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MIRON-NORDEN MODELS OF LAGRANGE TYPE

BY

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Dedicated to Academician Radu Miron at his 80th anniversary

Abstract. Acad. R. Miron was the first who introduced the notion of Norden space. The author of this paper, obtains a systematic theory of the operators $\overset{(12)}{D}G$, $\overset{(21)}{D}G$ with respect to a pseudoriemannian metric G. In the special case of G-conjugation he obtains a classification of all the Einstein or Einstein-Maxwell models. He also proves that for the special case of the Lagrange G-conjugation one can obtain Norden-Lagrange type models, which will be called Miron-Norden models of Lagrange type.

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1. Introduction. Let $\xi = (E, \pi, M)$ be a vector bundle over a connected and paracompact manifold M_n, with the type fiber, $F = R^m$ and $\dim E = n + m$.

Assuming E_{n+m} is endowed with a pseudoriemannian structure G on E, and the restriction of G to the vertical distribution $u \to V_uE$, on E, is a pseudo-riemannian structure, there exists an unique non-linear connection, N in ξ, such that

$$G(hX, vY) = 0, \quad X, Y \in \mathcal{X}(E)$$

where h and v are the projectors associated to N. It produces the following decomposition of G:

$$G = hG + vG$$

Using N, one obtains a horizontal distribution $u \to H_uE$ on E, such that $T_uE = H_uE \oplus V_uE,\ u \in E$.

Definition 1.1. *The symbol ε_X, defined by*

$$\varepsilon_X = \begin{cases} -1 & \text{if} \quad G(X,X) < 0 \\ 0 & \text{if} \quad G(X,X) = 0 \\ 1 & \text{if} \quad G(X,X) > 0 \end{cases}, \quad X \in T_uE$$

is called the pseudo-riemannian indicator of the vector X.

Definition 1.2. *The length of X, denoted by $|X| > 0$, is defined by*

$$|X|^2 = \varepsilon_X G(X,X).$$

A non null vector $U \in T_uE$ is called unit if $g(u,u) = \varepsilon_u$, $\varepsilon_u \in \{-1,1\}$.
If $\varepsilon_X < 0$ then X is said to be of spacelike type.
If $\varepsilon_X = 0$ then X is said to be a null vector.
If $\varepsilon_X > 0$ then X is said to be of time like type.
The sets $C_u^{(-)}$, $C_u^{(0)}$, $C_u^{(+)}$ are defined by

$$C_u^{(-)} = \{X \in T_uE \mid \varepsilon_X < 0\}, \quad C_u^{(0)} = \{X \in T_uE \mid \varepsilon_X = 0\},$$
$$C_u^{(+)} = \{X \in T_uE \mid \varepsilon_X > 0\}$$

and the sets $C^{(-)}$, $C^{(0)}$, $C^{(+)}$ are defined by

$$C^- : u \to C_u^-, \quad C^0 : u \to C_u^0, \quad C^+ : u \to C_u^+.$$

Definition 1.3. *Let be two distributions, $\underset{(r_1)}{\overset{(1)}{\mathcal{D}}} : u \to \underset{(r_1)}{\overset{(1)}{\mathcal{D}}}_u \subset T_uE$, $\underset{(r_2)}{\overset{(2)}{\mathcal{D}}} : u \to \underset{(r_2)}{\overset{(2)}{\mathcal{D}}} \subset T_uE$; $\dim \underset{(r_1)}{\overset{(1)}{\mathcal{D}}} = r_1$; $\dim \underset{(r_2)}{\overset{(1)}{\mathcal{D}}} = r_2$ such that $\underset{(r_1)}{\overset{(1)}{\mathcal{D}}} \perp \underset{(r_2)}{\overset{(2)}{\mathcal{D}}}$ and $r_1 + r_2 = n + m$. Then $\underset{(r_1)}{\overset{(1)}{\mathcal{D}}}, \underset{(r_2)}{\overset{(2)}{\mathcal{D}}}$ are called conjugated distributions on E.*

a) *If $\underset{(r_1)}{\overset{(1)}{\mathcal{D}}} \cap \underset{(r_2)}{\overset{(2)}{\mathcal{D}}} = \{0\}$, then $\underset{(r_1)}{\overset{(1)}{\mathcal{D}}}, \underset{(r_2)}{\overset{(2)}{\mathcal{D}}}$ are called the supplementary distributions on E.*

b) *If $\underset{(r_1)}{\overset{(1)}{\mathcal{D}}} \cap \underset{(r_2)}{\overset{(2)}{\mathcal{D}}} = \underset{(s)}{\overset{(0)}{\mathcal{D}}}$; $\dim \underset{(s)}{\overset{(0)}{\mathcal{D}}} = s \geq 1$, then $\underset{(s)}{\overset{(0)}{\mathcal{D}}}$ is called a null distribution, on E.*

If $X \in \underset{(s)}{\mathcal{D}}^{(0)}$, $X \neq 0$, we have $G(XX) = 0$.

From (1), (5) we have:

Proposition 1.1. *The horizontal distribution $u \to H_u E$, on E and the vertical distribution $u \to V_u E$ on E, are conjugated and supplementary distributions.*

Proposition 1.2. *In general case we have $s \leq \dfrac{n+m}{2}$.*

If $s < \dfrac{n+m}{2}$ then we have $\underset{(r)}{\overset{(0)}{\mathcal{D}}}{}_u \subset C_u^{(0)}$, $u \in E$.

Definition 1.4. *A d-linear connection D on E, is a linear connection D on E, which preserves by parallelism the horizontal and vertical distributions on E.*

One easily obtains

Proposition 1.3. *The conjugated horizontal and vertical distribution, on E, are conserved by the parallelism defined by a linear connection D on E, iff $hD_X vY = 0$; $vD_X hY = 0$; $X, Y \in \mathcal{X}(E)$.*

2. The operators $\overset{(12)}{D^h}$, $\overset{(12)}{D^v}$, $\overset{(21)}{D^h}$, $\overset{(21)}{D^v}$. In [3], [4] the author gives a very general theory of the operators $\overset{(12)}{D}$, $\overset{(21)}{D}$. In [3] a special attention is paid to the case when $\overset{(12)}{D}$, $\overset{(21)}{D}$ are associated to two linear d-connections on E, or $E = TM$, or E_{n+1}. The applications in the theory of Relativity are given. One obtains the classification of the Einstein models and of the Einstein-Maxwell models, as well.

In this paper we will present some properties of these operators on the almost Hermitian model of a generalized Lagrange Space introduced by Miron ([1]).

Let $(E = TM, \pi, M)$ be the tangent bundle over M_n and G a pseudo-riemannian structure, on $E = TM$ and $N = \overset{G}{N}$ a non-linear connection. We have $n = m$; $\dim(E = TM) = 2n$.

The natural almost complex structure, F, associated to $N = \overset{G}{N}$ is given by: $F(X^h) = -X^v$; $F(X^v) = X^h$, where X^h is the horizontal lift of $X \in \mathcal{X}(M)$ and X^v is the vertical lift of $X \in \mathcal{X}(M)$.

The pair (G, F) is an almost Hermitian structure on $\widetilde{TM} = TM \setminus \{0\}$ if $G(FX, FY) = G(X, Y)$, $X, Y \in \mathcal{X}(\widetilde{TM})$.

We have

Proposition 2.1. *If (G, F) is almost Hermitian structure then we have:*

$$G(X^h, Y^h) = G(X^v, Y^v)$$

$$(hG)(X^h, Y^h) = (vG)(X^v, Y^v), \quad X, Y \in \mathcal{X}(M).$$

Let $\{\delta/\delta x^i; \partial/\partial y^i\}$ be a local frame adapted to N and $\{dx^i, \delta y^i\}$ its dual. Then F is given by:

$$F(\delta, \delta x^i) = -\partial/\partial y^i, \quad F(\partial/\partial y^i) = \delta/\delta x^i.$$

When (G, F) is an almost Hermitian structure on $\widetilde{TM}$ we have:

$$G = g_{kr} dx^k \otimes dx^r + g_{is} \delta y^i \otimes \delta y^s, \quad k, r; l, s = \overline{1, n}$$

where (g_{kr}) is a d-tensor field, pseudo-riemannian.

The almost Hermitian space, $H^{2n} = (N; TM, G, F)$ is called by R. Miron, the almost Hermitian model of the generalized Lagrange space, $GL^n = (M, g_{rs}(xy))$.

If a linear d-connections $D(N)$ on $E = TM$ is a normal connection then we have ([6]) $D_X F(Y) = F(D_X Y)$, $X, Y \in \mathcal{X}(TM)$.

In a local frame adapted to N we have the local coefficients ([6])

$$D^h(\Gamma^l_{jk}(x, y), \Gamma^{n+i}_{n+j\ k} = \Gamma^i_{j\ k}), \quad D^v(\Gamma^i_{j\ n+k}(x, y), \Gamma^{n+i}_{n+j\ n+k} = \Gamma^i_{j\ n+k}).$$

We denote this connections by $\Gamma(N) = (\Gamma^i_{jk}, \Gamma^{n+i}_{n+j\ n+k})$. It is called the connection of the Lagrange spaces GL^n.

By (1.7) if X, of local components $\{X^i(x, y)\}$ is a d-vector field, we have $|X|^2 = \varepsilon_X g_{ik}(x, y) X^i X^k$ and the absolute energy on $\widetilde{TM}_1$ is given by $E(x, y) = \varepsilon_y g_{ik} y^i y^k$ and the energy is given by $\overline{E}(x, y) = \varepsilon_y E(x, y)$.

Definition 2.1. *Let be two normal connections* $\overset{(1)}{D}$, $\overset{(2)}{D}$. *The operators* $\overset{(12)}{D}{}^h G$, $\overset{(12)}{D}{}^v G$, *are defined by:*

$$(\overset{(12)}{D}{}^h_k G)_{ij} = \overset{(1)}{D}{}^h_k g_{ij} - \overset{(21)}{\tau}{}^r_{jk} g_{ir} \tag{2.1}$$

$$(\overset{(12)}{D}{}^v_{n+k} G)_{n+i\ m+j} = \overset{(1)}{D}{}^v_{n+k} g_{ij} - \overset{(21)}{\tau}{}^{n+r}_{n+j\ n+k} g_{ir} \tag{2.2}$$

where $\overset{(21)}{\tau} = \overset{(2)}{\Gamma} - \overset{(1)}{\Gamma}$.

The dual operators $\overset{(21)}{D}{}^h$, $\overset{(21)}{D}{}^v$ are defined by

$$(\overset{(21)}{D}{}^h_k G)_{ij} = \overset{(2)}{D}{}^h_k g_{ij} - \overset{(12)}{\tau}{}^r_{jk} g_{ir} \tag{2.3}$$

$$(\overset{(21)}{D}{}^v_{n+k} G)_{n+i\ n+j} = \overset{(2)}{D}{}^v_{n+k} g_{ij} - \overset{(12)}{\tau}{}^{n+r}_{n+j\ n+k} g_{ir} \tag{2.4}$$

where $\overset{(12)}{\tau} = -\overset{(21)}{\tau}$.

Proposition 2.2. *In general we have*

$$(\overset{(12)}{D}{}^h_k G)_{ij} \neq (\overset{(12)}{D}{}^h_k G)_{ji} \tag{2.5}$$

$$(\overset{(12)}{D}{}^v_{n+k} G)_{n+i\ n+j} \neq (\overset{(12)}{D}{}^v_{n+k} G)_{n+j\ n+i}, \qquad \overset{(12)}{D} G \neq \overset{(21)}{D} G.$$

The properties of $\overset{(12)}{D} G$, $\overset{(21)}{D} G$ in the general case are established by the author in [3], [4], [5].

Definition 2.2. *If the orthogonality of an arbitrary 1-dimensional distribution* $\underset{(1)}{\overset{(1)}{\mathcal{D}}}$, $\underset{(1)}{\overset{(2)}{\mathcal{D}}}$ $(\underset{(1)}{\overset{(1)}{\mathcal{D}}} \perp \underset{(1)}{\overset{(2)}{\mathcal{D}}})$ *is preserved by parallelism relative to* $\overset{(1)}{D}$ *and relative to* $\overset{(2)}{D}$ *then* $\overset{(1)}{D}$ *is said to be conjugated with* $\overset{(2)}{D}$ *or* $\overset{(1)}{D} \overset{G}{\sim} \overset{(2)}{D}$ *or*

$$G\underset{(1)}{L}{}^n = (M, g, \overset{(1)}{\Gamma}(N)) \overset{g}{\sim} G\underset{(2)}{L}{}^n = (M, g, \overset{(2)}{\Gamma}(N)).$$

The Definition 2.2 applied also to $\underset{(1)}{\mathcal{D}}{}^{(0)}$ because $X \in \underset{(1)}{\mathcal{D}}{}^{(0)}$ is selforthogonal $(g(X, X) = 0)$.

Example 2.1. If $\overset{(12)}{D} G = 0$ then $\overset{(21)}{D} G = 0$ and $G\underset{(1)}{L}{}^n \overset{g}{\sim} G\underset{(2)}{L}$.

Proposition 2.3. *If* $\overset{(12)}{D} G = 0$ *then we have*

$$\overset{(2)}{\Gamma}{}^s_{jk} = \overset{(1)}{\Gamma}{}^s_{jk} + g^{si} \overset{(1)}{D}{}^h_k g_{ij}; \qquad \overset{(2)}{\Gamma}{}^{n+s}_{n+j\ n+k} = \overset{(1)}{\Gamma}{}^{n+s}_{n+j\ n+k} + g^{si} \overset{(1)}{D}{}^v_{n+k} g_{ij}$$

and

$$\overset{(1)}{\Gamma}{}^{s}_{jk} = \overset{(2)}{\Gamma}{}^{s}_{jk} + g^{si}\overset{(2)}{D}{}^{h}_{k}g_{ij}; \quad \overset{(1)}{\Gamma}{}^{n+s}_{n+j\ n+k} = \overset{(2)}{\Gamma}{}^{n+s}_{n+j\ n+k} + g^{si}\overset{(2)}{D}{}^{v}_{n+k}g_{ij}.$$

We have: $\overset{m}{\Gamma} \overset{def}{=} \frac{1}{2}(\overset{(1)}{\Gamma} + \overset{(2)}{\Gamma}) = \overset{c}{\Gamma}$ (the canonical connection).

Proposition 2.4. *We have* $G\underset{(1)}{L}{}^{n} \overset{g}{\sim} G\underset{(2)}{L} \Leftrightarrow G\underset{(2)}{L}{}^{n} \overset{g}{\sim} G\underset{(1)}{L}{}^{n}$.

Proposition 2.5. *If* $G\underset{(1)}{L}{}^{n} \overset{g}{\sim} G\underset{(2)}{L}{}^{n}$, *then we have* $\overset{(12)}{D}G = \overset{(21)}{D}G$.

Proposition 2.6. *We have* $G\underset{(1)}{L}{}^{n} \overset{g}{\sim} G\underset{(2)}{L}{}^{n}$ *iff*

$$(\overset{(12)}{D}{}^{h}_{k}G)_{ij} = \alpha_k g_{ij}; \quad (\overset{(12)}{D}{}^{v}_{n+k}G)_{n+i\ n+j} = \beta_{n+k}g_{ij}$$

where α_k, β_{n+k} *are arbitrary d-covectors.*

Proposition 2.7. *If* $G\underset{(1)}{L}{}^{n} \overset{g}{\sim} G\underset{(2)}{L}{}^{n}$, *then* $\overset{(1)}{\Gamma}(N)$, $\overset{(2)}{\Gamma}(N)$ *cannot be both compatible with* $g_{ij}(x,y)$.

Proposition 2.8. *If* $\overset{(1)}{\Gamma}(N)$ *is the canonical connection* $\overset{(c)}{\Gamma}(N)$ *of* $G\underset{(1)}{L}{}^{n}$, *(*$\overset{(1)}{\Gamma}(N) = \overset{(c)}{\Gamma}(N)$*;* $G\underset{(1)}{L}{}^{n}$ *is noted by* $G^{(c)}L^n$*) and* $G\underset{(1)}{L}{}^{n} \overset{g}{\sim} G\underset{(2)}{L}{}^{n}$, *then for* $\overset{(2)}{\Gamma}(N)$ *we cannot have*

$$T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj} = 0 \quad \text{and} \quad T^{n+i}_{n+j\ n+k} = \Gamma^{n+i}_{n+j\ n+k} - \Gamma^{n+i}_{n+j\ n+k} = 0.$$

Another important result is given by

Proposition 2.9. *The set of conjugated spaces* $\{GL^n\}$ *is given by*

$$GL^n \overset{g}{\sim} G^{(c)}L^n, \text{ is true, but}$$

$$G\overline{L}^n \overset{g}{\sim} G^{(c)}L^n \text{ is not true.}$$

This classification plays an important role in a classification of the Einstein models or the classification of the Einstein-Maxwell models ([3], [4]).

Definition 2.3. *If* $G\underset{(1)}{L}{}^n \overset{g}{\sim} G\underset{(2)}{L}$ *and*

$$\overset{(1)}{T}{}^i_{jk} = 0, \quad \overset{(1)}{T}{}^{n+i}_{n+j\ n+k} = 0, \quad \overset{(2)}{T}{}^i_{jk} = 0, \quad \overset{(2)}{T}{}^{n+i}_{n+j\ n+k} = 0 \tag{2.6}$$

the set $\{M, g; \overset{(1)}{\Gamma}(N), \overset{(2)}{\Gamma}\}$ *is called the Miron-Norden space of the generalized Lagrange type.*

Proposition 2.10. *If* $\{M, g; \overset{(1)}{\Gamma}(N), \overset{(2)}{\Gamma}\}$ *is a Miron-Norden space, then* $\overset{(1)}{\Gamma}(N) \neq \overset{(c)}{\Gamma}(N)$, $\quad \overset{(2)}{\Gamma}(N) \neq \overset{(c)}{\Gamma}(N)$.

Proposition 2.11. *If* $\{M, g; \overset{(1)}{\Gamma}(N), \overset{(2)}{\Gamma}\}$ *is a Miron-Norden space then* $\overset{(2)}{\Gamma}(N)$ *is determined by* $(\overset{(1)}{\Gamma}(N), g)$ *as follows:*

$$\overset{(2)}{\Gamma}{}^r_{jk} = \overset{(1)}{\Gamma}{}^r_{jk} + g^{ri}\overset{(1)}{D}{}^h_k g_{ij} - \frac{1}{n-1} g^{is}(\overset{(1)}{D}{}^h_k g_{is} - \overset{(1)}{D}{}^h_s g_{ik})\delta^r_j \tag{2.7}$$

$$\begin{aligned}\overset{(2)}{\Gamma}{}^{n+r}_{n+j\ n+k} &= \overset{(1)}{\Gamma}{}^{n+r}_{n+j\ n+k} + g^{ri}\overset{(1)}{D}{}^v_{n+k} g_{ij} \\ &- \frac{1}{n-1} g^{is}(\overset{(1)}{D}{}^v_{n+k} g_{is} - \overset{(1)}{D}{}^v_{n+s} g_{ik})\delta^{n+r}_{n+j}.\end{aligned} \tag{2.8}$$

We obtain:

(a) $R(XY)Z^h = F(R(X,Y)Z^n)$

(b) $R(XY)Z^v = -F(R(X,Y)Z^h), \quad X, Y \in \mathcal{X}(TM),\ Z \in \mathcal{X}(M)$

where R is the curvature tensor of D.

From (a), (b) the essential local d-components of the curvature of the $\Gamma(N)$ are: R^i_{jkl}, R^i_{jkn+l}, $R^{n+i}_{n+j\ n+k\ n+l}$ ([1]).

Proposition 2.12. *If* $\{M, g, \overset{(1)}{\Gamma}(N), \overset{(2)}{\Gamma}(N)\}$ *is a Miron-Norden space, then we have*

$$\left(\overset{(2)}{R}{}^{n+r}_{n+j\ n+k\ n+l} - \frac{1}{n}\delta^r_j \overset{(2)}{B}_{n+k\ n+l}\right) g_{ri} \tag{2.9}$$

$$+\left(\overset{(1)}{R}{}^{n+r}_{n+i\ n+k\ n+l} - \frac{1}{n}\delta^r_i \overset{(1)}{B}_{n+k\ n+l}\right) g_{rj} = 0$$

where B is the d-tensor of Bianchi.

Proposition 2.13. *If $\{M, g, \overset{(1)}{\Gamma}(N), \overset{(2)}{\Gamma}(N)\}$ is a Miron Norden space and the horizontal distribution is integrable then we have*

$$\left(\overset{(2)}{R}{}^r_{jkl} - \frac{1}{n}\delta^r_j \overset{(2)}{B}_{kl}\right) g_{ri} + \left(\overset{(1)}{R}{}^r_{ikl} - \frac{1}{n}\delta^r_i \overset{(1)}{B}_{kl}\right) g_{rj} = 0 \tag{2.10}$$

where $B_{kl} = R^j_{jkl}$.

Proposition 2.14. *If $\{M, g, \overset{(1)}{\Gamma}(N), \overset{(2)}{\Gamma}(N)\}$ is a Miron-Norden space and $t = 0$ (t is the weak torsion of N) then we have*

$$\left\{\left(\overset{(2)}{R}{}^r_{jk\ n+l} - \frac{1}{n}\delta^r_j \overset{(2)}{B}_{k\ n+l}\right) - \left(\overset{(2)}{R}{}^r_{jl\ n+k} - \frac{1}{n}\delta^r_j \overset{(2)}{B}_{l\ n+k}\right)\right\} g_{ri} \tag{2.11}$$

$$+\left\{\left(\overset{(1)}{R}{}^r_{ik\ n+l} - \frac{1}{n}\delta^r_i \overset{(1)}{B}_{k\ n+l}\right) - \left(\overset{(1)}{R}{}^r_{il\ n+k} - \frac{1}{n}\delta^r_i \overset{(1)}{B}_{l\ n+k}\right)\right\} g_{rj} = 0.$$

Proposition 2.15. *If $\{M, g, \overset{(1)}{\Gamma}(N), \overset{(2)}{\Gamma}(N)\}$ is a Miron-Norden space and F is a complex structure then we have* (2.9), (2.10), (2.11) *and* (2.11) *is reduced to:*

$$\left(\overset{(2)}{R}{}^r_{jkn+l} - \frac{1}{n}\delta^r_j B_{kn+l}\right) g_{ir} + \left(\overset{(1)}{R}{}^r_{ikn+l} - \frac{1}{n}\delta^r_i \overset{(1)}{B}_{kn+l}\right) g_{rj} = 0.$$

Another results are given in [3], [4], [5].

The transformations $(\overset{(1)}{\Gamma}(N), \overset{(2)}{\Gamma}(N)) \to (\overset{(1)}{\overline{\Gamma}}(N), \overset{(2)}{\overline{\Gamma}}(N))$ which preserves the Miron-Norden space and its invariants, $\overset{(12)}{N}{}^{\,i}_{jkl}$, $\overset{(12)}{N}{}^{\,i}_{jk\ n+l}$, $\overset{(12)}{N}{}^{\,n+i}_{n+j\ n+k\ n+l}$ are also obtained [3], [4], [5].

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