ANALELE ŞTIINŢIFICE ALE UNIVERSITĂŢII "AL.I. CUZA" DIN IAŞI (S.N.)
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A CONVERGENCE RESULT FOR SECOND ORDER DIFFERENCE EQUATIONS OF MONOTONE TYPE

BY

G. APREUTESEI and N. APREUTESEI

Abstract. A continuous dependence on data result is established for a class of second order difference inclusions associated to a maximal monotone operator A in a Hilbert space. It is an extension of a theorem from another paper of the authors. The second boundary condition is here more general, namely it is defined with the aid of a monotone operator. One proves that the function that associates to A and to the other initial data, the solution of this boundary value problem is continuous in a specific sense. The sequence of operators is supposed to be convergent in the sense of the resolvent.

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1. Introduction

The goal of this paper is to study the continuous dependence on data for the second order difference problem

$$\begin{cases} u_{i+1} - (1+\theta_i)\, u_i + \theta_i u_{i-1} \in c_i A u_i + f_i, \quad 1 \leq i \leq N \\ u_0 = a,\ u_{N+1} - u_N \in -\beta\,(u_{N+1} - b) \end{cases}, \tag{1.1}$$

where N is a fixed positive integer, $A : D(A) \subseteq H \to H$ and $\beta : D(\beta) \subseteq H \to H$ are maximal monotone operators (possibly multivalued) in the real Hilbert space H, $D(A)$, $D(\beta)$ are their domains, while $a, b \in H$, $f_i \in H$, $c_i > 0$, $\theta_i \in (0,1)$, $1 \leq i \leq N$ are given elements. The above difference inclusion is the discrete variant of the second order differential equation

$$p(t)\, u'' + r(t)\, u' \in Au + f(t),\ t \in [0,T],$$

subject to the boundary conditions

$$u'(0) \in \alpha(u(0) - a),\ u'(T) \in -\beta(u(T) - b),$$

in the case when α is the subdifferential ∂j of the lower semicontinuous, convex, proper function $j : H \to (-\infty, +\infty]$, $j(x) = 0$ for $x = 0$ and $j(x) = +\infty$, otherwise. In this case the domain of α is $D(\alpha) = \{0\}$, so the boundary condition $u'(0) \in \alpha(u(0) - a)$ reduces to $u(0) = a$. This remark has suggested us the first boundary condition in (1.1). Existence theory, together with the asymptotic behavior and continuous dependence on data for the above differential equation can be found in [11], [1], [3], [20], [16] and many others. See also the monograph [8] and the references therein.

Similar results have been proved in the continuous case in [14] for first order differential equations and in [4], [6] for second order differential equations of the above type.

For second order difference inclusions with different boundary conditions, we mention the papers [2]. But unlike the paper [2], where the boundary conditions are bilocal, the second boundary condition from the present paper is defined with the aid of an one-to-one maximal monotone operator β. The domain of β is bounded, while A is supposed to be $\beta-$dissipative. Moreover, the proof of the main result is here simplified. Instead of the four auxiliary problems considered in [2], in the present paper we employ only two approximating problems, which are simpler than those from [2]. They are defined with the aid of the Yosida approximations of A. About maximal monotone operators theory, the reader may refer to [12], [13].

In [19], the authors obtain new convergence results for resolvent of maximal monotone operators and for generalized nonexpansive mappings in a Banach space. Strong convergence results for maximal monotone operators are also found in [15]. In [21] one proves strong and weak convergence theorems for a new resolvent of maximal monotone operators in a Banach space. The convergence theorem is next applied to a convex minimization problem. A convergence theorem for a generalized resolvent of maximal monotone operators in Banach spaces is proved in [18].

If in (1.1) we put $i \geq 1$ (this corresponds in the continuous case to $t \in [0, \infty)$), the continuous dependence on data was studied in [9].

Let $(\cdot, \cdot)$ and $||\cdot||$ be the scalar product and the norm in H and let $(a_i)_{0 \leq i \leq N}$ be the sequence of real numbers given by

$$a_0 = 1,\ a_i = \frac{1}{\theta_1 ... \theta_i},\ 1 \leq i \leq N.$$

Observe that

$$a_i\theta_i = \theta_{i-1},\ 1 \le i \le N. \tag{1.2}$$

Existence and asymptotic behavior of the solutions to problem (1.1) has been studied in [5] , [7] , [17] under different hypotheses on A and β. We recall briefly below such a result that we need in the sequel.

Without loss of generality we suppose that $0 \in A0$. If this condition is not satisfied, one replaces A in (1.1) by $\widetilde{A} = A + A^0 0$ and f_i by $\widetilde{f_i} = f_i - c_i A^0 0$, where $A^0 x$ is the element of least norm of the set Ax, i. e. $||A^0 x|| = \inf\{||y||,\ y \in Ax\}$.

Assume that the domain $D(\beta)$ of β is bounded in H and that A is $\beta-$dissipative, that is

$$(A_\lambda x - A_\lambda y, z) \le 0,\ (\forall)\, z \in -\beta(x-y) \text{ for } x - y \in D(\beta). \tag{1.3}$$

Definition 1.1. The multivalued operator A is called *one-to-one* if condition $Ax \cap Ay \neq \Phi$ implies that $x = y$.

The following result follows directly from the Theorem 3.2, [5] .

Theorem 1.1. *Let $A : D(A) \subseteq H \to H$, $\beta : D(\beta) \subseteq H \to H$ be maximal monotone operators H with $0 \in D(A) \cap D(\beta)$, $0 \in \beta(0)$, $a, b \in H$, $f_i \in H$, $c_i > 0$, $\theta_i \in (0,1)$, $1 \le i \le N$. If condition (1.3) holds and the domain $D(\beta)$ of β is bounded, then problem (1.1) has at least one solution $u = (u_i)_{1 \le i \le N} \in D(A)^N$. If A or β is one-to-one, the solution is unique.*

In the present paper we prove that the function that associates to $\{A, a, b, f_i\}$ the solution $u = (u_i)_{1 \le i \le N}$ of problem (1.1) is the continuos in the following sense. Consider a sequence of maximal monotone operators $A^n : D(A^n) \subseteq H \to H$ with $0 \in D(A^n)$, $a^n, b^n, f_i^n \in H$ sequences from H and the sequence of boundary value problems

$$\begin{cases} u_{i+1}^n - (1+\theta_i)\, u_i^n + \theta_i u_{i-1}^n \in c_i A^n u_i^n + f_i^n, & 1 \le i \le N \\ u_0^n = a^n,\ u_{N+1}^n - u_N^n \in -\beta\left(u_{N+1}^n - b^n\right). \end{cases} \tag{1.4}$$

Our goal is to prove that if $a^n \to a$, $b^n \to b$, $f_i^n \to f_i$ in H and $A^n \to A$ in the sense of resolvent, then $u_i^n \to u_i$ strongly in H, for all $1 \le i \le N$.

The structure of the paper is the following. In the next section, we formulate our hypotheses and the main result. Section 3 contains the proof of the main result. An application to difference-differential equations is given in the last section.

2. The main result

In this section, we give the main result of the paper, namely we show the convergence of the solution of problem (1.4) to the solution of problem (1.1). We work under the following hypotheses.

$(H1)$ $\beta : D(\beta) \subseteq H \to H$ is a nonlinear one-to-one maximal monotone operators in H, $0 \in D(\beta)$, $0 \in \beta(0)$;

$(H2)$ the domain $D(\beta)$ is bounded in H.

$(H3)$ $A : D(A) \subseteq H \to H$, $A^n : D(A^n) \subseteq H \to H$ are maximal monotone operators in H with $0 \in D(A^n) \cap D(A)$.

$(H4)$ A and A^n are $\beta-$dissipative, i. e. (1.3) holds for both A and A^n.

Hypotheses $(H1) - (H4)$ assure the existence and uniqueness of the solutions of problems (1.1) and (1.4).

In the sequel we impose some convergence conditions for the data:

$(H5)$ $a^n \to a$, $b^n \to b$, $f_i^n \to f_i$ in H;

$(H6)$ Sequence A^n converges to A in the sense of resolvent, i. e.

$$(I + \lambda A^n)^{-1} \xi \to (I + \lambda A)^{-1} \xi, \text{ as } n \to \infty, \ (\forall)\, \lambda > 0, \ (\forall)\, \xi \in H.$$

More details about different type of convergences can be found in [10]. The main result now can be stated.

Theorem 2.1. *Under the hypotheses* $(H1)$ *-*$(H6)$, *if* u_i *and* u_i^n *are the solutions to* (1.1) *and* (1.4), *respectively, we have*

$$u_i^n \to u_i \text{ strongly in} H, \text{ for } 1 \leq i \leq N.$$

In order to estimate the norm $||u_i^n - u_i||$, consider the auxiliary boundary value problems

$$(2.1) \quad \begin{cases} u_{i+1}^{\lambda} - (1+\theta_i)\, u_i^{\lambda} + \theta_i u_{i-1}^{\lambda} = c_i A_{\lambda} u_i^{\lambda} + \lambda u_i^{\lambda} + f_i, \quad 1 \leq i \leq N \\ u_0^{\lambda} = a, \ u_{N+1}^{\lambda} - u_N^{\lambda} \in -\beta\left(u_{N+1}^{\lambda} - b\right) \end{cases},$$

$$(2.2) \quad \begin{cases} u_{i+1}^{n\lambda} - (1+\theta_i)\, u_i^{n\lambda} + \theta_i u_{i-1}^{n\lambda} = c_i A_{\lambda}^n u_i^{n\lambda} + \lambda u_i^{n\lambda} + f_i^n, \quad 1 \leq i \leq N \\ u_0^{n\lambda} = a^n, \ u_{N+1}^{n\lambda} - u_N^{n\lambda} \in -\beta\left(u_{N+1}^{n\lambda} - b^n\right). \end{cases}$$

Since $c_iA_\lambda + \lambda I$ and $c_iA^n_\lambda + \lambda I$ are strongly monotone, problems (2.1) and (2.2) admit unique solutions in H (see Theorem 2.2 in [5]).

Remark 2.1. Denote by $J_\lambda = (I + \lambda A)^{-1}$, $J^n_\lambda = (I + \lambda A^n)^{-1}$ the resolvent of A and A^n respectively, and by $A_\lambda = (I - (I + \lambda A)^{-1})/\lambda$ and $A^n_\lambda = (I - (I + \lambda A^n)^{-1})/\lambda$ the associated Yosida approximations. Then, from the convergence $(H6)$ it follows that

$$J^n_\lambda \xi \to J_\lambda \xi,\ A^n_\lambda \xi \to A_\lambda \xi \text{ as } n \to \infty,\ (\forall)\, \lambda > 0,\ (\forall)\, \xi \in H. \tag{2.3}$$

Recall also that

$$\xi = J_\lambda \xi + \lambda A_\lambda \xi,\ \ (\forall)\, \xi \in H. \tag{2.4}$$

In the next section we give some auxiliary results about the boundedness of the solutions to (1.4) and (2.2). Next, we prove Theorem 2.1 with the aid of the solutions of problems (2.1), (2.2).

3. The proof of the main result

To deduce the convergence result from Theorem 2.1, observe that

$$||u^n_i - u_i|| \le ||u^n_i - u^{n\lambda}_i|| + ||u^{n\lambda}_i - u^\lambda_i|| + ||u^\lambda_i - u_i||,\ 1 \le i \le N,\ \lambda > 0. \tag{3.1}$$

We will show that each norm in the right-hand side tends to zero as $\lambda \to 0$ and $n \to \infty$. To do this, we first prove the boundedness with respect to n of the solutions to (1.4) and (2.2) . Like in the proof of Theorem 3.2 from [5] , we obtain the following result.

Lemma 3.1. *If $c_i > 0$, $\theta_i \in (0, 1)$, $1 \le i \le N$, $a^n, b^n \in H$, $f^n_i \in H$ are bounded sequences and $(H1)$ -$(H4)$ hold, then the solutions $(u^n_i)_{1\le i\le N}$ and $\left(u^{n\lambda}_i\right)_{1\le i\le N}$ of problems (1.4) and (2.2) are bounded in H for all $1 \le i \le N$.*

Now we give some estimates for each norm in the right-hand side of the inequality (3.1) .

Lemma 3.2. *Under the hypotheses of the previous section, the solutions of problems (1.1) , (1.4) , (2.1) , (2.2) satisfy the inequalities:*

$$||u^n_i - u^{n\lambda}_i|| \le C\sqrt{\lambda},\ ||u^\lambda_i - u_i|| \le C\sqrt{\lambda},$$

for all i, $1 \le i \le N$ and for some constant $C > 0$ independent of n and λ.

Proof. We show the first inequality. To do this, we multiply the difference between (1.4) and (2.2) by $a_i\left(u_i^n - u_i^{n\lambda}\right)$ and sum up from $i = 1$ to $i = N$. With the aid of (1.2) one finds

$$\begin{aligned}
&\sum_{i=1}^{N} a_i(u_{i+1}^n - u_{i+1}^{n\lambda} - u_i^n + u_i^{n\lambda}, u_i^n - u_i^{n\lambda}) \\
&- \sum_{i=1}^{N} a_{i-1}(u_i^n - u_i^{n\lambda} - u_{i-1}^n + u_{i-1}^{n\lambda}, u_i^n - u_i^{n\lambda}) \\
&= \sum_{i=1}^{N} c_i a_i(s_i^n - A_\lambda^n u_i^{n\lambda}, u_i^n - u_i^{n\lambda}) - \lambda \sum_{i=1}^{N} a_i(u_i^{n\lambda}, u_i^n - u_i^{n\lambda}),
\end{aligned}$$

where s_i^n is an arbitrary element from the set $A^n u_i^n$.

Since $A_\lambda^n u_i^{n\lambda} \in A^n(J_\lambda^n u_i^{n\lambda})$ and $s_i^n \in A^n u_i^n$, in view of the monotonicity of A we get $(s_i^n - A_\lambda^n u_i^{n\lambda}, u_i^n - J_\lambda^n u_i^{n\lambda}) \geq 0$. By (2.4) we have $(s_i^n - A_\lambda^n u_i^{n\lambda}, u_i^n - u_i^{n\lambda}) \geq -\lambda(s_i^n - A_\lambda^n u_i^{n\lambda}, A_\lambda^n u_i^{n\lambda})$, hence the above equality implies

$$\begin{aligned}
&\sum_{i=1}^{N} a_{i-1}||u_i^n - u_i^{n\lambda} - u_{i-1}^n + u_{i-1}^{n\lambda}||^2 - \lambda \sum_{i=1}^{N} c_i a_i(s_i^n - A_\lambda^n u_i^{n\lambda}, A_\lambda^n u_i^{n\lambda}) \\
&+ \lambda \sum_{i=1}^{N} a_i||u_i^n - u_i^{n\lambda}||^2 - \lambda \sum_{i=1}^{N} a_i(u_i^n, u_i^n - u_i^{n\lambda}) \\
&\leq a_N(u_{N+1}^n - u_{N+1}^{n\lambda} - u_N^n + u_N^{n\lambda}, u_N^n - u_N^{n\lambda}).
\end{aligned}$$

The boundary conditions and the monotonicity of β lead to

$$(u_{N+1}^n - u_{N+1}^{n\lambda} - u_N^n + u_N^{n\lambda}, u_{N+1}^n - u_{N+1}^{n\lambda}) \leq 0,$$

so

$$\begin{aligned}
&(u_{N+1}^n - u_{N+1}^{n\lambda} - u_N^n + u_N^{n\lambda}, u_N^n - u_N^{n\lambda}) \\
&= -||u_{N+1}^n - u_{N+1}^{n\lambda} - u_N^n + u_N^{n\lambda}||^2 \\
&+ (u_{N+1}^n - u_{N+1}^{n\lambda} - u_N^n + u_N^{n\lambda}, u_{N+1}^n - u_{N+1}^{n\lambda}) \leq 0.
\end{aligned} \tag{3.2}$$

Thus we get

$$\sum_{i=1}^{N} a_{i-1}||u_i^n - u_i^{n\lambda} - u_{i-1}^n + u_{i-1}^{n\lambda}||^2$$

$$
\begin{aligned}
&+\lambda\sum_{i=1}^{N} a_i||u_i^n-u_i^{n\lambda}||^2+\lambda\sum_{i=1}^{N} c_i a_i||A_\lambda^n u_i^{n\lambda}||^2\\
&\le\lambda\left(\sum_{i=1}^{N} c_i a_i||s_i^n||^2\right)^{1/2}\left(\sum_{i=1}^{N} c_i a_i||A_\lambda^n u_i^{n\lambda}||^2\right)^{1/2}\\
&+\lambda\left(\sum_{i=1}^{N} a_i||u_i^n||^2\right)^{1/2}\left(\sum_{i=1}^{N} a_i||u_i^n-u_i^{n\lambda}||^2\right)^{1/2}.
\end{aligned}
$$

Lemma 3.1, together with equation (1.4), assures the boundedness of s_i^n and thus,

$$
\begin{aligned}
&\sum_{i=1}^{N} a_{i-1}||u_i^n-u_i^{n\lambda}-u_{i-1}^n+u_{i-1}^{n\lambda}||^2+\lambda\sum_{i=1}^{N} a_i||u_i^n-u_i^{n\lambda}||^2\\
&+\lambda\sum_{i=1}^{N} c_i a_i||A_\lambda^n u_i^{n\lambda}||^2\le C_1\lambda\left(\sum_{i=1}^{N} c_i a_i||A_\lambda^n u_i^{n\lambda}||^2\right)^{1/2}\\
&+C_2\lambda\left(\sum_{i=1}^{N} a_i||u_i^n-u_i^{n\lambda}||^2\right)^{1/2},
\end{aligned}
\tag{3.3}
$$

for some constants $C_1, C_2 > 0$. Therefore, there exist $C_3, C_4 > 0$ such that

$$
\left\{\sum_{i=1}^{N} a_i||u_i^n-u_i^{n\lambda}||^2\le C_3,\ \sum_{i=1}^{N} c_i a_i||A_\lambda^n u_i^{n\lambda}||^2\le C_4\right.
$$

and consequently (3.3) gives

$$
\sum_{i=1}^{N} a_{i-1}||u_i^n-u_i^{n\lambda}-u_{i-1}^n+u_{i-1}^{n\lambda}||^2\le C_5\lambda. \tag{3.4}
$$

From this inequality and

$$
u_i^n-u_i^{n\lambda}=\sum_{k=1}^{i}\left(u_k^n-u_k^{n\lambda}-u_{k-1}^n+u_{k-1}^{n\lambda}\right),\ 1\le i\le N,
$$

we easily derive the desired estimate. Analogously one proves the second inequality. The Lemma is proved. □

Another estimate is obtained in the following auxiliary result.

Lemma 3.3. *The solutions of problems* (2.1), (2.2) *satisfy the estimate* $\lim_{n\to\infty} ||u_i^{n\lambda} - u_i^{\lambda}|| = 0$, *for all* $\lambda > 0$ *and* $1 \le i \le N$.

Proof. Subtract (2.1) from (2.2), multiply by $a_i\left(u_i^{n\lambda} - u_i^{\lambda}\right)$ and sum them up from $i = 1$ to $i = N$. With the aid of (1.2) one obtains that

$$\begin{aligned}
&\sum_{i=1}^{N} a_{i-1}||u_i^{n\lambda} - u_i^{\lambda} - u_{i-1}^{n\lambda} + u_{i-1}^{\lambda}||^2 \\
&+ \sum_{i=1}^{N} c_i a_i (A_\lambda^n u_i^{n\lambda} - A_\lambda^n u_i^{\lambda}, u_i^{n\lambda} - u_i^{\lambda}) \\
&+ \sum_{i=1}^{N} c_i a_i (A_\lambda^n u_i^{\lambda} - A_\lambda u_i^{\lambda}, u_i^{n\lambda} - u_i^{\lambda}) + \lambda \sum_{i=1}^{N} a_i ||u_i^{n\lambda} - u_i^{\lambda}||^2 \\
&= a_N (u_{N+1}^{n\lambda} - u_{N+1}^{\lambda} - u_N^{n\lambda} + u_N^{\lambda}, u_N^{n\lambda} - u_N^{\lambda}) \\
&- (u_1^{n\lambda} - u_1^{\lambda} - a^n + a, u_1^{n\lambda} - u_1^{\lambda}) - \sum_{i=1}^{N} a_i (f_i^n - f_i, u_i^{n\lambda} - u_i^{\lambda}).
\end{aligned} \tag{3.5}$$

But A_λ^n is monotone, so $\left(A_\lambda^n u_i^{n\lambda} - A_\lambda^n u_i^{\lambda}, u_i^{n\lambda} - u_i^{\lambda}\right) \ge 0$. Like in the proof of Lemma 3.2, we can easily deduce a inequality similar to (3.2) :

$$\left(u_{N+1}^{n\lambda} - u_{N+1}^{\lambda} - u_N^{n\lambda} + u_N^{\lambda}, u_N^{n\lambda} - u_N^{\lambda}\right) \le 0.$$

Next,

$$\begin{aligned}
&- (u_1^{n\lambda} - u_1^{\lambda} - a^n + a, u_1^{n\lambda} - u_1^{\lambda}) = -||u_1^{n\lambda} - u_1^{\lambda} \\
&- a^n + a||^2 - (u_1^{n\lambda} - u_1^{\lambda} - a^n + a, a^n - a) \le ||u_1^{n\lambda} - u_1^{\lambda} - a^n + a|| \cdot ||a^n - a|| \\
&\le \left(\sum_{i=1}^{N} a_{i-1} ||u_i^{n\lambda} - u_i^{\lambda} - u_{i-1}^{n\lambda} + u_{i-1}^{\lambda}||^2 \right)^{1/2} ||a^n - a||.
\end{aligned}$$

Thus, from (3.5) one derives that

$$\sum_{i=1}^{N} a_{i-1} ||u_i^{n\lambda} - u_i^{\lambda} - u_{i-1}^{n\lambda} + u_{i-1}^{\lambda}||^2 + \lambda \sum_{i=1}^{N} a_i ||u_i^{n\lambda} - u_i^{\lambda}||^2$$

$$\leq \left(\sum_{i=1}^{N} a_i ||u_i^{n\lambda} - u_i^{\lambda}||^2\right)^{1/2} \left[\left(\sum_{i=1}^{N} c_i^2 a_i ||A_\lambda^n u_i^\lambda - A_\lambda u_i^\lambda||^2\right)^{1/2} + \left(\sum_{i=1}^{N} a_i ||f_i^n - f_i||^2\right)^{1/2}\right]$$

$$+ \left(\sum_{i=1}^{N} a_{i-1} ||u_i^{n\lambda} - u_i^\lambda - u_{i-1}^{n\lambda} + u_{i-1}^\lambda||^2\right)^{1/2} ||a^n - a||.$$

It follows that

$$\sum_{i=1}^{N} a_{i-1} ||u_i^{n\lambda} - u_i^\lambda - u_{i-1}^{n\lambda} + u_{i-1}^\lambda||^2 \leq 2||a^n - a||^2$$

$$+ \frac{1}{\lambda}\left[\sum_{i=1}^{N} c_i^2 a_i ||A_\lambda^n u_i^\lambda - A_\lambda u_i^\lambda||^2 + \sum_{i=1}^{N} a_i ||f_i^n - f_i||^2\right],$$

$$\sum_{i=1}^{N} a_i ||u_i^{n\lambda} - u_i^\lambda||^2 \leq \frac{1}{2\lambda}||a^n - a||^2$$

$$+ 4\left[\sum_{i=1}^{N} c_i^2 a_i ||A_\lambda^n u_i^\lambda - A_\lambda u_i^\lambda||^2 + \sum_{i=1}^{N} a_i ||f_i^n - f_i||^2\right].$$

Now, the claim follows in view of hypotheses (H5), (H6) and (2.3).

Proof of Theorem 2.1. We pass to the superior limit as $n \to \infty$ and next to the limit as $\lambda \to 0$ in (3.1) , and make use of Lemmas 3.2, 3.3. This completes the proof. □

4. Applications to difference-differential equations

Let Ω be an open bounded subset of $\mathbb{R}^d$, $d \geq 1$, with the boundary $\partial\Omega$ smooth enough and let H be the Hilbert spaces $H = L^2(\Omega)$. Consider a sequence of maximal monotone operators in $\mathbb{R}$, $\gamma : D(\gamma) \subseteq \mathbb{R} \to \mathbb{R}$, $\gamma^n : D(\gamma^n) \subseteq \mathbb{R} \to \mathbb{R}$ such that $0 \in D(\gamma) \cap D(\gamma^n)$. Then the operators

$$Au = -\Delta u + \gamma(u),$$

$$D(A) = \{u \in H^2(\Omega),\ \partial u(x)/\partial\eta \in \gamma(u(x)),\ x \in \partial\Omega\}$$

and

$$A^n u = -\Delta u + \gamma^n(u)$$

$$D(A^n) = \{u \in H^2(\Omega),\ \partial u_i^n(x)/\partial\eta \in \gamma^n(u_i^n(x)),\ x \in \partial\Omega\}$$

are maximal monotone in $H = L^2(\Omega)$ ([12, p.89]). If $\gamma^n \to \gamma$, then A^n converges to A in the sense of the resolvent.

Assume that the operator $\beta : D(\beta) \subseteq L^2(\Omega) \to L^2(\Omega)$ is one-to-one maximal monotone in $L^2(\Omega)$, such that $0 \in D(\beta)$, $0 \in \beta(0)$, its domain is bounded, and condition $(H4)$ holds for both A and A^n. If $0 < c_i$, $0 < \theta_i < 1$, for $1 \le i \le N$, are given sequences and $a, a^n, b, b^n, f_i, f_i^n \in L^2(\Omega)$, then we compare the solutions of the boundary value problems

$$(4.1)\quad \begin{cases} u_{i+1}(x) - (1+\theta_i)u_i(x) + \theta_i u_{i-1}(x) \in -c_i\Delta u_i(x) & \\ \qquad +\gamma(u_i(x)) + f_i(x), & x \in \Omega \\ \partial u_i(x)/\partial\eta \in \gamma(u_i(x)), & x \in \partial\Omega \\ u_0(x) = a(x),\ u_{N+1}(x) - u_N(x) \in -\beta(u_{N+1}(x) - b(x)), & x \in \Omega \end{cases}$$

and

$$(4.2)\quad \begin{cases} u_{i+1}^n(x) - (1+\theta_i)u_i^n(x) + \theta_i u_{i-1}^n(x) \in -c_i\Delta u_i^n(x) & \\ \qquad +\gamma^n(u_i^n(x)) + f_i^n(x), & x \in \Omega \\ \partial u_i^n(x)/\partial\eta \in \gamma^n(u_i^n(x)), & x \in \partial\Omega \\ u_0^n(x)=a^n(x), u_{N+1}^n(x) - u_N^n(x) \in -\beta(u_{N+1}^n(x) - b^n(x)), & x \in \Omega, \end{cases}$$

for $1 \le i \le N$, where $\partial/\partial\eta$ is the outward normal derivative.

As a consequence of Theorem 1.1, we first remark that these problems admit unique solutions $u = (u_i)_{i\ge1}$, $u^n = (u_i^n)_{i\ge1} \in D(A)$. By virtue of Theorem 2.1, we have the following continuous dependence on data result.

Corollary 4.1. *In addition to the above hypotheses, suppose that $a^n \to a$, $b^n \to b$, $f_i^n \to f_i$ in $H = L^2(\Omega)$ and $\gamma^n \to \gamma$ in the sense of the resolvent. Then , $u_i^n \to u_i$ as $n \to \infty$ in H, for all $1 \le i \le N$.*

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Faculty of Mathematics,
"Al. I. Cuza" University of Iaşi,
11, Bd. Carol I, 700506, Iaşi,
ROMANIA
gapreutesei@yahoo.com

Department of Mathematics,
The "Gheorghe Asachi" Technical University of Iaşi,
11, Bd. Carol I, 700506, Iaşi,
ROMANIA
napreut@gmail.com