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ON THE TRAVELLER WAVE PROPAGATION PHENOMENON

BY

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Abstract. In the first part of this paper we present the determination of the constant C of the traveller wave potential introduced by Franz Joseph von Gerstner, obtaining the calculation equation of this constant from the Bernoulli equation, under the condition of its real values, establishing the connection between the wave height h and length λ, as well as the water static deep H in the channel.

Also, by writing the Bernoulli equation we determined the pressure variation function in a domain point by passing of the traveller wave, for the two real constant values corresponding to the radical signs.

As follows of the traveller wave potential constant determination, we obtain also the trajectory of a liquid particle for the real value of the constant C, having the negative sign before the radical.

We determined the same constant true value C from the calculus of the tide produced by the traveller wave, which rises up the water static level on the shore.

Further, one introduces the numerical solving of the heavy and ideal liquid traveller wave unsteady motion on an inclined plane bottom at the positive angle α, using the dimensionless equation system, in the case of a rectangular grid, studying the streamline stability conditions and presenting the error relaxation diagrams on the two calculus directions in a rectangular grid with two different steps.

Mathematics Subject Classification 2000: 76B07, 76B15.

Key words: Gerstner wave potential, Bernoulli equation for pressure variation, traveller wave tide, numerical solution stability.

1. The determination of the constant C of the traveller wave potential

The traveller wave potential, introduced by Franz Joseph von Gerstner in the year 1802 [1] and corresponding to the plane and unsteady motion of

the heavy and no viscous liquid with respect to the fixed absolute trihedron, being

$$(1) \qquad \Phi(X,Y,T) = F(Y)\sin(kX-\omega T), \text{ in which } k=\frac{2\pi}{\lambda} \text{ and } \omega=\frac{2\pi}{T_0},$$

from which, calculating the two velocity components:

$$(2) \quad U=\frac{\partial\Phi}{\partial X}=kF(Y)\cos(kX-\omega T) \text{ and } V=\frac{\partial\Phi}{\partial Y}=\frac{\partial F}{\partial Y}\sin(kX-\omega T),$$

that introduced in the mass conservation equation, he observes that the potential (1) is an harmonic function what permitted him to determine the wave amplitude function from an equation of Euler's type with constant coefficients, having a general solution of the form $F(Y)\approx e^{rY}$,

$$(3) \qquad \frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}=\Delta_{X,Y}\Phi=-k^2F(Y)+F''_{Y^2}=0\rightarrow r=\pm k.$$

The finale expression of Gerstner's potential becomes

$$(1') \quad \Phi(X,Y,T)=(Ae^{kY}+Be^{-kY})\sin(kX-\omega T)=C\,\mathrm{ch}\,kY\sin(kX-\omega T),$$

in which for the determination of one from the two constants, Gerstner annulled the vertical component of the liquid velocity on the plane bottom

$$(4) \quad V(X,Y,T)|_{Y=0}=k(Ae^{kY}-Be^{-kY})\sin(kX-\omega T)=0\rightarrow A{=}B{=}C/2.$$

This single constant C we have determined in [2] as function of the wave free surface height, deduced from the Bernoulli's equation, obtained by spatial integration of the motion equations, where $Y=H+\eta$ (fig.1), having the form

$$(5) \qquad \frac{1}{g}\frac{\partial\Phi}{\partial T}+\frac{U^2+V^2}{2g}+\frac{P}{\gamma}+Y=\mathrm{ct}(T).$$

in which the two velocity components are:

$$(6) \qquad \begin{aligned} U(X,Y,T)&=kC\mathrm{ch}kY\cos(kX-\omega T)=\frac{dX}{dT}\\ V(X,Y,T)&=kC\sin kY\sin(kX-\omega T)=\frac{dY}{dT}. \end{aligned}$$

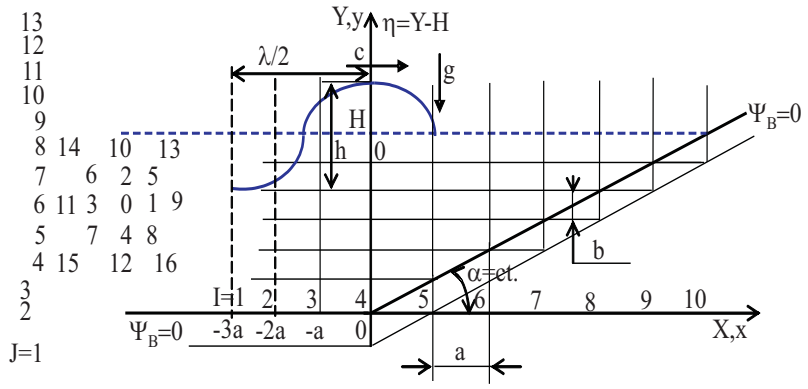

Figure 1: The axes position and the rectangular network knots numbering and disposition

Applying this equation for the same time moment $T = 0$ in the wave sole $X = -\lambda/2$, $Y = H - h/2$ and for the wave crest $X = 0$, $Y = H + h/2$, where we have the vertical velocity components equal with zero $V = 0$ and the pressure equal with that atmospheric P_0, we shall obtain:

$$\frac{\omega C}{g}\mathrm{ch}k\left(H-\frac{h}{2}\right)+\frac{k^2C^2}{2g}\mathrm{ch}^2k\left(H-\frac{h}{2}\right)+\frac{P_0}{\gamma}+H-\frac{h}{2}=\mathrm{ct}(0), \tag{7}$$

respectively

$$-\frac{\omega C}{g}\mathrm{ch}k\left(H+\frac{h}{2}\right)+\frac{k^2C^2}{2g}\mathrm{ch}^2k\left(H+\frac{h}{2}\right)+\frac{P_0}{\gamma}+H+\frac{h}{2}=\mathrm{ct}(0), \tag{8}$$

and subtracting the first relation (7) from the second (8) we obtain the equation of second degree

$$\begin{aligned}&k^2\left[\mathrm{ch}^2k\left(H+\frac{h}{2}\right)-\mathrm{ch}^2k\left(H-\frac{h}{2}\right)\right]C^2\\&-2\omega\left[\mathrm{ch}\,k\left(H+\frac{h}{2}\right)+\mathrm{ch}\,k\left(H-\frac{h}{2}\right)\right]C+2gh=0,\end{aligned} \tag{9}$$

from which we can calculate the two values of the constant $C(h,\lambda,H)$

$$C=\frac{\omega[\mathrm{ch}\,k\left(H+\frac{h}{2}\right)+\mathrm{ch}\,k\left(H-\frac{h}{2}\right)]}{k^2[\mathrm{ch}^2k\left(H+\frac{h}{2}\right)-\mathrm{ch}^2k\left(H-\frac{h}{2}\right)]} \tag{10}$$

$$\frac{\pm\omega\sqrt{[\mathrm{ch}\,k\left(H+\frac{h}{2}\right)+\mathrm{ch}\,k\left(H-\frac{h}{2}\right)]^2-\frac{2gh}{c^2}[\mathrm{ch}^2k\left(H+\frac{h}{2}\right)-\mathrm{ch}^2k\left(H-\frac{h}{2}\right)]}}{k^2[\mathrm{ch}^2k\left(H+\frac{h}{2}\right)-\mathrm{ch}^2k\left(H-\frac{h}{2}\right)]}.$$

To have real values for the two constants, we shall consider the expression under radical as positive, obtaining the link between the wave characteristics h, λ and the water height H in the channel, in the condition $k = 2gh/c^2 < 1$ [3]

$$\frac{\mathrm{ch}\,k\left(H+\frac{h}{2}\right)}{\mathrm{ch}\,k\left(H-\frac{h}{2}\right)}+1\geq\frac{2gh}{c^2}\left[\frac{\mathrm{ch}\,k\left(H+\frac{h}{2}\right)}{\mathrm{ch}\,k\left(H-\frac{h}{2}\right)}-1\right]$$

$$\rightarrow\frac{\mathrm{ch}\,k\left(H+\frac{h}{2}\right)}{\mathrm{ch}\,k\left(H-\frac{h}{2}\right)}\geq-\frac{1+k}{1-k}=-(1+2k+2k^2+\ldots), \tag{11}$$

in which introducing the notations: $\chi = h/\lambda$, $\kappa = 2gh/c^2$, $A = kH = 2\pi\aleph$, $B = k\frac{h}{2} = \pi\chi$, as the expression of the wave propagation velocity quadrate $c^2 = \frac{g\lambda}{2\pi}\mathrm{th}2\pi\aleph$ from the Gerstner theory [1], we shall obtain the necessary condition $\mathrm{ch}(A+B)+\mathrm{ch}(A-B)\geq\frac{2gh}{c^2}[\mathrm{ch}(A+B)-\mathrm{ch}(A-B)]$, or

$$\begin{aligned}\frac{\mathrm{ch}(A+B)+\mathrm{ch}(A-B)}{\mathrm{ch}(A+B)-\mathrm{ch}(A-B)}&=\frac{(e^A+e^{-A})(e^B+e^{-B})}{(e^A-e^{-A})(e^B-e^{-B})}\\&=\frac{\mathrm{ch}\,A\,\mathrm{ch}\,B}{\mathrm{sh}\,A\,\mathrm{sh}\,B}\\&=\frac{1}{\mathrm{th}\,A\,\mathrm{th}\,B}\geq\frac{2gh}{c^2}=\frac{4\pi\chi}{\mathrm{th}\,A}.\end{aligned}$$

and consequently

$$F(\pi\chi)=\pi\chi\mathrm{th}\,\pi\chi\leq\frac{1}{4}=0.25\leq 1, \tag{12}$$

represented graphically also in the figure 2. The expressions of the two velocity components are in this case:

$$\begin{aligned}U(X,Y,T)&=kC\mathrm{ch}\,kY\cos(kX-\omega T)=\frac{dX}{dT}\\V(X,Y,T)&=kC\mathrm{ch}\,kY\sin(kX-\omega T)=\frac{dY}{dT}.\end{aligned} \tag{13}$$

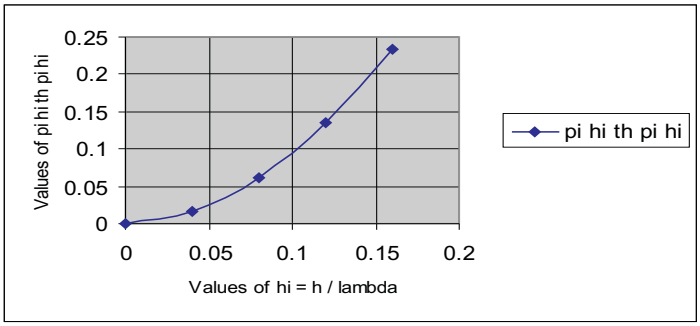

Figure 2: The condition of χ, to obtain the real values for the constant $C(h, \lambda, H)$

2. The pressure variation in the interior of the traveller wave on the horizontal bottom

Starting from the motion equations or from the Bernoulli's equation, of a heavy and ideal liquid

$$\begin{aligned} &\frac{\partial U}{\partial T} + \frac{\partial U}{\partial X}U + \frac{\partial U}{\partial Y}V + \frac{1}{\rho}P'_X = 0 \\ &\frac{\partial V}{\partial T} + \frac{\partial V}{\partial X}U + \frac{\partial V}{\partial Y} + \frac{1}{\rho}P'_Y + g = 0 \end{aligned} \tag{14}$$

and calculating the partial differential of the two velocity components from the equation (14) we obtain:

$$P'_X = -\rho\omega kC\,\mathrm{ch}\,kY\,\sin(kX-\omega T)+\rho k^3C^2\,\sin(kX-\omega T)\cos(kX-\omega T) \tag{15}$$

and

$$P'_Y = -\gamma + \rho\omega kC\,\mathrm{sh}\,kY\,\cos(kX-\omega T) - \rho k^3C^2\,\mathrm{sh}\,kY\,\mathrm{ch}\,kY. \tag{16}$$

Integrating partially these two equations we obtain the following pressure function expressions:

$$\begin{aligned} P(X,Y,T) &= F(Y,T) + \rho\omega C\,\mathrm{ch}\,kY\,\cos(kX-\omega T) \\ &\quad + \frac{\rho k^2C^2}{2}\sin^2(kX-\omega T) \end{aligned} \tag{15'}$$

$$\begin{aligned} P(X,Y,T) &= G(X,T) - \gamma Y + \rho\omega C\,\mathrm{ch}\,kY\,\cos(kX-\omega T) \\ &\quad - \frac{\rho k^2C^2}{2}\mathrm{sh}^2kY. \end{aligned} \tag{16'}$$

By derivation of the $(15')$ equation with respect to Y and $(16')$ with respect to X, we shall have

$$P'_Y = F'_Y + \rho\omega kC \operatorname{sh} kY \cos(kX - \omega T) \tag{15''}$$

and

$$P'_X = G'_X - \rho\omega kC \operatorname{ch} kY \sin(kX - \omega T), \tag{16''}$$

and by their introduction in the relations (16), respectively (15), we can determine the two differentials:

$$F'_Y = -\gamma - \rho k^3 C^2 \operatorname{sh} kY \operatorname{ch} kY \tag{17}$$

and

$$G'_X = \rho k^3 C^2 \sin(kX - \omega T) \cos(kX - \omega T). \tag{18}$$

By partial integration of the relations (17) and (18) we obtain the expression of these functions:

$$F(Y,T) = -\gamma Y - \frac{\rho k^2 C^2}{2} \operatorname{sh}^2 kY + F_0(T) \tag{17'}$$

and

$$G(X,T) = \frac{\rho k^2 C^2}{2} \operatorname{sh}^2(kX - \omega T) + G_0(T) \tag{18'}$$

and replacing these functions in the pressure function expressions from $(15')$ and $(16')$ we observe that we have $F_0(T) = G_0(T)$, the pressure function having the form

$$\begin{aligned} P(X,Y,T) = F_0(T) - \gamma Y - \frac{\rho k^2 C^2}{2} \operatorname{sh}^2 kY + \rho\omega C \operatorname{ch} kY \cos(kX - \omega T) \\ + \frac{\rho k^2 C^2}{2} \sin^2(kX - \omega T). \end{aligned} \tag{19}$$

We can determine this integration function on the liquid free surface, for which we have $Y = H$, the atmospheric pressure $P = P_0$ and the velocity component $U = 0$, setting the condition $\cos(kX - \omega T) = 0$, respectively $2\pi(\frac{X}{\lambda} - \frac{T}{T_0}) = \frac{\pi}{2} + N\pi$ and $\frac{X}{\lambda} - \frac{T}{T_0} = \frac{1}{4} + \frac{N}{2}$ for $T = 0$, resulting $P(\frac{\lambda}{4}, H, 0) =$

$P_0 = F_0(0) - \gamma H - \frac{\rho k^2 C^2}{2}(\mathrm{sh}^2 kH + 1)$, from which we deduce the value independent of time $F_0(0) = P_0 + \gamma H + \frac{\rho k^2 C^2}{2}(\mathrm{sh}^2 kH - 1)$, obtaining the expression of the pressure in wave inside

$$P = P_0 + \gamma(H - Y) + \rho\omega C \mathrm{ch}\, kY \cos(kX - \omega T) + \frac{\rho k^2 C^2}{2}[\mathrm{sh}^2 kH - 1 - \mathrm{sh}^2 kY + \sin^2(kX - \omega T)]. \tag{19'}$$

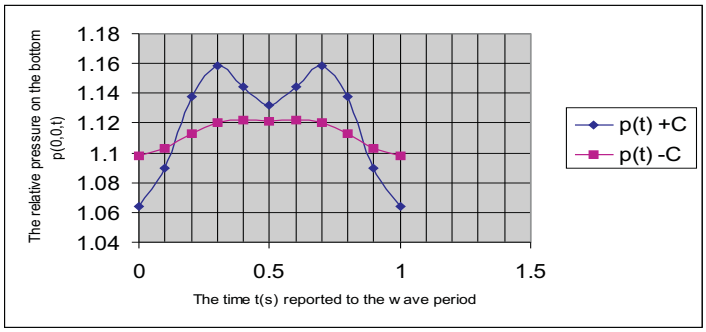

Figure 3: The pressure variation of the traveller wave on the channel bottom for the two constant C values

For the dimensionless pressure values representation in a point $X = Y = 0$ in the figure 3, we introduced the followings dimensionless variables and functions, taking as characteristic magnitudes: the water level in channel H, the wave period T_0, and the atmospheric pressure P_0 on wave free surface

$$x = \frac{X}{H},\ y = \frac{Y}{H},\ t = \frac{T}{T_0} \text{ and } u = \frac{U}{c},\ v = \frac{V}{c},\ p = \frac{P}{P_0},\ \Psi = \frac{\Psi}{Hc}, \tag{20}$$

the formula (19′) becomes

$$p(x,y,t) = 1 + \frac{\gamma H}{P_0}(1-y) + \frac{2\rho\pi^2 C^2}{\lambda^2 P_0}\left[\mathrm{sh}^2 2\pi\frac{H}{\lambda} - \mathrm{sh}^2 2\pi\frac{H}{\lambda}y + \sin 2\pi\left(\frac{H}{\lambda}x - t\right) - 1\right] + \frac{2\pi\rho C}{T_0 P_0}\mathrm{ch}2\pi\frac{H}{\lambda}y\cos 2\pi\left(\frac{H}{\lambda}x - t\right) \tag{21}$$

and for the adequate form

$$p(0,0,t) = 1 + \frac{\gamma H}{P_0} + \frac{2\rho\pi^2 C^2}{\lambda^2 P_0}\left[\mathrm{sh}^2 2\pi\frac{H}{\lambda} + \sin(-2\pi t) - 1\right] + \frac{2\pi\rho C}{T_0 P_0}\cos(-2\pi t). \tag{21'}$$

3. The liquid particle trajectory on the wave free surface

By integration with respect to the time of the two velocity components (13) we obtain:

$$X - X_0 = -\frac{kC}{\omega}\mathrm{ch}kY\sin(kX - \omega T) \text{ and } Y - Y_0 = \frac{kC}{\omega}\mathrm{sh}kY\cos(kX - \omega T)$$

and eliminating the time between these relations, we obtained the liquid particle trajectory (fig. 4), considering that: $Y_0 = H = 1m, X_0 = 0, \lambda = 4m$, $C = 0.05222$, h=0.1m and $T_0 = \lambda/c = 1.671s$, the wave velocity being $c = 2.394m/s$ from the traveller wave Gerstner theory c^2=gkth2$\pi\aleph$ [1].

$$\frac{(X - X_0)^2}{\frac{k^2C^2}{\omega^2}\mathrm{ch}^2 kY_0} + \frac{(Y - Y_0)^2}{\frac{k^2C^2}{\omega^2}\mathrm{sh}^2 kY_0} = 1 \rightarrow Y - H = \pm\frac{kC}{\omega}\mathrm{sh}kH\sqrt{1 - \frac{X^2\omega^2}{k^2C^2\mathrm{ch}^2 kH}}. \tag{22}$$

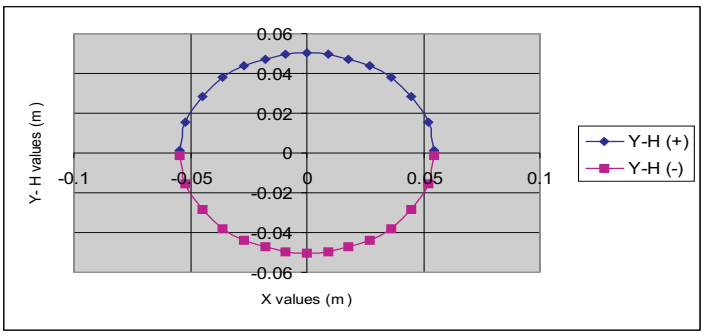

Figure 4: The liquid particle trajectory on the wave free surface

4. The water tidal calculation, produced by the traveller wave

To calculate the water flux brought by a traveller wave in a section, for instance $X = 0$, we shall present in fig. 5, the flow rates dividing the wave period T_0 in 20 intervals using the formula

$$(23) \qquad Q_n(0, Y_n, \delta T) = \int_{Y=0}^{H+\frac{h}{2}\sum_i^n \delta Y_i} U_n(0, Y_n, \delta T)\delta Y$$
$$= C\cos\left(\frac{n\pi}{10}\right)\mathrm{sh}k\left(H + \frac{h}{2} - \sum_i^n \delta Y_i\right),$$

in which we considered the horizontal velocity component and the vertical velocity component (13) under the forms

$$(24) \qquad U_n(0, H + \frac{h}{2} - \sum_{i=1}^{n=20} \delta Y_i, \frac{nT_0}{20}) = kC\mathrm{ch}\, kY\cos\left(\frac{n\pi}{10}\right)$$

and

$$(25) \qquad V_{n-1}\left(0, H + \frac{h}{2} - \sum_{i=1}^{n=20} \delta Y_i, \frac{iT_0}{20}\right) = kC\mathrm{sh}\, kY\sin\left(-\frac{i\pi}{10}\right),$$

the height of the wave crest being diminished in each moment with the

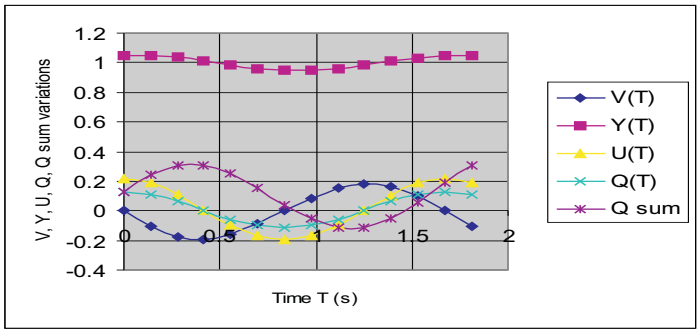

Figure 5: The water tide flux transported by a traveller wave

quantity

$$(26) \qquad \delta Y_i = V_{n-1}\cdot\delta T = \left[kC\mathrm{sh}\, kY\sin\left(-\frac{i\pi}{10}\right)\right]\frac{T_0}{20}$$

in which case we proved that the only C value with physical signification is that, having the negative sign before the radical, as in the case of wave trajectory calculation.

5. Numerical solving of the ideal liquid traveller wave motion on an inclined plane bottom

Starting from the Leonhard Euler flow equations:

$$\frac{\partial U}{\partial T}+\frac{\partial U}{\partial X}U+\frac{\partial U}{\partial Y}V+\frac{1}{\rho}\frac{\partial P}{\partial X}=0, \tag{27}$$

$$\frac{\partial V}{\partial T}+\frac{\partial V}{\partial X}U+\frac{\partial V}{\partial Y}V+\frac{1}{\rho}\frac{\partial P}{\partial Y}+g=0, \tag{28}$$

$$\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}=0, \tag{29}$$

for more generality of the numerical solution, we shall consider the dimensionless form of these equations taking as characteristic magnitudes: the water deep H in the channel, the wave period T_0 and the wave propagation velocity c. With the new dimensionless variables and functions (20), the dimensionless equation system being:

$$\mathrm{Sh}\,u'_t+u'_su+u'_yv+\mathrm{Eu}\,p'_x=0, \tag{27$'$}$$

$$\mathrm{Sh}\,v'_t+v'_xu+v'_yv+\mathrm{Eu}\,p'_Y+\frac{1}{\mathrm{Fr}}=0, \tag{28$'$}$$

the mass conservation equation being an invariant

$$u'_x+v'_y=0, \tag{29$'$}$$

in which we have denoted by: $\mathrm{Sh}=H/cT_0$ the number of Strouhal, $\mathrm{Eu}=P_0/\rho c_2$ the number of Euler and with $\mathrm{Fr}=c_2/gH$ the number of Froude.

To eliminate the pressure function, being unknown on the all domain boundaries [4], [6], in virtue of Schwarz commutative relation of the partial mixed differential of second order $p'_{xy}=p''_{yx}$, we shall partially differentiate the first motion equation (27$'$) with respect to the y variable and the second motion equation (28$'$) with respect to the x variable, obtaining by subtraction the equation

$$\mathrm{Sh}(u''_{ty}-v''_{tx})+u(u''_{xy}-v''_{x^2})+v(u''_{y^2}-v''_{xy})+u'_xu'_y+u'_yv'_y-v'_xu'_x-v'_yv'_x=0. \tag{30}$$

To eliminate the mass conservation equation (29′), totally unstable in the iterative numerical calculus [4], [6], we have introduced the streamline function by the relations

$$(31)\quad u=\varphi'_x=\psi'_y \text{ and } v=\varphi'_y=-\psi'_{x'},\rightarrow u'_x+v'_y{=}\Delta_{x,y}\varphi=\psi''_{yx}-\psi''_{xy}{=}0,$$

in which case the equation (30) becomes

$$(30')\qquad \mathrm{Sh}(\psi''_{x^2}+\psi''_{y^2})'_t+\psi'_y(\psi'''_{xy^2}+\psi'''_{x^3})-\psi'_x(\psi'''_{y^3}{+}\psi'''_{x^2y})=0.$$

Developing the streamline function in Cazacu's proper series [7], which are better as Taylor's series concerning the numerical solution stability and from which we can calculate the expressions for the two different grid steps for instance $b=\delta Y/H=0.1$ and $a=\delta X/H=2b=0.2$:

$$-v_0,u_0=\psi'_{x,y}=\frac{1}{a,b}\left[\frac{2}{3}(\psi_{1,2}-\psi_{3,4})-\frac{1}{12}(\psi_{9,10}-\psi_{11,12})\right],$$
$$\psi''_{x^2,y^2}=\frac{1}{a^2,b^2}\left[\frac{4}{3}(\psi_{1,2}+\psi_{3,4})-\frac{5}{2}\psi_0-\frac{1}{12}(\psi_{9,10}-\psi_{11,12})\right]$$

or simpler

$$(32)\qquad \begin{aligned}
\psi''_{x^2,y^2}&=\frac{1}{a^2,b^2}[\psi_{1,2}-2\psi_0+\psi_{3,4}],\\
\psi'''_{x^2,y^2}&=\frac{1}{a^3,b^3}\left[\frac{2}{3}(\psi_{9,10}-\psi_{11,12})-\frac{4}{3}(\psi_{1,2}+\psi_{3,4})\right],\\
\psi'''_{x^2y}&=\frac{1}{a^2b}\left[\frac{2}{3}(\psi_5+\psi_6-\psi_7-\psi_8)-\frac{4}{3}(\psi_2-\psi_4)\right],\\
\psi'''_{xy^2}&=\frac{1}{ab^2}\left[\frac{2}{3}(\psi_5-\psi_6-\psi_7+\psi_8)-\frac{4}{3}(\psi_1-\psi_3)\right]
\end{aligned}$$

that introduced in the relation (30'), in which we denoted the time step with δt and the streamline function at the past time with an exponent negative, we obtain the following algebraic streamline equation

$$\begin{aligned}
0=&\frac{\mathrm{Sh}}{\delta t}\left[\frac{1}{a^2}(\psi_1-2\psi_0+\psi_3-\psi_1+2\psi_0-\psi_3)\right.\\
&\left.+\frac{1}{b^2}(\psi_2-2\psi_0+\psi_4-\psi_2+2\psi_0-\psi_4)\right]\\
&+\frac{1}{b}\left[\frac{2}{3}(\psi_2-\psi_4)-\frac{1}{12}(\psi_{10}-\psi_{12})\right]\left\{\frac{1}{ab^2}\left[\frac{2}{3}(\psi_5-\psi_6-\psi_7+\psi_8)\right.\right.
\end{aligned}$$

$$-\frac{4}{3}(\psi_1-\psi_3)\Big]+\frac{1}{a^3}\left[\frac{2}{3}(\psi_9-\psi_{11})-\frac{4}{3}(\psi_1+\psi_3)\right]\Big\}$$
$$-\frac{1}{a}\left[\frac{2}{3}(\psi_1-\psi_3)-\frac{1}{12}(\psi_9-\psi_{11})\right]\left\{\frac{1}{b^3}\left[\frac{2}{3}(\psi_{10}-\psi_{12})-\frac{4}{3}(\psi_2+\psi_4)\right]\right.$$
$$\left.+\frac{1}{a^2b}\left[\frac{2}{3}(\psi_5+\psi_6-\psi_7-\psi_8)-\frac{4}{3}(\psi_2-\psi_4)\right]\right\}$$

from whose linear part we can extract the streamline value 0 in the point zero

$$2\psi_0\left(\frac{1}{a^2}+\frac{1}{b^2}\right)=\frac{1}{a^2}(\psi_1+\psi_3-\psi_1^-+2\psi_0^--\psi_3^-)$$
$$+\frac{1}{b^2}(\psi_2+\psi_4-\psi_2^-+2\psi_0^--\psi_4^-)$$
$$+\frac{\delta t}{\mathrm{Sh}}\left\{\frac{1}{b}\left[\frac{2}{3}(\psi_2-\psi_4)-\frac{1}{12}(\psi_{10}-\psi_{12})\right]\left\{\frac{1}{ab^2}\left[\frac{2}{3}(\psi_5-\psi_6-\psi_7+\psi_8)\right.\right.\right.$$
$$\left.-\frac{4}{3}(\psi_1-\psi_3)\right]+\frac{1}{a^3}\left[\frac{2}{3}(\psi_9-\psi_{11})-\frac{4}{3}(\psi_1+\psi_3)\right]\Big\}$$
$$-\frac{1}{a}\left[\frac{2}{3}(\psi_1-\psi_3)-\frac{1}{12}(\psi_9-\psi_{11})\right]\left\{\frac{1}{b^3}\left[\frac{2}{3}(\psi_{10}-\psi_{12})\right.\right.$$
$$\left.\left.-\frac{4}{3}(\psi_2+\psi_4)\right]+\frac{1}{a^2b}\left[\frac{2}{3}(\psi_5+\psi_6-\psi_7-\psi_8)-\frac{4}{3}(\psi_2-\psi_4)\right]\right\}. \tag{33}$$

To study the numerical solution stability we introduce in (33) the velocity components u_0 and v_0

$$2\psi_0\left(\frac{1}{a^2}+\frac{1}{b^2}\right)=\frac{1}{a^2}(\psi_1+\psi_3-\psi_1^-+2\psi_0^--\psi_3^-)$$
$$+\frac{1}{b^2}(\psi_2+\psi_4-\psi_2^-+2\psi_0^--\psi_4^-)$$
$$+\frac{\delta t}{\mathrm{Sh}}\left\{u_0\left\{\frac{1}{ab^2}\left[\frac{2}{3}(\psi_5-\psi_6-\psi_7+\psi_8)-\frac{4}{3}(\psi_1-\psi_3)\right]\right.\right.$$
$$\left.+\frac{1}{a^3}\left[\frac{2}{3}(\psi_9-\psi_{11})-\frac{4}{3}(\psi_1+\psi_3)\right]\right\}$$
$$+v_0\left\{\frac{1}{b^3}\left[\frac{2}{3}(\psi_{10}-\psi_{12})-\frac{4}{3}(\psi_2+\psi_4)\right]\right.$$
$$\left.\left.+\frac{1}{a^2b}\left[\frac{2}{3}(\psi_5+\psi_6-\psi_7-\psi_8)-\frac{4}{3}(\psi_2-\psi_4)\right]\right\}\right\} \tag{33$'$}$$

and considering the error propagation in the two directions [4], [6] we can write the relations:

$$\delta\psi_{n+1}^{\pm x} = \frac{\delta\psi_n}{2(\frac{1}{a^2}+\frac{1}{b^2})}\left[\frac{1}{a^2} \pm \frac{4}{3ab^2}\frac{u_0\delta t}{\mathrm{Sh}} - \frac{4}{3a^3}\frac{u_0\delta t}{\mathrm{Sh}}\right] \mp \frac{\delta\psi_{n-1}}{2(\frac{1}{a^2}+\frac{1}{b^2})}\frac{2}{3a^3}\frac{u_0\delta t}{\mathrm{Sh}}, \tag{34}$$

respectively

$$\delta\psi_{n+1}^{\pm y} = \frac{\delta\psi_n}{2(\frac{1}{a^2}+\frac{1}{b^2})}\left[\frac{1}{b^2} \pm \frac{4}{3a^2b}\frac{v_0\delta t}{\mathrm{Sh}} - \frac{4}{3b^3}\frac{v_0\delta t}{\mathrm{Sh}}\right] \mp \frac{\delta\psi_{n-1}}{2(\frac{1}{a^2}+\frac{1}{b^2})}\frac{2}{3b^3}\frac{v_0\delta t}{\mathrm{Sh}}, \tag{35}$$

which for the coefficients stability condition, to be lower as the unity [4]

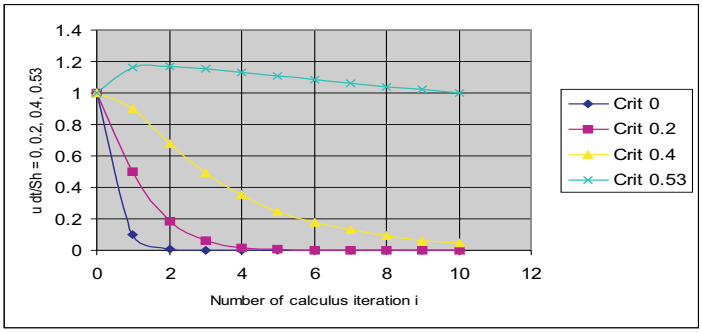

Figure 6: The monotone relaxation diagram on $+x$ direction for different Criteria $u_0\delta t/\mathrm{Sh}$ values

and taking the better stability iterative calculus condition on $+x$ and $+y$ directions we represented the first monotone relaxation in the figure 6, in the particular case when $a = 0.2$ and $b = 0.1$

$$\delta\psi_{n+1}^{+x} = \left(0.1 + 2\frac{u_0\delta t}{\mathrm{Sh}}\right)\delta\psi_n - 0.333\frac{u_0\delta t}{\mathrm{Sh}}\delta\psi_{n-1} \to \frac{u_0\delta t}{\mathrm{Sh}} \prec 0.54 \to \delta t \prec 0.54\mathrm{Sh}/u_0, \tag{34'}$$

and for the second oscillating relaxation diagram, which is represented in

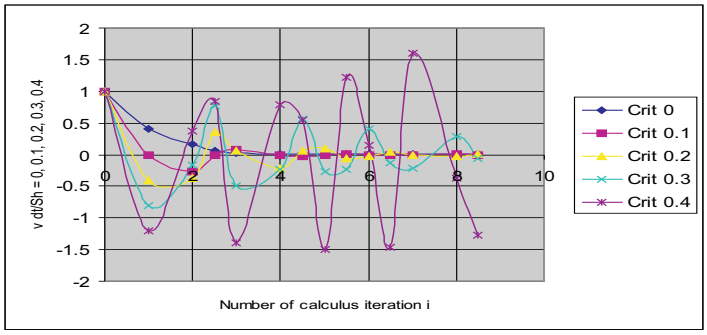

Figure 7: The oscillating relaxation diagram on $+y$ direction for different Criteria $v_0\delta t/\mathrm{Sh}$ values

the figure 7.

$$\delta\psi_{n+1}^{+y} = \left(0.4 - 4\frac{v_0\delta t}{\mathrm{Sh}}\right) d\psi_n - 2.667\frac{v_0\delta t}{\mathrm{Sh}}\delta\psi_{n-1}$$

$$\rightarrow -0.009 \prec \frac{v_0\delta t}{\mathrm{Sh}} \rightarrow \delta t \succ -0.009\mathrm{Sh}/v_0. \tag{35'}$$

6. Conclusions

The obtained results are very interesting even in the case of the heavy and ideal liquid, concerning the traveller wave propagation on a horizontal or inclined bottom, which have take in evidence the wave unsteady motion particularities by the numerical calculus: of the constant C value, the pressure variation by the traveller wave passing, the liquid particle trajectory on the wave free surface, the traveller wave tidal effect and the stability of the streamline function by numerical solving.

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