sigma^j_\alpha}{\partial x^i} - \sigma^i_\alpha\frac{\partial\sigma^j_\beta}{\partial x^i}\right) + L^\varepsilon_{\alpha\beta}\mathcal{N}_{\varepsilon\gamma}$$

and using (1), the second term is given by $N_{j\gamma}\sigma^j_\varepsilon L^\varepsilon_{\beta\alpha} = -N_{\varepsilon\gamma}L^\varepsilon_{\alpha\beta}$ which ends the proof. □

Remark 1. A σ^1-related connection N on TE^* determines a connection $\mathcal{N}$ on $\mathcal{T}E^*$ with the coefficients $\mathcal{N}_{\alpha\beta} = \sigma^i_\alpha N_{i\beta}$ and curvature $\mathcal{R}_{\alpha\beta\gamma} = \sigma^i_\alpha\sigma^j_\beta R_{ij\gamma}$. Converse it is not true, because σ is only injective.

3.1. Hamiltonian dynamics. In the following we consider a regular Hamiltonian $\mathcal{H} : E^* \to \mathbb{R}$, i.e. the matrix

$$g^{\alpha\beta}(x,\mu) = \frac{\partial^2\mathcal{H}}{\partial\mu_\alpha\partial\mu_\beta}$$

is nondegenerate. Then, since ω_E is a symplectic section on $(\mathcal{T}E^*, [\cdot,\cdot], \sigma^1)$ and $d\mathcal{H} \in \Gamma(\mathcal{T}E^*)$, there exists a unique section $\rho_\mathcal{H} \in \Gamma(\mathcal{T}E^*)$ such that

$$i_{\rho_\mathcal{H}}\omega_E = d\mathcal{H},$$

called the *Hamilton section*. With respect to the local basis $\{\mathcal{X}_\alpha, \mathcal{P}^\alpha\}$, the local expression of $\rho_\mathcal{H}$ is

$$\rho_\mathcal{H} = \frac{\partial\mathcal{H}}{\partial\mu_\alpha}\mathcal{X}_\alpha - \left(\sigma^i_\alpha\frac{\partial\mathcal{H}}{\partial x^i} + \mu_\gamma L^\gamma_{\alpha\beta}\frac{\partial\mathcal{H}}{\partial\mu_\beta}\right)\mathcal{P}^\alpha \tag{12}$$

Thus, the vector field $\sigma^1(\rho_\mathcal{H})$ on E^* is given by [5]

$$\sigma^1(\rho_\mathcal{H}) = \sigma^i_\alpha\frac{\partial\mathcal{H}}{\partial\mu_\alpha}\frac{\partial}{\partial x^i} - \left(\sigma^i_\alpha\frac{\partial\mathcal{H}}{\partial x^i} + \mu_\gamma L^\gamma_{\alpha\beta}\frac{\partial\mathcal{H}}{\partial\mu_\beta}\right)\frac{\partial}{\partial\mu_\alpha}$$

and consequently, the integral curves of $\rho_\mathcal{H}$ (i.e. the integral curves of the vector field $\sigma^1(\rho_\mathcal{H})$) satisfy the Hamilton equations

$$\frac{dx^i}{dt} = \sigma^i_\alpha\frac{\partial\mathcal{H}}{\partial\mu_\alpha}, \quad \frac{d\mu_\alpha}{dt} = -\sigma^i_\alpha\frac{\partial\mathcal{H}}{\partial x^i} - \mu_\gamma L^\gamma_{\alpha\beta}\frac{\partial\mathcal{H}}{\partial\mu_\beta} \tag{13}$$

Let us consider a linear mapping $\mathcal{J}:\mathcal{T}E^*\to\mathcal{T}E^*$ locally given by

$$\mathcal{J}(\mathcal{X}_\alpha)=g_{\alpha\beta}\mathcal{P}^\beta,\quad \mathcal{J}(\mathcal{P}^\alpha)=0,$$

where $g^{\alpha\beta}g_{\alpha\gamma}=\delta^\beta_\gamma$ and we obtain

Theorem 2. *The following properties hold:*
1) $\mathcal{J}$ *is globally defined on* $\mathcal{T}E^*$.
2) $im\mathcal{J}=\ker\mathcal{J}=V\mathcal{T}E^*,\mathcal{J}\mathcal{C}=0$
3) $\mathcal{J}$ *is a almost tangent structure,* $\mathcal{J}\circ\mathcal{J}=0$.
4) $\mathcal{J}$ *is an integrable structure.*

Proof. The first relations result by straightforward computation. For 4) we consider the Nijenjuis tensor $N_{\mathcal{J}}(\rho,\nu)=[\mathcal{J}\rho,\mathcal{J}\nu]-\mathcal{J}[\mathcal{J}\rho,\nu]-\mathcal{J}[\rho,\mathcal{J}\nu]$ and we have

$$N_{\mathcal{J}}(\mathcal{X}_\alpha,\mathcal{X}_\beta)=\left(g_{\alpha\gamma}\frac{\partial g_{\beta\varepsilon}}{\partial\mu^\gamma}-g_{\beta\gamma}\frac{\partial g_{\alpha\varepsilon}}{\partial\mu^\gamma}\right)\mathcal{P}^\varepsilon,\ N_{\mathcal{J}}(\mathcal{X}_\alpha,\mathcal{P}^\beta)=N_{\mathcal{J}}(\mathcal{P}^\alpha,\mathcal{P}^\beta)=0.$$

From $g_{\alpha\gamma}g^{\alpha\beta}=\delta^\beta_\gamma$ we get $\frac{\partial g_{\alpha\gamma}}{\partial\mu_\varepsilon}g^{\alpha\beta}=-g_{\alpha\gamma}\frac{\partial g^{\alpha\beta}}{\partial\mu_\varepsilon}=-g_{\alpha\gamma}\frac{\partial g^{\alpha\varepsilon}}{\partial\mu_\beta}=\frac{\partial g_{\alpha\gamma}}{\partial\mu_\beta}g^{\alpha\varepsilon}$, so $\frac{\partial g_{\alpha\gamma}}{\partial\mu_\varepsilon}g^{\alpha\beta}=\frac{\partial g_{\alpha\gamma}}{\partial\mu_\beta}g^{\alpha\varepsilon}$ and, by multiplication with $g_{\varepsilon\delta}$ it results $\frac{\partial g_{\alpha\gamma}}{\partial\mu_\varepsilon}g^{\alpha\beta}g_{\varepsilon\delta}=\frac{\partial g_{\delta\gamma}}{\partial\mu_\beta}$ or, by multiplication with $g_{\beta\theta}$, leads to $\frac{\partial g_{\theta\gamma}}{\partial\mu_\varepsilon}g_{\varepsilon\delta}=\frac{\partial g_{\delta\gamma}}{\partial\mu_\varepsilon}g_{\varepsilon\theta}$. □
From $ker\mathcal{J}=im\mathcal{J}=V\mathcal{T}E^*$ we get

$$\mathcal{J}h=\mathcal{J},\ h\mathcal{J}=0,\ \mathcal{J}v=0,\ v\mathcal{J}=\mathcal{J}$$

and using the fact that $\mathcal{N}=2h-id$, it results

Remark 2. Let $\mathcal{N}$ be a bundle morphism of $\tau_1:\mathcal{T}E^*\to E^*$, smooth on $\mathcal{T}E^*\backslash\{0\}$. Then $\mathcal{N}$ is a connection on$\mathcal{T}E^*$ if and only if there exists an almost tangent structure $\mathcal{J}$ on $\mathcal{T}E^*$such that

$$\mathcal{J}\mathcal{N}=\mathcal{J},\quad \mathcal{N}\mathcal{J}=-\mathcal{J}.$$

Now, we express the non-homogeneity of nonlinear connection.

Definition 2. *The tension of a nonlinear connection* $\mathcal{N}$ *is a section* $\mathcal{T}$ *on* $\mathcal{T}E^*$ *given by the relation*

$$\mathcal{T}=\frac{1}{2}\mathcal{L}_{\mathcal{C}}\mathcal{N}$$

where $\mathcal{L}$ *is Lie derivative on Lie algebroid.*

In the local coordinates we get the expression

$$\mathcal{T} = \left(\frac{\partial N_{\alpha\beta}}{\partial \mu_\gamma} \mu_\gamma - N_{\alpha\beta} \right) \mathcal{X}^\alpha \otimes \mathcal{P}^\beta$$

A nonlinear connection on $\mathcal{T}E^*$ is said to be homogeneous if $\mathcal{N}$ is homogeneous of degree 1, and therefore the tension $\mathcal{T}$ vanishes. A homogeneous nonlinear connection $\mathcal{N}$ is said to be linear connection if $\mathcal{N}$ is C^1 on the null section. If $\mathcal{N}$ is a linear connection, then the function $\mathcal{N}_{\alpha\beta}$ are linear on μ, then we may write $\mathcal{N}_{\alpha\beta}(x,\mu) = \Gamma^\gamma_{\alpha\beta}(x)\mu_\gamma$.

Definition 3. *The torsion of a connection $\mathcal{N}$ is the vector valued two form $t = [\mathcal{J}, h]$ where h is horizontal projector and $[\mathcal{J}, h]$ is the Frolicher-Nijenhuis bracket.*

By direct computation we get [4]

Proposition 3. *t is a semibasic vector-valued form. Its local expression is*

$$t = \frac{1}{2} t_{\alpha\beta\gamma} \mathcal{X}^\alpha \wedge \mathcal{X}^\beta \otimes \mathcal{P}^\gamma,$$

where

$$t_{\alpha\beta\gamma} = g_{\alpha\epsilon} \frac{\partial \mathcal{N}_{\beta\gamma}}{\partial \mu_\varepsilon} - g_{\beta\epsilon} \frac{\partial \mathcal{N}_{\alpha\gamma}}{\partial \mu_\varepsilon} + \delta^*_\alpha(g_{\beta\gamma}) - \delta^*_\beta(g_{\alpha\gamma}) - L^\epsilon_{\alpha\beta} g_{\epsilon\gamma}.$$

Let us consider the linear mapping $\mathcal{F} : \mathcal{T}E^* \to \mathcal{T}E^*$ given by

$$\mathcal{F}(h\rho) = \mathcal{J}\rho, \quad \mathcal{F}(\mathcal{J}\rho) = -h\rho,$$

where $\rho \in \Gamma(\mathcal{T}E^*)$, and h is the horizontal projector of connection $\mathcal{N}$.

Corollary 2. *The mapping $\mathcal{F}$ has the properties:*
a) *$\mathcal{F}$ is an almost complex structure $\mathcal{F} \circ \mathcal{F} = -id$.*
b) *Locally it is given by*

$$\mathcal{F} = g_{\alpha\beta} \mathcal{P}^\beta \otimes \mathcal{X}^\alpha - g^{\alpha\beta} \delta^*_\alpha \otimes \delta \mathcal{P}_\beta$$

Proof. It results by definition that

$$(\mathcal{F} \circ \mathcal{F})(h\rho) = \mathcal{F}(\mathcal{J}(\rho)) = -h\rho, \quad (\mathcal{F} \circ \mathcal{F})(\mathcal{J}\rho) = \mathcal{F}(-h\rho) = -\mathcal{J}\rho,$$

and

$$\mathcal{F}(\delta^*_\alpha) = g_{\alpha\beta} \mathcal{P}^\beta, \quad \mathcal{F}(\mathcal{P}^\alpha) = -g^{\alpha\beta} \delta^*_\beta,$$

which concludes the proof. □

Proposition 4. *The almost complex structure is integrable if and only if the torsion and curvature vanish*

$$t_{\alpha\beta\gamma} = 0, \quad \mathcal{R}_{\alpha\beta\gamma} = 0.$$

Proof. Let $\mathbf{N}_{\mathcal{F}}$ be the Nijenjuis tensor of the almost complex structure. We get

$$\mathbf{N}_{\mathcal{F}}(\delta^*_\alpha, \delta^*_\beta) = \mathbf{N}^\gamma_{\alpha\beta}\delta^*_\gamma + \mathbf{N}_{\alpha\beta(\gamma)}\mathcal{P}^\gamma,$$

$$\mathbf{N}_{\mathcal{F}}(\delta^*_\alpha, \mathcal{P}^\beta) = \mathbf{N}^{(\beta)\gamma}_{\alpha}\delta^*_\gamma + \mathbf{N}^{(\beta)}_{\alpha(\gamma)}\mathcal{P}^\gamma,$$

$$\mathbf{N}_{\mathcal{F}}(\mathcal{P}^\alpha, \mathcal{P}^\beta) = -g^{\varepsilon\alpha}g^{\beta\tau}\mathbf{N}_{\mathcal{F}}(\delta^*_\alpha, \delta^*_\beta),$$

where

$$\mathbf{N}^\gamma_{\alpha\beta} = g^{\varepsilon\gamma}t_{\alpha\beta\varepsilon}, \quad \mathbf{N}_{\alpha\beta(\gamma)} = g_{\alpha\varepsilon}\frac{\partial g_{\beta\gamma}}{\partial \mu_\varepsilon} - g_{\beta\varepsilon}\frac{\partial g_{\alpha\gamma}}{\partial \mu_\varepsilon} - \mathcal{R}_{\alpha\beta\gamma},$$

$$\mathbf{N}_{\alpha\beta(\gamma)} = \mathbf{N}^{(\varepsilon)\tau}_{\alpha} g_{\varepsilon\beta}g_{\gamma\tau}, \quad \mathbf{N}^\gamma_{\alpha\beta} = \mathbf{N}^{(\varepsilon)}_{\alpha(\tau)}g^{\tau\gamma}g_{\beta\varepsilon},$$

which ends the proof. □

Let us consider the section

$$\rho = \frac{\partial \mathcal{H}}{\partial \mu_\alpha}\mathcal{X}_\alpha + \rho_\alpha \mathcal{P}^\alpha$$

which is called the *semi-Hamilton section.* We have

Proposition 5. *The semi-Hamilton section satisfies the relation*

$$\mathcal{J}[\rho, \mathcal{J}\nu] = -\mathcal{J}\nu, \tag{14}$$

for every section ν of $\mathcal{T}E^*$.

Proof. For $\nu = \nu^\alpha \mathcal{X}_\alpha + \nu_\alpha \mathcal{P}^\alpha$ we get $\mathcal{J}\nu = \nu^\alpha g_{\alpha\beta}\mathcal{P}^\beta$ and by straightforward computation it results

$$\mathcal{J}[\rho, \mathcal{J}\nu] = -\nu^\alpha g_{\alpha\beta}g_{\gamma\varepsilon}\frac{\partial^2 H}{\partial \mu_\gamma \partial \mu_\beta}\mathcal{P}^\varepsilon = -\nu^\alpha g_{\alpha\varepsilon}\mathcal{P}^\varepsilon$$

which concludes the proof. □

As a particular case from [4] we have

Proposition 6. *If ρ is the semi-Hamilton section then*

$$\mathcal{N} = -\mathcal{L}_\rho \mathcal{J}$$

is a nonlinear connection on $\mathcal{T}E^*$.

Proof. The relations

$$\mathcal{J}\mathcal{N}(\nu) = \mathcal{J}(-[\rho, \mathcal{J}\nu] + \mathcal{J}[\rho, \nu]) = -\mathcal{J}([\rho, \mathcal{J}(\nu)]) = \mathcal{J}\nu,$$

$$\mathcal{N}\mathcal{J}(\nu) = -[\rho, \mathcal{J}^2\nu] + \mathcal{J}[\rho, \mathcal{J}\nu] = \mathcal{J}[\rho, \mathcal{J}\nu] = -\mathcal{J}\nu,$$

and remark 2 end the proof. □

Corollary 3. *The nonlinear connection $\mathcal{N} = -\mathcal{L}_{\rho_\mathcal{H}}\mathcal{J}$ has the coefficients given by*

$$\begin{aligned} \mathcal{N}_{\alpha\beta} &= \frac{1}{2}(\sigma^i_\gamma\{g_{\alpha\beta}, \mathcal{H}\} - \frac{\partial^2\mathcal{H}}{\partial x^i \partial \mu_\varepsilon}(\sigma^i_\beta g_{\alpha\varepsilon} + \sigma^i_\alpha g_{\beta\varepsilon}) \\ &\quad + \mu_\gamma L^\gamma_{\alpha\beta} + \frac{\partial \mathcal{H}}{\partial \mu_\delta}(g_{\alpha\varepsilon}L^\varepsilon_{\delta\beta} + g_{\beta\varepsilon}L^\varepsilon_{\delta\alpha})) \end{aligned}$$

where $\rho_\mathcal{H}$ is the Hamilton section and

$$\{g_{\alpha\beta}, \mathcal{H}\} = \frac{\partial g_{\alpha\beta}}{\partial \mu_\gamma}\frac{\partial \mathcal{H}}{\partial x^i} - \frac{\partial g_{\alpha\beta}}{\partial x^i}\frac{\partial \mathcal{H}}{\partial \mu_\gamma}.$$

Remark 3. In the case of standard Lie algebroids ($E = TM$, $\sigma = id$, $L^\alpha_{\beta\gamma} = 0$) the canonical nonlinear connection on cotangent bundle is obtained (see Miron [9]).

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Received: 15.X.2007

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