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HOLOMORPHIC HYPERSURFACES OF A HOLOMORPHICALLY CONFORMALLY FLAT ANTI-KÄHLER MANIFOLD

BY

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Dedicated to Academician Radu Miron at his 80th anniversary

Abstract. The aim of the present paper is to investigate the holomorphic hypersurface of HC-flat (holomorphically conformally flat) anti-Kähler manifold. The obtained results correspond to well known theorem of E. Cartan and J. Schouten, saying that a hypersurface of conformally flat Riemannian manifold of dim ≥ 5, is itself conformally flat if and only if it is quasi-umbilical.

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The object of the paper. It is well known ([12], [13]) that any complex hypersurface of a Kähler manifold is minimal. For the holomorphic hypersurface of an anti-Kähler manifold, this is not true. The aim of the present paper is to investigate the holomorphic hypersurfaces of a HC-flat (holomorphicly conformally flat) anti-Kähler manifold. The obtained results are given in the sections 4 and 5.

They correspond to the well-known theorem of CARTAN and SCHOUTEN saying that a hypersurface of a conformally flat Riemannian manifold of dimension ≥ 5, is itself conformally flat if and only if it is quasi-umbilical ([1], [2], [11]).

In the sections 1 and 2 we give some preliminary remarks.

In the section 3 we prove two lemmas for the latter use.

In the last section, we consider HC-flat holomorphic hypersurfaces of a manifold of constant totally real sectional curvature.

1. Preliminaries. By an anti-Kähler manifold [7] we mean a triple $(\tilde{M}, G, F)$, where $\tilde{M}$ is a connected differentiable manifold, $F = (F_B^A)$ is $(1,1)$-tensor field and $G = (G_{AB})$ is a pseudo-Riemannian metric on $\tilde{M}$ satisfying

$$F_B^A F_C^B = -\delta_C^A, \quad F_A^E F_B^D G_{ED} = -G_{AB}, \quad \tilde{\nabla}_C F_B^A = 0, \tag{1.1}$$

where $\tilde{\nabla}$ is the Levi–Civita connection and the capital Latin indices $A, B, C, \ldots$ run over the range $\{1, 2, \ldots, 2m\}$, $2m = \dim \tilde{M}$. If we put $F_{AB} = F_A^C G_{CB}$, then

$$F_{AB} = F_{BA}. \tag{1.2}$$

In some papers $(\tilde{M}, G, F)$ is named B-manifold ([3], [4], [8], [9]) and in the others – the Kähler manifold with Norden metric ([5], [6]).

The manifold $(\tilde{M}, G, F)$ is orientable and even dimensional. The metric G of such a manifold is indefinite and the signature is (m, m). Also $\operatorname{tr} F = 0$.

We denote by

R_{ABCD} – the components of the Riemmannian curvature tensor,

$\tilde{\rho}_{AB} = \tilde{R}^C{}_{ABC}$ – the Ricci tensor,

$\overset{*}{\rho}_{AB} = F_A^D \tilde{\rho}_{DB}$ – the second Ricci tensor,

$\tilde{\varkappa} = G^{AB} \tilde{\rho}_{AB}$ – the scalar curvature,

$\overset{*}{\varkappa} = F^{AB} \tilde{\rho}_{AB} = G^{AB} \overset{*}{\rho}_{AB}$ – the second scalar curvature.

Since $\tilde{\nabla} F = 0$, the curvature tensor and the Ricci tensors satisfy

$$F_A^L F_B^M \tilde{\mathcal{R}}_{LMCD} = -\tilde{\mathcal{R}}_{ABCD}, \tag{1.3}$$

$$F_A^L F_B^M \tilde{\rho}_{LM} = -\tilde{\rho}_{AB}. \tag{1.4}$$

Let $T_p(\tilde{M})$ be the tangent vector space of $\tilde{M}$ at $p \in \tilde{M}$.

Every nondegenerate with respect to G 2-plane π in $T_p(\tilde{M})$ has two sectional curvatures

$$\mathcal{K}_1(\pi, p) = \frac{\tilde{R}(X, Y, Y, X)}{G(X, X)G(Y, Y) - G(X, Y)^2}$$

and

$$\mathcal{K}_2(\pi, p) = \frac{\tilde{R}(X, Y, Y, FX)}{G(X, X)G(Y, Y) - G(X, Y)^2},$$

where $\{X, Y\}$ is a basis of π.

A 2-plane π in $T_p(\tilde{M})$ is said to be totally real if $\pi \neq F\pi$ and $\pi \perp F\pi$ with respect to G.

The manifold $\tilde{M}$ ($m \geq 2$) is of pointwise totally real sectional curvatures, i.e.,

$$\mathcal{K}_1(\pi, p) = k_1(p), \quad \mathcal{K}_2(\pi, p) = k_2(p)$$

for an arbitrary nondegenerate totally real 2-plane π in $T_p(\tilde{M})$, if and only if ([3], [4])

(1.5)
$$\begin{aligned}\tilde{R}_{ABCD} = {} & \frac{\tilde{\varkappa}(p)}{4m(m-1)}(G_{AD}G_{BC} - G_{AC}G_{BD} - F_{AD}F_{BC} + F_{AC}F_{BD}) \\ & - \frac{\overset{\tilde{*}}{\varkappa}(p)}{4m(m-1)}(G_{AD}F_{BC} + G_{BC}F_{AD} - G_{AC}F_{BD} - G_{BD}F_{AC}).\end{aligned}$$

Both functions $\tilde{\varkappa}$ and $\overset{\tilde{*}}{\varkappa}$ are constant if $\tilde{M}$ is connected and $m \geq 3$. It is known ([8], [9], [10]) that the holomorphically conformal curvature tensor of $(\tilde{M}, G, F)$, $m \geq 3$, is

(1.6)
$$\begin{aligned}(H\tilde{C})_{ABCD} = {} & \tilde{R}_{ABCD} - \frac{1}{2(m-2)}\big(G_{AD}\tilde{\rho}_{BC} \\ & + G_{BC}\tilde{\rho}_{AD} - G_{AC}\tilde{\rho}_{BD} - G_{BD}\tilde{\rho}_{AC} \\ & - F_{AD}\overset{\tilde{*}}{\rho}_{BC} - F_{BC}\overset{\tilde{*}}{\rho}_{AD} + F_{AC}\overset{\tilde{*}}{\rho}_{BD} + F_{BD}\overset{\tilde{*}}{\rho}_{AC}\big) \\ & + \frac{\tilde{\varkappa}}{4(m-1)(m-2)}\big(G_{AD}G_{BC} - G_{AC}G_{BD} - F_{AD}F_{BC} + F_{AC}F_{BD}\big) \\ & - \frac{\overset{\tilde{*}}{\varkappa}}{4(m-1)(m-2)}\big(G_{AD}F_{BC} + G_{BC}F_{AD} - G_{AC}F_{BD} - G_{BD}F_{AC}\big).\end{aligned}$$

2. Holomorphic hypersurfaces. Let $(\tilde{M}, G, F)$ be an anti-Kähler manifold and $\dim \tilde{M} = 2n+2$. A submanifold M, $\dim M = 2n$, is said to be a *holomorphic hypersurface* (*h-hypersurface* in short) of $\tilde{M}$ if the restriction of G on M has the maximal rank and $FT_p(M) = T_p(M)$, $p \in M$. Suppose that M is expressed in each neighborhood $\tilde{U}$ of $\tilde{M}$ by the equations

$$x^A = x^A(u^a)$$

where x^A are the local coordinates of $\tilde{M}$ in $\tilde{U}$ and u^a are the local coordinates in $U = \tilde{U} \cap M$. Lowercase Latin indices $a, b, c, \ldots, i, j, k, \ldots$ run over

the range $\{1, 2, \dots, 2n\}$. We have in U $2n$ local vector fields B_a having the components

$$B_a^A = \frac{\partial x^A}{\partial u^a}$$

and spanning the tangent space of M at each point of U.

Since the complex structure F leaves invariant the tangent space of M at each point of M, FB_b is a linear combination of B_a in U, that is

$$FB_b = f_b^a B_a, \quad \text{i.e.,} \quad F_D^A B_b^D = f_b^a B_a^A, \tag{2.1}$$

where the functions f_a^b are the components of a tensor field f of type $(1, 1)$ defined globally on M.

Applying the operator F on both sides of (2.1) and taking into account (1.1), we find

$$B_b^E = -f_b^a f_a^d B_d^E,$$

that is

$$f^2 = -\operatorname{Id}, \quad \text{i.e.,} \quad f_b^a f_a^c = -\delta_b^c \quad \text{and} \quad \operatorname{tr} f = 0. \tag{2.2}$$

The Riemannian metric g induced on M has the components

$$g_{ab} = G_{AB} B_a^A B_b^B, \tag{2.3}$$

in each neigbourhood U of M. Thus, we obtain

$$F_C^E F_B^D G_{ED} B_a^C B_b^B = -G_{CB} B_a^C B_b^B,$$

or

$$f_a^e f_b^d B_e^E B_d^D G_{ED} = -g_{ab},$$

that is

$$f_a^e f_b^d g_{ed} = -g_{ab}, \tag{2.4}$$

where from, putting

$$f_{ab} = f_a^e g_{eb}$$

and using (2.3), we find

$$f_{ab} = f_{ba}. \tag{2.5}$$

Concerning the tensor F_{CD}, we have

$$F_{CD}B^C_a B^D_b = f_{ab}. \tag{2.6}$$

There exist locally vector fields $N_{1|}$ and $N_{2|}$, normal to M and such that [5]

$$\begin{gathered} G_{AB}N^A_{1|}N^B_{1|} = -G_{AB}N^A_{2|}N^B_{2|}, \quad G_{AB}N^A_{1|}N^B_{2|} = 0, \\ F^A_B N^B_{1|} = -N^A_{2|}, \quad F^A_B N^B_{2|} = N^A_{1|}. \end{gathered} \tag{2.7}$$

Denoting by h and k the second fundamental forms corresponding to $N_{1|}$ and $N_{2|}$ respectively, we have

$$\nabla_d B^A_b = h_{db}N^A_{1|} - k_{db}N^A_{2|}, \tag{2.8}$$

where the operator ∇ has the following meaning

$$\nabla_d B^A_b = \partial_d B^A_b + \left\{ \begin{matrix} \tilde{A} \\ DE \end{matrix} \right\} B^D_d B^E_b - \left\{ \begin{matrix} a \\ db \end{matrix} \right\} B^A_a$$

and $\left\{ \begin{matrix} \tilde{A} \\ DE \end{matrix} \right\}$ and $\left\{ \begin{matrix} a \\ db \end{matrix} \right\}$ are the Christoffel symbols relative to the metrics G and g.

Differentiating (2.1) covariantly along M and using $\tilde{\nabla}F = 0$, we obtain

$$F^A_B \nabla_d B^B_b = (\nabla_d f^a_b)B^A_a + f^a_b \nabla_d B^A_a.$$

Now, in view of (2.7) and (2.8), we have

$$(\nabla_d f^a_b)B^A_a + (k_{db} + f^a_b h_{da})N^A_{1|} + (h_{db} - f^a_b k_{da})N^h_{2|} = 0,$$

wherefrom, taking into account that B^A_a, $N^A_{1|}$ and $N^A_{2|}$ are linearly independent, we get

$$\nabla_d f^a_b = 0. \tag{2.9}$$

The relations (2.2), (2.4) and (2.9) show that the *h-hypersurface* (M, g, f) *of an anti-Kähler manifold is itself an anti-Kähler manifold.*

As for the second fundamental forms, h and k, we have

$$h_{ij} = f^a_i k_{aj}, \quad k_{ij} = -f^a_i h_{aj}, \tag{2.10}$$

$$f^a_i f^b_j h_{ab} = -h_{ij}, \quad f^a_i f^b_j k_{ab} = -k_{ij}. \tag{2.11}$$

Also, we shall use the second fundamental forms of order 2 and 3, namely

$$h^2_{ij} = h_{ia}h_{jb}g^{ab}, \quad h^3_{ij} = h^2_{ia}h_{jb}g^{ab}.$$

It is easy to see that they satisfy the following conditions

$$\begin{aligned} &h^2_{ij} = h^2_{ji}, \quad f^a_i f^b_j h^2_{ab} = -h^2_{ij}, \quad \text{i.e.,} \quad f^a_i h^2_{aj} = f^a_j h^2_{ia}, \\ &h^3_{ij} = h^3_{ji}, \quad f^a_i f^b_j h^3_{ab} = -h^3_{ij}, \quad \text{i.e.,} \quad f^a_i h^3_{aj} = f^a_j h^3_{ia}. \end{aligned} \tag{2.12}$$

3. Two lemmas. Let A and D be two symmetric $(0,2)$ tensors at a point of a manifold (M, g, f) satisfying

$$f^a_i f^b_j A_{ab} = -A_{ij}, \quad f^a_i f^b_j D_{ab} = -D_{ij}. \tag{3.1}$$

We define the tensor $Q(A, D)$ by the formula

$$\begin{aligned} Q(A,D)_{ijrs} &= A_{ir}D_{js} + A_{jr}D_{is} - A_{is}D_{jr} - A_{js}D_{ir} \\ &\quad - f^a_i A_{ar} f^b_j D_{bs} - f^a_j A_{ar} f^b_i D_{bs} + f^a_i A_{as} f^b_j D_{br} + f^a_j A_{as} f^b_i D_{br}. \end{aligned} \tag{3.2}$$

Then

$$Q(A, D) = -Q(D, A)$$

because of which $Q(A, A) = 0$.

Lemma 1. *Let A and D be two symmetric $(0,2)$ tensors satisfying (3.1). If*

$$Q(A, D)_{ijrs} = 0, \tag{3.3}$$

then

$$D_{is} = \delta A_{is} + \bar{\delta} f^a_i A_{as}, \tag{3.4}$$

where δ and $\bar{\delta}$ are some scalars.

Proof. Let X be a vector such that

$$X^A X^B A_{ab} = \omega \neq 0. \tag{3.5}$$

We put

$$\begin{aligned} \bar{X}^i &= f^i_a X^a, \qquad & \bar{\omega} &= \bar{X}^a X^b A_{ab}, \\ \eta &= X^a X^b D_{ab}, & \bar{\eta} &= \bar{X}^a X^b D_{ab}. \end{aligned} \tag{3.6}$$

Transvecting (3.3) with $X^i X^r$ and symmetrizing the resulting equality, we get

$$\omega D_{js} - \bar{\omega} f^a_j D_{as} = \eta A_{js} - \bar{\eta} f^a_j A_{as}.$$

It follows

$$\bar{\omega} D_{js} + \omega f^a_j D_{as} = \bar{\eta} A_{js} + \eta f^a_j A_{as}.$$

These two relations imply

$$D_{js} = \frac{\omega\eta + \bar{\omega}\bar{\eta}}{\omega^2 + \bar{\omega}^2} A_{js} + \frac{\bar{\omega}\eta - \omega\bar{\eta}}{\omega^2 + \bar{\omega}^2} f^c_j A_{as}.$$

But this is just the relation (3.4). □

Lemma 2. *Let A be a symmetric $(0,2)$ tensor satisfying* (3.1) *and*

$$A_{im}A_{jl} - A_{il}A_{jm} - f^a_i A_{am} f^b_j A_{bl} + f^a_i A_{al} f^b_j A_{bm} = EG_{ijlm} + H f^a_i G_{ajlm}, \tag{3.7}$$

where E and H are some scalars and

$$G_{ijlm} = g_{im}g_{jl} - g_{il}g_{jm} - f_{im}f_{jl} + f_{il}f_{jm}.$$

Then

$$A_{ij} = Cg_{ij} + Bf_{ij} + \tau V_{ij} + \sigma \bar{V}_{ij}, \tag{3.8}$$

where C, B, τ and σ are some scalars,

$$V_{ij} = V_i V_j - \bar{V}_i \bar{V}_j, \quad \bar{V}_{ij} = f^a_i V_{aj} = \bar{V}_i V_j + V_i \bar{V}_j,$$

V is a vector and $\bar{V} = fV$.

Proof. Besides the notations (3.5) and (3.6) we shall use

$$V_i = X^a A_{ai}, \quad \bar{V}_i = \bar{X}^a A_{ai}.$$

Then

$$f^a_i V_a = \bar{V}_i, \quad f^a_i \bar{V}_a = -V_i.$$

Now, we transvect (3.7) with X^iX^m. The resulting equality, using the notations

$$X_{ij} = X_iX_j - \bar{X}_i\bar{X}_j, \quad \bar{X}_{ij} = f_i^aX_{aj} = \bar{X}_iX_j + X_i\bar{X}_j,$$
$$P = EX_aX^a + HX_a\bar{X}^a, \quad Q = HX_aX^a - EX_a\bar{X}^a,$$

can be expressed in the form

$$\omega A_{ij} - \bar{\omega}f_i^aA_{aj} = Pg_{ij} + Qf_{ij} - EX_{ij} - H\bar{X}_{ij}.$$

wherefrom we obtain

$$\bar{\omega}A_{ij} + \omega f_i^aA_{aj} = -Qg_{ij} + Pf_{ij} + HX_{ij} - E\bar{X}_{ij}.$$

These two relations imply

$$\begin{aligned} A_{ij} = {} & \frac{\omega P - \bar{\omega}Q}{\omega^2+\bar{\omega}^2}g_{ij} + \frac{\bar{\omega}P + \omega Q}{\omega^2+\bar{\omega}^2}f_{ij} + \frac{1}{\omega^2+\bar{\omega}^2}(\omega V_{ij} + \bar{\omega}\bar{V}_{ij}) \\ & + \frac{\bar{\omega}H - \omega E}{\omega^2+\bar{\omega}^2}X_{ij} - \frac{\omega H + \bar{\omega}E}{\omega^2+\bar{\omega}^2}\bar{X}_{ij}, \end{aligned} \tag{3.9}$$

because of which we have

$$\begin{aligned} & V_kA_{ij} - V_jA_{ik} - \bar{V}_kf_i^aA_{aj} + \bar{V}_jf_i^aA_{ak} \\ & \quad = \frac{\omega P - \bar{\omega}Q}{\omega^2+\bar{\omega}^2}(V_kg_{ij} - V_jg_{ik} - \bar{V}_kf_{ij} + \bar{V}_jf_{ik}) \\ & \quad + \frac{\bar{\omega}P + \omega Q}{\omega^2+\bar{\omega}^2}(V_kf_{ij} - V_jf_{ik} + \bar{V}_kg_{ij} - \bar{V}_jg_{ik}) \\ & \quad + \frac{\bar{\omega}H - \omega E}{\omega^2+\bar{\omega}^2}(V_kX_{ij} - V_jX_{ik} - \bar{V}_k\bar{X}_{ij} + \bar{V}_jX_{ik}) \\ & \quad - \frac{\omega H + \bar{\omega}E}{\omega^2+\bar{\omega}^2}(V_k\bar{X}_{ij} - V_j\bar{X}_{ik} + \bar{V}_kX_{ij} - \bar{V}_jX_{ik}). \end{aligned} \tag{3.10}$$

On the other hand, transvecting (3.7) with X^i and then changing the indices, we obtain

$$\begin{aligned} V_kA_{ij} - V_jA_{ik} - \bar{V}_kf_i^aA_{aj} + \bar{V}_jf_i^aA_{ak} = {} & E(X_kg_{ij} - X_jg_{ik} - \bar{X}_kf_{ij} + \bar{X}_jf_{ik}) \\ & + H(X_kf_{ij} + \bar{X}_kg_{ij} - X_jf_{ik} - \bar{X}_kg_{ij}). \end{aligned}$$

Comparing this and (3.10), we find

$$\begin{aligned} & E(X_kg_{ij} - X_jg_{ik} - \bar{X}_kf_{ij} + \bar{X}_jf_{ik}) \\ & + H(X_kf_{ij} + \bar{X}_kg_{ij} - X_jf_{ik} - \bar{X}_jg_{ij}) \end{aligned}$$

$$
\begin{aligned}
&= \frac{\omega P - \bar\omega Q}{\omega^2+\bar\omega^2}(V_k g_{ij} - V_j g_{ik} - \bar V_k f_{ij} + \bar V_j f_{ik})\\
&+ \frac{\bar\omega P + \omega Q}{\omega^2+\bar\omega^2}(V_k f_{ij} - V_j f_{ik} + \bar V_k g_{ij} - \bar V_j g_{ik})\\
&+ \frac{\bar\omega H - \omega E}{\omega^2+\bar\omega^2}(V_k X_{ij} - V_j X_{ik} - \bar V_k \bar X_{ij} + \bar V_j \bar X_{ik})\\
&- \frac{\omega H + \bar\omega E}{\omega^2+\bar\omega^2}(V_k \bar X_{ij} - V_j \bar X_{ik} + \bar V_k X_{ij} - \bar V_j X_{ik}),
\end{aligned}
$$

wherefrom, transvecting with g^{ij}, we get

$$
\begin{aligned}
&\Big[(n-1)E + \frac{\bar\omega H - \omega E}{\omega^2+\bar\omega^2} X_a V^a - \frac{\omega H + \bar\omega E}{\omega^2+\bar\omega^2} X_a \bar V^a\Big] X_k\\
&+ \Big[(n-1)H - \frac{\bar\omega H - \omega E}{\omega^2+\bar\omega^2} X_a \bar V^a - \frac{\omega H + \bar\omega E}{\omega^2+\bar\omega^2} X_a V^a\Big] \bar X_k\\
&= \frac{1}{\omega^2+\bar\omega^2}[(n-1)(\omega P - \bar\omega Q) + (\bar\omega H - \omega E) X_a X^a - (\omega H + \bar\omega E) X_a \bar X^a] V_k\\
&+ \frac{1}{\omega^2+\bar\omega^2}[(n-1)(\bar\omega P + \omega Q) - (\bar\omega H - \omega E) X_a \bar X^a - (\omega H + \bar\omega E) X_a x^a] \bar V_k.
\end{aligned}
$$

This can be written in the form $KX_k + \bar K \bar X_k = LV_k + \bar L \bar V_k$, wherefrom it follows $-\bar K X_k + K \bar X_k = -\bar L V_k + L \bar V_k$.

These two relations yield

$$(K^2 + \bar K^2) X_k = (KL + \bar K \bar L) V_k + (K \bar L - \bar K L) \bar V_k,$$

that is

$$X_k = \alpha V_k + \beta \bar V_k$$

and therefore

$$\bar X_k = -\beta V_k + \alpha \bar V_k.$$

Next we have

$$
\begin{aligned}
X_{ij} &= (\alpha^2 - \beta^2) V_{ij} + 2\alpha\beta \bar V_{ij},\\
\bar X_{ij} &= -2\alpha\beta V_{ij} + (\alpha^2 - \beta^2) \bar V_{ij}.
\end{aligned}
$$

Substituting this into (3.9), we obtain

$$
\begin{aligned}
A_{ij} = \frac{1}{\omega^2+\bar\omega^2}\big\{&(\omega P - \bar\omega Q) g_{ij} + (\bar\omega P + \omega Q) f_{ij}\\
&+ [\omega + (\alpha^2 - \beta^2)(\bar\omega H - \omega E) + 2\alpha\beta(\omega H + \bar\omega E)] V_{ij}\\
&+ [\bar\omega + 2\alpha\beta(\bar\omega H - \omega E) - (\alpha^2 - \beta^2)(\omega H + \bar\omega E)] \bar V_{ij}\big\}
\end{aligned}
$$

i.e., we obtain the relation of the form (3.8). □

4. HC-flat h-hypersurfaces of HC-flat manifold. The Gauss equation of the h-hypersurface (M, g, f) of an anti-Kähler manifold $(\tilde{M}, G, F)$ is

$$\tilde{R}_{ABCD}B_i^A B_j^B B_l^C B_m^D = R_{ijlm} - (h_{im}h_{jl} - h_{il}h_{jm}) + (k_{im}k_{jl} - k_{il}k_{jm}), \tag{4.1}$$

where R_{ijlm} are the components of the curvature tensor of M. We denote by ρ_{ij} the Ricci tensor of M and put

$$\overset{*}{\rho}_{ij} = f_i^a \rho_{aj}, \quad \varkappa = \rho_{ij}g^{ij}, \quad \overset{*}{\varkappa} = \overset{*}{\rho}_{ij}g^{ij} = \rho_{ij}f^{ij}.$$

Now we suppose that the ambient manifold $\tilde{M}$ is HC-flat, i.e.,

$$(H\tilde{C})_{ABCD} = 0. \tag{4.2}$$

Taking into account that $m = n + 1$, the condition (4.2), according to (1.6), is

$$\begin{aligned}
\tilde{R}_{ABCD} = {} & \frac{1}{2(n-1)}(G_{AD}\tilde{\rho}_{BC} + G_{BC}\tilde{\rho}_{AD} - G_{AC}\tilde{\rho}_{BD} - G_{BD}\tilde{\rho}_{AC} \\
& \qquad - F_{AD}\overset{\tilde{*}}{\rho}_{BC} - F_{BC}\overset{\tilde{*}}{\rho}_{AD} + F_{AC}\overset{\tilde{*}}{\rho}_{BD} + F_{BD}\overset{\tilde{*}}{\rho}_{AC}) \\
& - \frac{\tilde{\varkappa}}{4n(n-1)}(G_{AD}G_{BC} - G_{AC}G_{BD} - F_{AD}F_{BC} + F_{AC}F_{BD}) \\
& + \frac{\overset{\tilde{*}}{\varkappa}}{4n(n-1)}(G_{AD}F_{BC} + G_{BC}F_{AD} - G_{AC}F_{BD} - G_{BD}F_{AC}).
\end{aligned}$$

Substituting this into (4.1), we get

$$\begin{aligned}
R_{ijlm} = {} & \frac{1}{2(n-1)}(g_{im}\tilde{\rho}_{BC}B_j^B B_l^C + g_{jl}\tilde{\rho}_{AD}B_i^A B_m^D \\
& \qquad - g_{il}\tilde{\rho}_{BD}B_j^B B_m^D - g_{jm}\tilde{\rho}_{AC}B_i^A B_l^C \\
& \qquad - f_{im}\overset{\tilde{*}}{\rho}_{BC}B_j^B B_l^C - f_{jl}\overset{\tilde{*}}{\rho}_{AD}B_i^A B_m^D \\
& \qquad + f_{il}\overset{\tilde{*}}{\rho}_{BD}B_j^B B_m^D + f_{jm}\overset{\tilde{*}}{\rho}_{AC}B_i^A B_l^C) \\
& - \frac{\tilde{\varkappa}}{4n(n-1)}G_{ijlm} + \frac{\overset{\tilde{*}}{\varkappa}}{4n(n-1)}f_i^a G_{ajlm} \\
& + h_{im}h_{jl} - h_{il}h_{jm} - k_{im}k_{jl} + k_{il}k_{jm}.
\end{aligned} \tag{4.3}$$

Transvecting (4.3) with g^{im} and using (1.4), (2.1), (2.11) and (2.12), we obtain

$$\begin{aligned}\rho_{jl} - \operatorname{tr} h\, h_{jl} - \operatorname{tr} k f^a_j h_{al} + 2h^2_{jl}\\ = \frac{n-2}{n-1}\tilde\rho_{BC}B^B_jB^C_l + \Big(\frac{1}{2(n-1)}\tilde\rho_{AD}B^A_iB^D_mg^{im} - \frac{\tilde\varkappa}{2n}\Big)g_{jl}\\ - \Big(\frac{1}{2(n-1)}\overset{\tilde *}{\rho}_{AD}B^A_iB^D_mg^{im} - \frac{\overset{\tilde *}{\varkappa}}{2n}\Big)f_{jl}.\end{aligned}$$

From now on, we suppose $n > 2$. Putting

$$\begin{aligned}\frac{1}{2(n-1)}\tilde\rho_{AD}B^A_iB^D_mg^{im} - \frac{\tilde\varkappa}{2n} = s,\\ \frac{1}{2(n-1)}\overset{\tilde *}{\rho}_{AD}B^A_iB^D_mg^{im} - \frac{\overset{\tilde *}{\varkappa}}{2n} = t,\end{aligned}$$

the preceding relation can be rewritten in the form

$$\tilde\rho_{BC}B^B_jB^C_l = \frac{n-1}{n-2}\big(\rho_{jl} - sg_{jl} + tf_{jl} - \operatorname{tr} h\, h_{jl} - \operatorname{tr} k f^a_j h_{al} + 2h^2_{jl}\big). \tag{4.4}$$

Transvecting (4.4) with f^j_k and using (2.1), we find

$$\overset{\tilde *}{\rho}_{DC}B^D_kB^C_l = \frac{n-1}{n-2}\big(\overset{*}{\rho}_{kl} - sf_{kl} - tg_{kl} - \operatorname{tr} hf^a_kh_{al} + \operatorname{tr} k\, h_{kl} + 2f^a_kh^2_{al}\big). \tag{4.5}$$

We substitute (4.4) and (4.5) into (4.3) and get

$$\begin{aligned}R_{ijlm} - \frac{1}{2(n-2)}\big(&g_{im}\rho_{jl} + g_{jl}\rho_{im} - g_{il}\rho_{jm} - g_{jm}\rho_{il}\\ &- f_{im}\overset{*}{\rho}_{jl} - f_{jl}\overset{*}{\rho}_{im} + f_{il}\overset{*}{\rho}_{jm} + f_{jm}\overset{*}{\rho}_{il}\big)\\ + \frac{\varkappa}{4(n-1)(n-2)}G_{ijlm} - \frac{\overset{*}{\varkappa}}{4(n-1)(n-2)}f^a_iG_{ajlm}&\\ = \Big(\frac{\varkappa}{4(n-1)(n-2)} - \frac{s}{n-2} - \frac{\tilde\varkappa}{4n(n-1)}\Big)G_{ijlm}&\\ + \Big(-\frac{\overset{*}{\varkappa}}{4(n-1)(n-2)} + \frac{t}{n-2} + \frac{\overset{\tilde *}{\varkappa}}{4n(n-1)}\Big)f^a_iG_{ajlm}&\\ - \frac{1}{2(n-2)}\big(\operatorname{tr} h\, h_{jl} + \operatorname{tr} kf^a_jh_{al} - 2h^2_{jl}\big)g_{im}&\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{2(n-2)}\big(\operatorname{tr} h\, h_{im} + \operatorname{tr} k f_i^a h_{am} - 2h_{im}^2\big)g_{jl}\\
&+ \frac{1}{2(n-2)}\big(\operatorname{tr} h\, h_{jm} + \operatorname{tr} k f_j^a h_{am} - 2h_{jm}^2\big)g_{il}\\
&+ \frac{1}{2(n-2)}\big(\operatorname{tr} h\, h_{il} + \operatorname{tr} k f_i^a h_{al} - 2h_{il}^2\big)g_{jm}\\
&+ \frac{1}{2(n-2)}\big(\operatorname{tr} h f_j^a h_{al} - \operatorname{tr} k\, h_{jl} - 2f_j^a h_{al}^2\big)f_{im}\\
&+ \frac{1}{2(n-2)}\big(\operatorname{tr} h f_i^a h_{am} - \operatorname{tr} k\, h_{im} - 2f_i^a h_{am}^2\big)f_{jl}\\
&- \frac{1}{2(n-2)}\big(\operatorname{tr} h f_j^a h_{am} - \operatorname{tr} k\, h_{jm} - 2f_j^a h_{am}^2\big)f_{il}\\
&- \frac{1}{2(n-2)}\big(\operatorname{tr} h f_i^a h_{al} - \operatorname{tr} k\, h_{il} - 2f_i^a h_{al}^2\big)f_{jm}\\
&\qquad + h_{im}h_{jl} - h_{il}h_{jm} - k_{im}k_{jl} + k_{il}k_{jm}.
\end{aligned}
$$

The expression on the left-hand side is the holomorphically conformal curvature tensor of (M, g, f), so that we have

$$
\begin{aligned}
2(n-2)(CH)_{ijlm} &= \lambda G_{ijlm} + \mu f_i^a G_{ajlm}\\
&\quad + \operatorname{tr} h\big(- h_{jl}g_{im} - h_{im}g_{jl} + h_{jm}g_{il} + h_{il}g_{jm}\\
&\qquad + f_j^a h_{al}f_{im} + f_i^a h_{am}f_{jl} - f_j^a h_{am}g_{il} - f_i^a h_{al}f_{jm}\big)\\
&\quad + \operatorname{tr} k\big(- f_j^a h_{al}g_{im} - f_i^a h_{am}g_{jl} + f_j^a h_{am}g_{il} + f_i^a h_{al}g_{jm}\big)\\
&\qquad - h_{jl}f_{im} - h_{im}f_{jl} + h_{jm}f_{il} + h_{il}f_{jm}\big)\\
&\quad + 2\big(h_{jl}^2 g_{im} + h_{im}^2 g_{jl} - h_{jm}^2 g_{il} - h_{il}^2 g_{jm}\\
&\qquad - f_j^a h_{al}^2 f_{im} - f_i^a h_{am}^2 f_{jl} + f_j^a h_{am}^2 f_{il} + f_i^a h_{al}^2 f_{jm}\big)\\
&\quad + 2(n-2)\big(h_{im}h_{jl} - h_{il}h_{jm} - f_i^a h_{am}f_j^b h_{bl} + f_i^a h_{al}f_j^b h_{bm}\big),
\end{aligned}
\tag{4.6}
$$

where

$$
\begin{aligned}
\lambda &= 2(n-2)\Big[\frac{\varkappa}{4(n-1)(n-2)} - \frac{\tilde{\varkappa}}{4n(n-1)} - \frac{s}{n-2}\Big],\\
\mu &= 2(n-2)\Big[- \frac{\overset{*}{\varkappa}}{4(n-1)(n-2)} + \frac{\tilde{\overset{*}{\varkappa}}}{4n(n-1)} + \frac{t}{n-2}\Big].
\end{aligned}
$$

Thus

Proposition. *The necessary and the sufficient condition for a hypersurface of a HC-flat manifold to be itself HC-flat is*

(4.7)
$$\begin{aligned}\lambda G_{ijlm} + \mu f_i^a G_{ajlm} + \operatorname{tr} h\big(&- h_{jl}g_{im} - h_{im}g_{jl} + h_{jm}g_{il} + h_{il}g_{jm}\\ &+ f_j^a h_{al} f_{im} + f_i^a h_{am} f_{jl} - f_j^a h_{am} f_{il} - f_i^a h_{al} f_{jm}\big)\\ + \operatorname{tr} k\big(&- f_j^a h_{al} g_{im} - f_i^a h_{am} g_{jl} + f_j^a h_{am} g_{il} + f_i^a h_{al} g_{jm}\big)\\ &- h_{jl} f_{im} - h_{im} f_{jl} + h_{jm} f_{il} + h_{il} f_{jm}\big)\\ + 2\big(h_{jl}^2 g_{im} &+ h_{im}^2 g_{jl} - h_{jm}^2 g_{il} - h_{il}^2 g_{jm}\\ &- f_j^a h_{al}^2 f_{im} - f_i^a h_{am}^2 f_{jl} + f_j^a h_{am}^2 f_{il} + f_i^a h_{al}^2 f_{jm}\big)\\ + 2(n-2)\big(h_{im}h_{jl} &- h_{il}h_{jm} - f_i^a h_{am} f_j^b h_{bl} + f_i^a h_{al} f_j^b h_{bm}\big) = 0,\end{aligned}$$

where λ and μ are some scalar functions.

5. The main theorem. Using (4.7), we can determine h_{ij}. To do that, we transvect (4.7) with $h^i{}_t$ and find

$$\begin{aligned}&\lambda\big(h_{tm}g_{jl} - h_{tl}g_{jm} - f_m^a h_{at} f_{jl} + f_l^a h_{at} f_{jm}\big)\\ &+ \mu\big(h_{tm}f_{jl} + g_{jl}f_m^a h_{at} - h_{tl}f_{jm} - g_{jm}f_l^a h_{at}\big)\\ &+ \operatorname{tr} h\big(- h_{jl}h_{tm} - h_{tm}^2 g_{jl} + h_{jm}h_{tl} + h_{tl}^2 g_{jm}\\ &\quad + f_j^a h_{al} f_m^b h_{bt} + f_m^a h_{at}^2 f_{jl} - f_j^a h_{am} f_t^b h_{bl} - f_l^a h_{at}^2 f_{jm}\big)\\ &+ \operatorname{tr} k\big(- f_j^a h_{al} h_{tm} - f_m^a h_{at}^2 g_{jl} + f_j^a h_{am} h_{tl} + f_l^a h_{at}^2 g_{jm}\\ &\quad - h_{jl} f_m^a h_{at} - h_{tm}^2 f_{jl} + h_{jm} f_l^a h_{at} + h_{lt}^2 f_{jm}\big)\\ &+ 2\big(h_{jl}^2 h_{tm} + h_{tm}^3 g_{jl} - h_{jm}^2 h_{tl} - h_{tl}^3 g_{jm}\\ &\quad - f_j^a h_{al}^2 f_m^b h_{bt} - f_m^a h_{at}^3 f_{jl} + f_j^a h_{am}^2 f_l^b h_{bt} + f_l^a h_{at}^3 f_{jm}\big)\\ &+ 2(n-2)\big(h_{tm}^2 h_{jl} - h_{tl}^2 h_{jm} - f_m^a h_{at}^2 f_j^b h_{bl} + f_l^a h_{at}^2 f_j^b h_{bm}\big) = 0,\end{aligned} \tag{5.1}$$

wherefrom, transvecting with g^{tm} and taking into account that $\operatorname{tr} k = -\operatorname{tr}(fh)$, we obtain

$$h_{jl}^3 = a g_{jl} + \bar{a} f_{jl} + b h_{jl} + \bar{b} f_j^a h_{al} + c h_{jl}^2 + \bar{c} f_j^a h_{al}^2, \tag{5.2}$$

where

$$a = \frac{1}{4n}\big[\lambda \operatorname{tr} h - \mu \operatorname{tr} k - \operatorname{tr} h \operatorname{tr} h^2 - \operatorname{tr} k \operatorname{tr}(fh^2) + 2 \operatorname{tr} h^3\big],$$

$$\bar a = \frac{1}{4n}\big[\mu \operatorname{tr} h + \lambda \operatorname{tr} k + \operatorname{tr} h \operatorname{tr}(fh^2) - \operatorname{tr} k \operatorname{tr} h^2 - 2\operatorname{tr}(fh^3)\big],$$
$$b = \frac{1}{4n}\big[-2\lambda - (\operatorname{tr} h)^2 + (\operatorname{tr} k)^2 + 2(n-2)\operatorname{tr} h^2\big],$$
$$\bar b = \frac{1}{4n}\big[-2\mu - 2\operatorname{tr} h \operatorname{tr} k - 2(n-2)\operatorname{tr}(fh^2)\big],$$
$$c = \frac{3\operatorname{tr} h}{2n}, \quad \bar c = \frac{3\operatorname{tr} k}{2n}.$$

On the other hand, symmetrizing (5.1) with respect to t and j, and using (5.2), we obtain

$$\begin{aligned}
&g_{jl}\Big[(\lambda+2b)h_{tm} + (\mu+2\bar b)f_t^a h_{am} - \tfrac{n-3}{n}\operatorname{tr} h\, h^2_{tm} - \tfrac{n-3}{n}\operatorname{tr} k f_t^a h^2_{am}\Big]\\
&+g_{tl}\Big[(\lambda+2b)h_{jm} + (\mu+2\bar b)f_j^a h_{am} - \tfrac{n-3}{n}\operatorname{tr} h\, h^2_{jm} - \tfrac{n-3}{n}\operatorname{tr} k f_j^a h^2_{am}\Big]\\
&-g_{jm}\Big[(\lambda+2b)h_{tl} + (\mu+2\bar b)f_t^a h_{al} - \tfrac{n-3}{n}\operatorname{tr} h\, h^2_{tl} - \tfrac{n-3}{n}\operatorname{tr} k f_t^a h^2_{al}\Big]\\
&-g_{tm}\Big[(\lambda+2b)h_{jl} + (\mu+2\bar b)f_j^a h_{al} - \tfrac{n-3}{n}\operatorname{tr} h\, h^2_{jl} - \tfrac{n-3}{n}\operatorname{tr} k f_j^a h^2_{al}\Big]\\
&-f_{jl}\Big[(\lambda+2b)f_t^a h_{am} - (\mu+2\bar b)h_{tm} - \tfrac{n-3}{n}\operatorname{tr} h f_t^a h^2_{am} + \tfrac{n-3}{n}\operatorname{tr} k\, h^2_{tm}\Big]\\
&-f_{tl}\Big[(\lambda+2b)f_j^a h_{am} - (\mu+2\bar b)h_{jm} - \tfrac{n-3}{n}\operatorname{tr} h f_j^a h^2_{am} + \tfrac{n-3}{n}\operatorname{tr} k\, h^2_{jm}\Big]\\
&+f_{jm}\Big[(\lambda+2b)f_t^a h_{al} - (\mu+2\bar b)h_{tl} - \tfrac{n-3}{n}\operatorname{tr} h f_t^a h^2_{al} + \tfrac{n-3}{n}\operatorname{tr} k\, h^2_{tl}\Big]\\
&+f_{tm}\Big[(\lambda+2b)f_j^a h_{al} - (\mu+2\bar b)h_{jl} - \tfrac{n-3}{n}\operatorname{tr} h f_j^a h^2_{al} + \tfrac{n-3}{n}\operatorname{tr} k\, h^2_{jl}\Big]\\
&+2(n-3)\big[h^2_{tm}h_{jl} - h^2_{lt}h_{jm} + h^2_{jm}h_{lt} - h^2_{jl}h_{tm}\\
&-f_t^a h^2_{am} f_j^b h_{bl} + f_l^a h^2_{at} f_j^b h_{bm} - f_j^a h^2_{am} f_l^b h_{bt} + f_j^a h^2_{al} f_t^b h_{bm}\big] = 0
\end{aligned} \tag{5.3}$$

wherefrom, transvecting with g^{jl} we get

$$\begin{aligned}
&(\lambda+2b)h_{tm} + (\mu+2\bar b)f_t^a h_{am} - \frac{n-3}{n}\operatorname{tr} h\, h^2_{tm} - \frac{n-3}{n}\operatorname{tr} k f_t^a h^2_{am}\\
&= pg_{tm} + qf_{tm} - \frac{n-3}{n}\big[\operatorname{tr} h\, h^2_{tm} - \operatorname{tr} h^2 h_{tm} - \operatorname{tr}(fh) f_t^a h^2_{am} + \operatorname{tr}(fh^2) f_t^a h_{tm}\big],
\end{aligned} \tag{5.4}$$

where p and q are some scalar functions.

The relation (5.4) implies

$$(5.5)\quad (\lambda+2b)f_t^a h_{am}-(\mu+2\bar{b})h_{tm}-\frac{n-3}{n}\operatorname{tr}h f_t^a h_{am}^2+\frac{n-3}{n}\operatorname{tr}k\,h_{tm}^2$$
$$=-qg_{tm}+pf_{tm}-\frac{n-3}{n}\big[\operatorname{tr}h f_t^a h_{am}^2-\operatorname{tr}h^2 f_t^a h_{am}+\operatorname{tr}(fh)h_{tm}^2-\operatorname{tr}(fh^2)h_{tm}\big].$$

Substituting (5.4) and (5.5) into (5.3), and supposing $n\neq 3$, we get

$$\begin{aligned}
&\frac{\operatorname{tr}h^2}{2n}\big(g_{jl}h_{tm}+g_{tl}h_{jm}-g_{jm}h_{tl}-g_{tm}h_{jl}\\
&\qquad -f_{jl}f_t^a h_{am}-f_{tl}f_m^a h_{aj}+f_{jm}f_t^a h_{al}+f_{tm}f_j^a h_{al}\big)\\
&-\frac{\operatorname{tr}h}{2n}\big(g_{jl}h_{tm}^2+g_{tl}h_{jm}^2-g_{jm}h_{tl}^2-g_{tm}h_{jl}^2\\
&\qquad -f_{jl}f_t^a h_{am}^2-f_{tl}f_m^a h_{aj}^2+f_{jm}f_t^a h_{al}^2+f_{tm}f_j^a h_{al}^2\big)\\
&-\frac{\operatorname{tr}(fh^2)}{2n}\big(g_{jl}f_t^a h_{am}+g_{tl}f_j^a h_{am}-g_{jm}f_t^a h_{al}-g_{tm}f_j^a h_{al}\\
&\qquad +f_{jl}h_{tm}+f_{tl}h_{mj}-f_{jm}h_{tl}-f_{tm}h_{jl}\big)\\
&+\frac{\operatorname{tr}(fh)}{2n}\big(g_{jl}f_m^a h_{at}^2+g_{tl}f_j^a h_{am}^2-g_{jm}f_t^a h_{al}^2-g_{tm}f_j^a h_{al}^2\\
&\qquad +f_{jl}h_{mt}^2+f_{tl}h_{jm}^2-f_{jm}h_{tl}^2-f_{tm}h_{lj}^2\big)\\
&\qquad +h_{jl}h_{tm}^2+h_{tl}h_{jm}^2-h_{jm}h_{tl}^2-h_{tm}h_{jl}^2\\
&\qquad -f_j^a h_{al}f_t^b h_{bm}^2-f_t^a h_{al}f_j^b h_{bm}^2+f_j^a h_{am}f_t^b h_{bl}^2+f_t^a h_{am}f_j^b h_{bl}^2=0.
\end{aligned}$$

This relation, in view of (3.2), can be expressed in the form

$$(5.6)\quad \frac{\operatorname{tr}h^2}{2n}Q(g,h)-\frac{\operatorname{tr}h}{2n}Q(g,h^2)-\frac{\operatorname{tr}(fh^2)}{2n}Q(f,h)+\frac{\operatorname{tr}(fh)}{2n}Q(f,h^2)$$
$$+Q(h,h^2)=0.$$

Now, let us consider the expression

$$\begin{aligned}
&Q\Big(h-\frac{\operatorname{tr}h}{2n}g+\frac{\operatorname{tr}(fh)}{2n}f,\ h^2-\frac{\operatorname{tr}h^2}{2n}g+\frac{\operatorname{tr}(fh^2)}{2n}f\Big)\\
&\quad=Q(h,h^2)-\frac{\operatorname{tr}h^2}{2n}Q(h,g)+\frac{\operatorname{tr}(fh^2)}{2n}Q(h,f)\\
&\quad-\frac{\operatorname{tr}h}{2n}Q(g,h^2)+\frac{\operatorname{tr}h}{2n}\frac{\operatorname{tr}h^2}{2n}Q(g,g)-\frac{\operatorname{tr}h}{2n}\frac{\operatorname{tr}(fh^2)}{2n}Q(g,f)\\
&+\frac{\operatorname{tr}(fh)}{2n}Q(f,h^2)-\frac{\operatorname{tr}(fh)}{2n}\frac{\operatorname{tr}h^2}{2n}Q(f,g)+\frac{\operatorname{tr}(fh)}{2n}\frac{\operatorname{tr}(fh^2)}{2n}Q(f,f).
\end{aligned}$$

But

$$Q(g,g) = Q(g,f) = Q(f,g) = Q(f,f) = 0.$$

Therefore

$$\begin{aligned} &Q\Big(h - \frac{\operatorname{tr} h}{2n}g + \frac{\operatorname{tr}(fh)}{2n}f,\ h^2 - \frac{\operatorname{tr} h^2}{2n}g + \frac{\operatorname{tr}(fh^2)}{2n}f\Big) \\ &= Q(h,h^2) + \frac{\operatorname{tr} h^2}{2n}Q(g,h) - \frac{\operatorname{tr}(fh^2)}{2n}Q(f,h) - \frac{\operatorname{tr} h}{2n}Q(g,h^2) + \frac{\operatorname{tr}(fh)}{2n}Q(f,h^2) \end{aligned}$$

wherefrom, according to (5.6), it follows

$$Q\Big(h - \frac{\operatorname{tr} h}{2n}g + \frac{\operatorname{tr}(fh)}{2n}f,\ h^2 - \frac{\operatorname{tr} h^2}{2n}g + \frac{\operatorname{tr}(fh^2)}{2n}f\Big) = 0. \tag{5.7}$$

Putting

$$\begin{aligned} A &= h - \frac{\operatorname{tr} h}{2n}g + \frac{\operatorname{tr}(fh)}{2n}f, \\ D &= h^2 - \frac{\operatorname{tr} h^2}{2n}g + \frac{\operatorname{tr}(fh^2)}{2n}f \end{aligned}$$

and applying Lemma 1, we get from (5.7)

$$\begin{aligned} h^2 - \frac{\operatorname{tr} h^2}{2n}g + \frac{\operatorname{tr}(fh^2)}{2n}f = \delta\Big(h - \frac{\operatorname{tr} h}{2n}g + \frac{\operatorname{tr}(fh)}{2n}f\Big) \\ + \bar{\delta}\Big(fh - \frac{\operatorname{tr} h}{2n}f - \frac{\operatorname{tr}(fh)}{2n}g\Big). \end{aligned}$$

In the other words,

$$h_{ij}^2 = \gamma g_{ij} + \bar{\gamma} f_{ij} + \delta h_{ij} + \bar{\delta} f_i^a h_{aj}, \tag{5.8}$$

where γ, $\bar{\gamma}$, δ and $\bar{\delta}$ are some scalar functions.

Substituting (5.8) into (4.7), we obtain

$$\begin{aligned} &\frac{\lambda + 4\gamma}{2(n-2)}G_{ijlm} + \frac{\mu + 4\bar{\gamma}}{2(n-2)}f_i^a G_{ajlm} \\ &\quad + \frac{2\delta - \operatorname{tr} h}{2(n-2)}\big(g_{im}h_{jl} + g_{jl}h_{im} - g_{il}h_{jm} - g_{jm}h_{il} \\ &\qquad - f_{im}f_j^a h_{al} - f_{jl}f_i^a h_{am} + f_{il}f_j^a h_{am} + f_{jm}f_i^a h_{al}\big) \\ &\quad + \frac{2\bar{\delta} - \operatorname{tr} k}{2(n-2)}\big(f_{im}h_{jl} + f_{jl}h_{im} - f_{il}h_{jm} - f_{jm}h_{il} \\ &\qquad + g_{im}f_j^a h_{al} + g_{jl}f_i^a h_{am} - g_{il}f_j^a h_{am} - g_{jm}f_i^a h_{al}\big) \\ &\qquad + h_{im}h_{jl} - h_{il}h_{jm} - f_i^a h_{am}f_j^b h_{bl} + f_i^a h_{al}f_j^b h_{bm} = 0. \end{aligned} \tag{5.9}$$

If we put

$$A_{ij} = h_{ij} + \frac{2\delta - \operatorname{tr} h}{2(n-2)} g_{ij} + \frac{2\bar{\delta} - \operatorname{tr} k}{2(n-2)} f_{ij}, \tag{5.10}$$

we can rewrite (5.9) as follows

$$\begin{aligned} A_{im}A_{jl} - A_{il}A_{jm} - f_i^a A_{am} f_j^b A_{bl} + f_i^a A_{al} f_j^b A_{bm} \\ = \left[\Big(\frac{2\delta - \operatorname{tr} h}{2(n-2)} \Big)^2 - \Big(\frac{2\bar{\delta} - \operatorname{tr} k}{2(n-2)} \Big)^2 - \frac{\lambda + 4\gamma}{2(n-2)} \right] G_{ijlm} \\ - \left[\frac{(2\delta - \operatorname{tr} h)(2\bar{\delta} - \operatorname{tr} k)}{4(n-2)^2} - \frac{\mu + 4\bar{\gamma}}{2(n-2)} \right] f_i^a G_{ajlm}. \end{aligned}$$

Now, according to Lemma 2, (3.8) holds, and in view of (5.10), we finally obtain

$$h_{ij} = \phi g_{ij} + \psi f_{ij} + \tau V_{ij} + \sigma \bar{V}_{ij}, \tag{5.11}$$

where ϕ, ψ, τ and σ are some scalar functions.

Conversely, if (5.11) holds, then (4.7) holds, too. Indeed, if (5.11) holds, then

$$\begin{aligned} f_i^a h_{aj} &= -\psi g_{ij} + \phi f_{ij} - \sigma V_{ij} + \tau \bar{V}_{ij}, \\ h_{ij}^2 &= (\phi^2 - \psi^2) g_{ij} + 2\phi\psi f_{ij} + T V_{ij} + S \bar{V}_{ij}, \\ f_i^a h_{aj}^2 &= -2\phi\psi g_{ij} + (\phi^2 - \psi^2) f_{ij} - S V_{ij} + T \bar{V}_{ij}, \end{aligned} \tag{5.12}$$

where

$$\begin{aligned} T &= 2(\phi\tau - \psi\sigma) + (\tau^2 - \sigma^2) V_a V^a + 2\tau\sigma V_a \bar{V}^a, \\ S &= 2(\phi\sigma + \psi\tau) + 2\tau\sigma V_a V^a - (\tau^2 - \sigma^2) V_a \bar{V}^a. \end{aligned}$$

Substituting this into left-hand side of (4.7), we get

$$\begin{aligned} &[\lambda - 2(\phi \operatorname{tr} h - \psi \operatorname{tr} k) + 2n(\phi^2 - \psi^2)] G_{ijlm} \\ &+ [\mu - 2(\psi \operatorname{tr} h + \phi \operatorname{tr} k) + 4n\phi\psi] f_i^a G_{ajlm} \\ &+ [2T - (\tau \operatorname{tr} h - \sigma \operatorname{tr} k) + 2(n-2)(\phi\tau - \psi\sigma)] \\ &\quad \times [V_{jl} g_{im} + V_{im} g_{jl} - V_{jm} g_{il} - V_{il} g_{jm} \\ &\quad - \bar{V}_{jl} f_{im} - \bar{V}_{im} f_{jl} + \bar{V}_{jm} f_{il} + \bar{V}_{il} f_{jm}] \\ &+ [2S - (\sigma \operatorname{tr} h + \tau \operatorname{tr} k) + 2(n-2)(\phi\sigma + \psi\tau)] \\ &\quad \times [\bar{V}_{jl} g_{im} + \bar{V}_{im} g_{jl} - \bar{V}_{jm} g_{il} - \bar{V}_{il} g_{jm} \\ &\quad + V_{jl} f_{im} + V_{im} f_{jl} - V_{jm} f_{il} - V_{il} f_{jm}]. \end{aligned} \tag{5.13}$$

The relations (5.11) and (5.12) yield

$$\begin{aligned}\operatorname{tr} h &= 2n\phi + 2\tau V_a V^a + 2\sigma V_a \bar{V}^a,\\ \operatorname{tr} k &= 2n\psi - 2\tau V_a \bar{V}^a + 2\sigma V_a V^a.\end{aligned}$$

Thus

$$\begin{aligned}\tau \operatorname{tr} h - \sigma \operatorname{tr} k &= 2n(\tau\phi - \sigma\psi) + 2(\tau^2 - \sigma^2) V_a V^a + 4\tau\sigma V_a \bar{V}^a,\\ \sigma \operatorname{tr} h + \tau \operatorname{tr} k &= 2n(\phi\sigma + \psi\tau) + 4\tau\sigma V_a V^a - 2(\tau^2 - \sigma^2) V_a \bar{V}^a,\end{aligned}$$

and therefore

$$\begin{aligned}2T - (\tau \operatorname{tr} h - \sigma \operatorname{tr} k) + 2(n-2)(\phi\tau - \psi\sigma) &= 0,\\ 2S - (\sigma \operatorname{tr} h + \tau \operatorname{tr} k) + 2(n-2)(\phi\sigma + \psi\tau) &= 0.\end{aligned}$$

So (5.13), that is left-hand side of (4.7), reduces to

$$\begin{aligned}&[\lambda - 2(\phi \operatorname{tr} h - \psi \operatorname{tr} k) + 2n(\phi^2 - \psi^2)] G_{ijlm}\\ &+ [\mu - 2(\psi \operatorname{tr} h + \phi \operatorname{tr} k) + 4n\phi\psi] f_i^a G_{ajlm}.\end{aligned}$$

On the other words, if (5.11) holds, then (4.6) becomes

$$\begin{aligned}2(n-2)(CH)_{ijlm} = {}&[\lambda - 2(\phi \operatorname{tr} h - \psi \operatorname{tr} k) + 2n(\phi^2 - \psi^2)] G_{ijlm}\\ &+ [\mu - 2(\psi \operatorname{tr} h + \phi \operatorname{tr} k) + 4n\phi\psi] f_i^a G_{ajlm}.\end{aligned}$$

But $(CH)_{ijlm} g^{im} = 0$, while

$$G_{ijlm} g^{im} = 2(n-1) g_{jl}, \quad f_i^a G_{ajlm} g^{im} = 2(n-1) f_{jl}$$

and we have

$$\begin{aligned}&[\lambda - 2(\phi \operatorname{tr} h - \psi \operatorname{tr} k) + 2n(\phi^2 - \psi^2)] g_{jl}\\ &+ [\mu - 2(\psi \operatorname{tr} h + \phi \operatorname{tr} k) + 4n\phi\psi] f_{jl} = 0.\end{aligned}$$

It follows

$$\begin{aligned}\lambda &= 2(\phi \operatorname{tr} h - \psi \operatorname{tr} k) - 2n(\phi^2 - \psi^2),\\ \mu &= 2(\psi \operatorname{tr} h + \phi \operatorname{tr} k) - 4n\phi\psi,\end{aligned}$$

and therefore $(CH)_{ijlm} = 0$.

Finally, we remark that if $n = 3$, then (5.4) reduces to

$$(\lambda + 2b)h_{tm} + (\mu + 2\bar{b})f_t^a h_{am} = pg_{tm} + qf_{tm}$$

This, together with

$$-(\mu + 2b)h_{tm} + (\lambda + 2b)f_t^a h_{am} = -qg_{tm} + pf_{tm}$$

implies

$$h_{ij} = \phi g_{ij} + \psi f_{ij}. \tag{5.14}$$

Thus, we can state

Theorem 1. *Let (M, g, f) be a holomorphic hypersurface of dimension $2n$, $n > 3$ of a HC-flat anti-Kähler manifold. Let h and k be the second fundamental forms with respect to the normals $N_{1|}$ and $N_{2|}$ satisfying* (2.7). *Then, in order that (M, g, f) be itself HC-flat, it is necessary and sufficient that h satisfies* (5.11), *where ϕ, ψ, τ and σ are some scalar functions,*

$$V_{ij} = V_i V_j - \bar{V}_i \bar{V}_j, \quad \bar{V}_{ij} = f_i^a V_{aj} = \bar{V}_i V_j + V_i \bar{V}_j, \tag{5.15}$$

V is a vector field, $\bar{V} = fV$ and $k_{ij} = -f_i^a h_{aj}$.

If $n = 3$, then (5.14) *holds.*

6. HC-flat h-hypersurfaces of a manifold of constant totally real sectional curvature. In this section we consider the special case when the ambient manifold $\tilde{M}$ is of constant totally real sectional curvature. Such a manifold is HC-flat and we can apply Theorem 1.

On one hand, if $\tilde{M}$ is of constant totally real sectional curvature, according to (1.5), (2.3) and (2.6), (4.1) gives

$$\begin{aligned} R_{ijlm} = {} & \frac{\tilde{\varkappa}}{4m(m-1)} G_{ijlm} - \frac{\overset{*}{\varkappa}}{4m(m-1)} f_i^a G_{ajlm} \\ & + h_{im}h_{jl} - h_{il}h_{jm} - k_{im}k_{jl} + k_{il}k_{jm}. \end{aligned} \tag{6.1}$$

On the other hand, for a HC-flat h-hypersurface, (5.11) holds, because of which, and taking into account that

$$\begin{aligned} V_{im}V_{jl} - V_{il}V_{jm} - \bar{V}_{im}\bar{V}_{jl} + \bar{V}_{il}\bar{V}_{jm} = 0, \\ V_{im}\bar{V}_{jl} + \bar{V}_{im}V_{jl} - V_{il}\bar{V}_{jm} - \bar{V}_{il}V_{jm} = 0, \end{aligned}$$

we have

$$\begin{aligned}h_{im}h_{jl}-h_{il}h_{jm}-k_{im}k_{jl}+k_{il}k_{jm}&=(\phi^2-\psi^2)G_{ijlm}+2\phi\psi f_i^aG_{ajlm}\\&\quad+(\phi\tau-\psi\sigma)(g_{im}V_{jl}+g_{jl}V_{im}-g_{il}V_{jm}-g_{jm}V_{il}\\&\qquad-f_{im}\bar V_{jl}-f_{jl}\bar V_{im}+f_{il}\bar V_{jm}+f_{jm}\bar V_{il})\\&\quad+(\phi\sigma+\psi\tau)(g_{im}\bar V_{jl}+g_{jl}\bar V_{im}-g_{il}\bar V_{jm}-g_{jm}\bar V_{il}\\&\qquad+f_{im}V_{jl}+f_{jl}V_{im}-f_{il}V_{jm}-f_{jm}V_{il}).\end{aligned}$$

Thus, (6.1) can be expressed in the form

$$\begin{aligned}R_{ijlm}&=\lambda G_{ijlm}+\mu f_i^aG_{ajlm}\\&\quad+\xi(g_{im}V_{jl}+g_{jl}V_{im}-g_{il}V_{jm}-g_{jm}V_{il}\\&\qquad-f_{im}\bar V_{jl}-f_{jl}\bar V_{im}+f_{il}\bar V_{jm}+f_{jm}\bar V_{il})\\&\quad+\theta(g_{im}\bar V_{jl}+g_{jl}\bar V_{im}-g_{il}\bar V_{jm}-g_{jm}\bar V_{il}\\&\qquad+f_{im}V_{jl}+f_{jl}V_{im}-f_{il}V_{jm}-f_{jm}V_{il}),\end{aligned}\tag{6.2}$$

where

$$\lambda=\frac{\tilde\varkappa}{4n(n+1)}+\phi^2-\psi^2,\quad \mu=-\frac{\overset{\sim}{*}\varkappa}{4n(n+1)}+2\phi\psi,$$
$$\xi=\phi\tau-\psi\sigma,\quad \theta=\phi\sigma+\psi\tau.$$

Conversely, if (6.2) holds, then

$$\begin{aligned}\rho_{jl}&=2\big[(n-1)\lambda+\xi V_aV^a+\theta V_a\bar V^a\big]g_{jl}\\&\quad+2\big[(n-1)\mu-\xi V_a\bar V^a+\theta V_aV^a\big]f_{jl}+2(n-2)\big(\xi V_{jl}+\theta\bar V_{jl}\big)\\f_j^a\rho_{al}&=\overset{*}{\rho}_{jl}=-2\big[(n-1)\mu-\xi V_a\bar V^a+\theta V_aV^a\big]g_{jl}\\&\quad+2\big[(n-1)\lambda+\xi V_aV^a+\theta V_a\bar V^a\big]f_{jl}+2(n-2)\big[-\theta V_{jl}+\xi\bar V_{jl}\big],\\\varkappa&=4(n-1)\big[n\lambda+2\xi V_aV^a+2\theta V_a\bar V^a\big],\\\overset{*}{\varkappa}&=-4(n-1)\big[n\mu+2\theta V_aV^a-2\xi V_a\bar V^a\big],\end{aligned}$$

and we easily find that

$$\begin{aligned}(CH)_{ijlm}&=R_{ijlm}-\frac{1}{2(n-2)}\big(g_{im}\rho_{jl}+g_{jl}\rho_{im}-g_{il}\rho_{jm}-g_{jm}\rho_{il}\\&\qquad-f_{im}\overset{*}{\rho}_{jl}-f_{jl}\overset{*}{\rho}_{im}+f_{il}\overset{*}{\rho}_{jm}+f_{jm}\overset{*}{\rho}_{il}\big)\\&\quad+\frac{\varkappa}{4(n-1)(n-2)}G_{ijlm}-\frac{\overset{*}{\varkappa}}{4(n-1)(n-2)}f_i^aG_{ajlm}=0.\end{aligned}$$

But, if $n = 3$, then $\tau = \sigma = 0$.

Therefore $\xi = \theta = 0$ and (6.2) reduces to $R_{ijlm} = \lambda G_{ijlm} + \mu f_i^a G_{ajlm}$ i.e., (M, g, f) is itself a manifold of constant totally real sectional curvatures.

Thus, we can state

Theorem 2. *Let (M, g, f) be a h-hypersurface of the anti-Kähler manifold of constant totally real sectional curvatures. If $n > 3$ $(2n = \dim M)$, then (M, g, f) is CH-flat if and only if its curvature tensor has the form* (6.2)*, where V_{ij} and $\bar{V}_{ij}$ have the meaning* (5.15)*, and λ, μ, ξ and θ are some scalar functions. If $n = 3$, then (M, g, f) is CH-flat if and only if it is itself a manifold of constant totally real sectional curvatures.*

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