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THE HOLOMORPHIC φ-SECTIONAL CURVATURE OF TANGENT SPHERE BUNDLES WITH SASAKIAN STRUCTURES

BY

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Abstract. We study the holomorphic φ-sectional curvature of the natural diagonal tangent sphere bundles with Sasakian structures, determined by Druţă-Romaniuc and Oproiu. After finding the explicit expressions for the components of the curvature tensor field and of the curvature tensor field corresponding to the Sasakian space forms, we find that there are no tangent sphere bundles of natural diagonal lift type of constant holomorphic φ-sectional curvature.

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Key words: natural lift, tangent sphere bundle, Sasakian structure, holomorphic φ-sectional curvature.

1. Introduction

The geometry of the tangent sphere bundles of constant radius r, endowed with the metrics induced by some Riemannian metrics from the ambient tangent bundle, was studied by authors like Abbassi, Boeckx, Calvaruso, Kowalski, Munteanu, Park, Sekigawa, Sekizawa (see [1], [2], [4]-[7], [9], [12]-[14], [16]). In the most part of the papers, such as [4], [5] and [16], the metric considered on the tangent bundle TM was the Sasaki metric, but Boeckx noticed that the unit tangent bundle equipped with the induced Cheeger-Gromoll metric is isometric to the tangent sphere bundle $T_{\frac{1}{\sqrt{2}}}M$ with the induced Sasaki metric. Then, in 2000, Kowalski and Sekizawa showed how the geometry of the tangent sphere bundles depends on the radius (see [10]).

The tangent sphere bundles T_rM of constant radius r are hypersurfaces of the tangent bundles, consisting of the tangent vectors which have the norm equal to r only. Every almost Hermitian structure on the tangent bundle induces an almost contact structure on the tangent sphere bundle of constant radius r. Important results in this direction may be found in the recent surveys [1] and [9].

In the present paper we study the conditions under which the tangent sphere bundles of natural diagonal lift type with Sasakian structures, determined in [7], have constant holomorphic φ-sectional curvature, i.e. they are Sasakian space forms. We prove that this conditions are never satisfied, so there are no natural diagonal tangent sphere bundles of constant radius r, which are also Sasakian space forms.

The manifolds, tensor fields and other geometric objects considered in this paper are assumed to be differentiable of class C^∞ (i.e. smooth). The Einstein summation convention is used throughout this paper, the range of the indices h, i, j, k, l, m, r being always $\{1, \dots, n\}$.

2. Preliminary results

Let (M, g) be a smooth n-dimensional Riemannian manifold and denote its tangent bundle by $\tau : TM \to M$. The total space TM has a structure of $2n$-dimensional smooth manifold, induced from the smooth manifold structure of M. This structure is obtained by using local charts on TM induced from usual local charts on M. If $(U, \varphi) = (U, x^1, \dots, x^n)$ is a local chart on M, then the corresponding induced local chart on TM is $(\tau^{-1}(U), \Phi) = (\tau^{-1}(U), x^1, \dots, x^n,\ y^1, \dots, y^n)$, where the local coordinates $x^i, y^j,\ i, j = 1, \dots, n$, are defined as follows. The first n local coordinates of a tangent vector $y \in \tau^{-1}(U)$ are the local coordinates in the local chart (U, φ) of its base point, i.e. $x^i = x^i \circ \tau$, by an abuse of notation. The last n local coordinates $y^j,\ j = 1, \dots, n$, of $y \in \tau^{-1}(U)$ are the vector space coordinates of y with respect to the natural basis in $T_{\tau(y)}M$ defined by the local chart (U, φ). Due to this special structure of differentiable manifold for TM, it is possible to introduce the concept of M-tensor field on it (see [11]).

Denote by $\dot{\nabla}$ the Levi Civita connection of the Riemannian metric g on M. Then we have the direct sum decomposition

$$TTM = VTM \oplus HTM \tag{2.1}$$

of the tangent bundle to TM into the vertical distribution $VTM = \mathrm{Ker}\ \tau_*$ and the horizontal distribution HTM defined by $\dot{\nabla}$ (see [17]). The vertical and horizontal lifts of a vector field X on M will be denoted by X^V and X^H respectively. The set of vector fields $\{\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}\}$ on $\tau^{-1}(U)$ defines a local frame field for VTM, and for HTM we have the local frame field $\{\frac{\delta}{\delta x^1}, \dots, \frac{\delta}{\delta x^n}\}$, where $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \Gamma^h_{0i}\frac{\partial}{\partial y^h}$, $\quad \Gamma^h_{0i} = y^k\Gamma^h_{ki}$, and $\Gamma^h_{ki}(x)$ are the Christoffel symbols of g.

The set $\{\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\}_{i,j=\overline{1,n}}$, denoted also by $\{\partial_i, \delta_j\}_{i,j=\overline{1,n}}$, defines a local frame on TM, adapted to the direct sum decomposition (2.1). Extensive literature concerning canonical frame structures on TM defined by canonical N-connections can be found in [3].

Consider the energy density of the tangent vector y with respect to the Riemannian metric g

$$t = \frac{1}{2}\|y\|^2 = \frac{1}{2}g_{\tau(y)}(y, y) = \frac{1}{2}g_{ik}(x)y^iy^k, \quad y \in \tau^{-1}(U).$$

Obviously, we have $t \in [0, \infty)$ for every $y \in TM$.

The second author considered in [15] an (1,1)-tensor field J on the tangent bundle TM, obtained as natural 1-st order lift (see [8]) of the metric g from the base manifold to the tangent bundle TM:

$$\begin{aligned} JX^H_y &= a_1(t)X^V_y + b_1(t)g_{\tau(y)}(X, y)y^V_y, \\ JX^V_y &= -a_2(t)X^H_y - b_2(t)g_{\tau(y)}(X, y)y^H_y, \end{aligned}$$

$\forall X \in \mathcal{T}^1_0(M)$, $\forall y \in TM$, a_1, a_2, b_1, b_2 being smooth functions of the energy density.

The above (1,1)-tensor field J defines an almost complex structure on the tangent bundle if and only if

$$a_2 = \frac{1}{a_1}, \ b_2 = -\frac{b_1}{a_1(a_1 + 2tb_1)}.$$

Then it was considered on TM a Riemannian metric $\widetilde{G}$ of natural diagonal lift type:

$$\begin{aligned} &\widetilde{G}(X^H_y, Y^H_y) = c_1(t)g_{\tau(y)}(X, Y) + d_1(t)g_{\tau(y)}(X, y)g_{\tau(y)}(Y, y), \\ &\widetilde{G}(X^V_y, Y^V_y) = c_2(t)g_{\tau(y)}(X, Y) + d_2(t)g_{\tau(y)}(X, y)g_{\tau(y)}(Y, y), \\ &\widetilde{G}(X^V_y, Y^H_y) = \widetilde{G}(X^H_y, X^V_y) = 0, \end{aligned} \tag{2.2}$$

$\forall X, Y \in \mathcal{T}^1_0(TM)$, $\forall y \in TM$, where $c_1,\ c_2,\ d_1,\ d_2$ are smooth functions of the energy density on TM. The conditions for $\widetilde{G}$ to be a Riemannian metric on TM (i.e. to be positive definite) are $c_1 > 0,\ c_2 > 0,\ c_1 + 2td_1 > 0,\ c_2 + 2td_2 > 0$ for every $t \geq 0$.

The Riemannian metric $\widetilde{G}$ is almost Hermitian with respect to the almost complex structure J if and only if

$$\frac{c_1}{a_1} = \frac{c_2}{a_2} = \lambda, \quad \frac{c_1 + 2td_1}{a_1 + 2tb_1} = \frac{c_2 + 2td_2}{a_2 + 2tb_2} = \lambda + 2t\mu,$$

where $\lambda > 0,\ \mu > 0$ are functions of t.

The symmetric matrix of type $2n \times 2n$

$$\begin{pmatrix} \widetilde{G}^{(1)}_{ij} & 0 \\ 0 & \widetilde{G}^{(2)}_{ij} \end{pmatrix} = \begin{pmatrix} c_1(t)g_{ij} + d_1(t)g_{0i}g_{0j} & 0 \\ 0 & c_2(t)g_{ij} + d_2(t)g_{0i}g_{0j} \end{pmatrix},$$

associated to the metric $\widetilde{G}$ in the adapted frame $\{\delta_j, \partial_i\}_{i,j=\overline{1,n}}$, has the inverse

$$\begin{pmatrix} \widetilde{H}^{kl}_{(1)} & 0 \\ 0 & \widetilde{H}^{kl}_{(2)} \end{pmatrix} = \begin{pmatrix} p_1(t)g^{kl} + q_1(t)y^k y^l & 0 \\ 0 & p_2(t)g^{kl} + q_2(t)y^k y^l \end{pmatrix},$$

where g^{kl} are the entries of the inverse matrix of $(g_{ij})_{i,j=\overline{1,n}}$, and $p_1,\ q_1,\ p_2,\ q_2$, are some real smooth functions of the energy density. More precisely, they may be expressed as rational functions of $c_1,\ d_1,\ c_2,\ d_2$:

$$p_1 = \frac{1}{c_1},\ p_2 = \frac{1}{c_2},\ q_1 = -\frac{d_1}{c_1(c_1 + 2td_1)},\ q_2 = -\frac{d_2}{c_2(c_2 + 2td_2)}.$$

Let us recall some results proved in the papers [6] and [7].

Proposition 2.1 ([6])**.** *The Levi-Civita connection $\widetilde{\nabla}$ associated to the Riemannian metric $\widetilde{G}$ from the tangent bundle TM has the form*

$$\begin{cases} \widetilde{\nabla}_{X^V} Y^V = Q(X^V, Y^V),\ \widetilde{\nabla}_{X^H} Y^V = (\dot{\nabla}_X Y)^V + P(Y^V, X^H), \\ \widetilde{\nabla}_{X^V} Y^H = P(X^V, Y^H),\ \widetilde{\nabla}_{X^H} Y^H = (\dot{\nabla}_X Y)^H + S(X^H, Y^H), \end{cases}$$
$$\forall X, Y \in \mathcal{T}^1_0(M),$$

where the M-tensor fields $Q,\ P,\ S$, have the following components with respect to the adapted frame $\{\partial_i, \delta_j\}_{i,j=\overline{1,n}}$:

$$Q^h_{ij} = \frac{1}{2}(\partial_i \widetilde{G}^{(2)}_{jk} + \partial_j \widetilde{G}^{(2)}_{ik} - \partial_k \widetilde{G}^{(2)}_{ij})\widetilde{H}^{kh}_{(2)},$$

$$P^h_{ij} = \frac{1}{2}(\partial_i \widetilde{G}^{(1)}_{jk} + R^l_{0jk}\widetilde{G}^{(2)}_{li})\widetilde{H}^{kh}_{(1)},\ S^h_{ij} = -\frac{1}{2}(\partial_k \widetilde{G}^{(2)}_{ij} + R^l_{0ij}\widetilde{G}^{(2)}_{lk})\widetilde{H}^{kh}_{(2)}, \tag{2.3}$$

R^h_{kij} being the components of the curvature tensor field of the Levi Civita connection $\dot\nabla$ on the base manifold (M, g) and $R^h_{0ij} = y^k R^h_{kij}$. The vector fields $Q(X^V, Y^V)$ and $S(X^H, Y^H)$ are vertically valued, while the vector field $P(Y^V, X^H)$ is horizontally valued.

Using the relations (2.3), we may easily prove that the M-tensor fields Q, P, S, have invariant expressions of the forms

$$\begin{aligned}
Q(X^V, Y^V) &= \frac{c_2'}{2c_2}[g(y, X)Y^V + g(y, Y)X^V] - \frac{c_2' - 2d_2}{2(c_2 + 2td_2)}g(X, Y)y^V \\
&\quad + \frac{c_2 d_2' - 2c_2' d_2}{2c_2(c_2 + 2td_2)}g(y, X)g(y, Y)y^V, \\
P(X^V, Y^H) &= \frac{c_1'}{2c_1}g(y, X)Y^H + \frac{d_1}{2c_1}g(y, Y)X^H + \frac{d_1}{2(c_1 + 2td_1)}g(X, Y)y^H \\
&\quad + \frac{c_1 d_1' - c_1' d_1 - d_1^2}{2c_1(c_1 + 2td_1)}g(y, X)g(y, Y)y^H \\
&\quad - \frac{c_2}{2c_1}(R(X, y)Y)^H - \frac{c_2 d_1}{2c_1(c_1 + 2td_1)}g(X, R(Y, y)y)y^H, \\
S(X^H, Y^H) &= -\frac{d_1}{2c_2}[g(y, X)\ Y^V + g(y, Y)\ X^V] - \frac{c_1'}{2(c_2 + 2td_2)}g(X, Y)\ y^V \\
&\quad - \frac{c_2 d_1' - 2d_1 d_2}{2c_2(c_2 + 2td_2)}g(y, X)g(y, Y)y^V - \frac{1}{2}(R(X, Y)y)^V,
\end{aligned}$$

for every vector fields $X, Y \in \mathcal{T}^1_0(M)$ and every tangent vector $y \in TM$.

Since from now on we shall work on the subset T_rM of TM consisting of spheres of constant radius r, we shall consider only the tangent vectors y for which the energy density t is equal to $\frac{r^2}{2}$, and the coefficients from the definition (2.2) of the metric $\widetilde{G}$ become constant. So we may consider them constant from the beginning. Then the M-tensor fields involved in the expression of the Levi-Civita connection become simpler:

$$\begin{aligned}
Q(X^V, Y^V) &= \frac{d_2}{c_2 + r^2 d_2}g(X, Y)y^V, \\
P(X^V, Y^H) &= \frac{d_1}{2c_1}g(Y, y)X^H \\
&\quad + \frac{d_1}{2c_1(c_1 + r^2 d_1)}[c_1 g(X, Y) - d_1 g(X, y)g(Y, y)]y^H
\end{aligned}$$

$$- \frac{c_2}{2c_1}(R(X,y)Y)^H - \frac{c_2 d_1}{2c_1(c_1+r^2d_1)} g(X,R(Y,y)y)y^H,$$

$$S(X^H,Y^H) = -\frac{d_1}{2c_2}[g(X,y)Y^V + g(Y,y)X^V]$$

$$+ \frac{d_1 d_2}{c_2(c_2+2td_2)} g(y,X)g(y,Y)y^V - \frac{1}{2}(R(X,Y)y)^V.$$

3. Sasakian structures on tangent sphere bundles of constant radius r

Let $T_rM = \{y \in TM : g_{\tau(y)}(y,y) = r^2\}$, with $r \in (0,\infty)$, and the projection $\overline{\tau} : T_rM \to M$, $\overline{\tau} = \tau \circ i$, where i is the inclusion map of T_rM into TM.

Now we present the construction of a generator system for the tangent bundle to T_rM. The horizontal lift of any vector field on M is tangent to T_rM, but the vertical lift is not always tangent to T_rM. The tangential lift of a vector X to $(p,y) \in T_rM$ is tangent to T_rM and is defined by

$$X^T_y = X^V_y - \frac{1}{r^2} g_{\tau(y)}(X,y)y^V_y.$$

The tangential lift of the tangent vector $y \in T_rM$, vanishes, i.e. $y^T_y = 0$.

The tangent bundle TT_rM is spanned by δ_i and $\partial^T_j = \partial_j - \frac{1}{r^2} g_{0j} y^k \partial_k$, $i,j,k = \overline{1,n}$, although $\{\partial^T_j\}_{j=\overline{1,n}}$ are not independent. They fulfill the relation

$$y^j \partial^T_j = 0,$$

and in any point $y \in T_rM$ they span an $(n-1)$-dimensional subspace of T_yT_rM.

Denote by G' the metric on T_rM induced from the metric $\widetilde{G}$ defined on TM. If we consider only the tangent vectors $y \in T_rM$, the energy density t defined by them is equal to $\frac{r^2}{2}$, and the coefficients from the definition (2.2) of the metric $\widetilde{G}$ become constants. Thus, the metric G' from T_rM has the form

$$\begin{cases} G'(X^H_y, Y^H_y) = c_1 g_{\tau(y)}(X,Y) + d_1 g_{\tau(y)}(X,y) g_{\tau(y)}(Y,y), \\ G'(X^T_y, Y^T_y) = c_2[g_{\tau(y)}(X,Y) - \frac{1}{r^2} g_{\tau(y)}(X,y) g_{\tau(y)}(Y,y)], \\ G'(X^H_y, Y^T_y) = G'(Y^T_y, X^H_y) = 0, \end{cases} \tag{3.4}$$

for every vector fields X,Y on M and every tangent vector y, where $c_1,\ d_1,\ c_2$ are constants. The conditions for G to be positive are $c_1 > 0,\ c_2 > 0,\ c_1 + r^2 d_1 > 0$.

Theorem 3.1 ([7]). *The almost contact metric structure (φ, ξ, η, G) on T_rM, with φ, ξ, η, and G given respectively by*

$$\varphi X^H = a_1 X^T, \varphi X^T = -a_2 X^H + \frac{a_2}{r^2} g(X,y) y^H, \tag{3.5}$$

$$\xi = \frac{1}{2\lambda r^2 \alpha} y^H,\ \eta(X^T) = 0,\ \eta(X^H) = 2\alpha\lambda g(X,y), \tag{3.6}$$

$$G = \alpha G' \tag{3.7}$$

for every tangent vector fields $X, Y \in \mathcal{T}_0^1(M)$, and every tangent vector $y \in T_rM$, where $\alpha = \frac{c_1 + r^2 d_1}{4r^2\lambda^2}$ and the metric G' is given by (3.4)*, is a contact metric structure, and it is Sasakian if and only if the base manifold has constant sectional curvature $c = \frac{a_1^2}{r^2}$.*

Proposition 3.2 ([7]). *The Levi-Civita connection ∇, associated to the Riemannian metric G on the tangent sphere bundle T_rM of constant radius r has the expression*

$$\begin{cases} \nabla_{\partial_i^T} \partial_j^T = A_{ij}^h \partial_h^T,\ \nabla_{\delta_i} \partial_j^T = \Gamma_{ij}^h \partial_h^T + B_{ji}^h \delta_h, \\ \nabla_{\partial_i^T} \delta_j = B_{ij}^h \delta_h,\ \nabla_{\delta_i} \delta_j = \Gamma_{ij}^h \delta_h + C_{ij}^h \partial_h^T, \end{cases}$$

where the M-tensor fields involved as coefficients have the expressions:

$$\begin{aligned} A_{ij}^h &= -\frac{1}{r^2} g_{0j} \delta_i^h, \quad C_{ij}^h = -\frac{d_1}{2c_2}(\delta_j^h g_{0i} - \delta_i^h g_{0j}) - \frac{1}{2} R_{0ij}^h, \\ B_{ij}^h &= P_{ij}^h - \frac{1}{r^2} g_{i0} P_{0j}^h = \frac{d_1}{2c_1} \delta_i^h g_{0j} + \frac{d_1}{2(c_1 + r^2 d_1)} (g_{ij} - \frac{2c_1 + r^2 d_1}{r^2 c_1} g_{0i} g_{0j}) y^h \\ &\quad - \frac{c_2}{2c_1} R_{jik}^h y^k - \frac{c_2 d_1}{2c_1(c_1 + r^2 d_1)} R_{ikjl} y^h y^k y^l, \end{aligned}$$

where $g_{0j} = g_{j0} = y^i g_{ij}$ and $P_{0j}^h = y^i P_{ij}^h$.

We mention that A and C have these quite simple expressions, since they are the coefficients of the tangential part of the Levi-Civita connection. All the terms containing $y^h \partial_h^T = 0$ have been cancelled.

The invariant form of the Levi-Civita connection from T_rM is

$$\begin{cases}
\nabla_{X^T}Y^T &= -\frac{1}{r^2}g(Y,y)X^T,\\
\nabla_{X^T}Y^H &= \frac{d_1}{2c_1}g(Y,y)\ X^H + \frac{d_1}{2(c_1+d_1r^2)}g(X,Y)y^H - \frac{c_2}{2c_1}(R(X,y)Y)^H\\
&\quad -\frac{d_1(2c_1+r^2d_1)}{2r^2c_1(c_1+r^2d_1)}g(X,y)g(Y,y)y^H\\
&\quad -\frac{c_2d_1}{2c_1(c_1+r^2d_1)}g(X,R(Y,y)y)y^H,\\
\nabla_{X^H}Y^T &= (\dot\nabla_XY)^T + \frac{d_1}{2c_1}g(X,y)Y^H + \frac{d_1}{2(c_1+r^2d_1)}g(X,Y)y^H\\
&\quad -\frac{c_2}{2c_1}(R(Y,y)X)^H - \frac{d_1(2c_1+r^2d_1)}{2r^2c_1(c_1+r^2d_1)}g(X,y)g(Y,y)y^H\\
&\quad -\frac{c_2d_1}{2c_1(c_1+r^2d_1)}g(Y,R(X,y)y)y^H,\\
\nabla_{X^H}Y^H &= (\dot\nabla_XY)^H - \frac{d_1}{2c_2}[g(X,y)\ Y^T - g(Y,y)\ X^T] - \frac{1}{2}(R(X,Y)y)^T.
\end{cases}$$

In the case where A, B, C are those given in Proposition 3.2, it can be shown (see e.g. [6]) that their covariant derivatives are:

$$\begin{cases}
\dot\nabla_k A^h_{ij} = 0, \quad \dot\nabla_k C^h_{ij} = -\frac{1}{2}\dot\nabla_k R^h_{lij}y^l,\\
\dot\nabla_k B^h_{ij} = -\frac{c_2}{2c_1}\dot\nabla_k R^h_{jil}y^l - \frac{c_2d_1}{2c_1(c_1+r^2d_1)}\dot\nabla_k R_{iljr}y^ly^ry^h,
\end{cases}$$

where $\dot\nabla_k R^h_{kij}$, $\dot\nabla_k R_{hkij}$ are the usual local coordinate expressions of the covariant derivatives of the curvature tensor field and the Riemann-Christoffel tensor field of the Levi Civita connection $\dot\nabla$ from the base manifold M. If the base manifold (M,g) is locally symmetric then, obviously, $\dot\nabla_k A^h_{ij} = 0$, $\dot\nabla_k B^h_{ij} = 0$, $\dot\nabla_k C^h_{ij} = 0$.

In the paper [6] we obtained by a standard straightforward computation the horizontal and tangential components of the curvature tensor field K, denoted by sequences of H and T, to indicate horizontal, or tangential argument on a certain position. For example, we may write

$$K(\delta_i,\delta_j)\partial^T_k = HHTH^h_{kij}\delta_h + HHTT^h_{kij}\partial^T_h.$$

The expressions of the non-zero M-tensor fields which appear as coefficients are

$$
\begin{aligned}
HHHH^h_{kij} &= R^h_{kij} + B^h_{li}C^l_{jk} - B^h_{lj}C^l_{ik} + B^h_{lk}R^l_{0ij},\\
HHHT^h_{kij} &= \frac{1}{2}(\dot\nabla_j R^h_{lik}y^l - \dot\nabla_i R^h_{ljk}y^l),\\
HHTH^h_{kij} &= \frac{c_2}{2c_1}(\dot\nabla_j R^h_{ikl} - \dot\nabla_i R^h_{jkl})y^l\\
&\quad + \frac{c_2 d_1}{2c_1(c_1+r^2d_1)}(\dot\nabla_j R_{ilkm} - \dot\nabla_i R_{jlkm})y^l y^m y^h\\
HHTT^h_{kij}\partial^T_h &= (R^h_{kij} + C^h_{il}B^l_{kj} - C^h_{jl}B^l_{ki} + A^h_{lk}R^l_{0ij})\partial^T_h,\\
TTHH^h_{kij} &= \partial^T_i B^h_{jk} - \partial^T_j B^h_{ik} + B^h_{il}B^l_{jk} - B^h_{jl}B^l_{ik} - \frac{1}{r^2}(g_{i0}B^h_{jk} - g_{j0}B^h_{ik}),\\
TTTT^h_{kij} &= \frac{1}{r^2}(g_{jk}\delta^h_i - g_{ik}\delta^h_j) + \frac{1}{r^4}(\delta^h_j g_{0i}g_{0k} - \delta^h_i g_{0j}g_{0k}),\\
THHH^h_{kij} &= \frac{c_2}{2c_1}\dot\nabla_j R^h_{kil}y^l + \frac{c_2 d_1}{2c_1(c_1+r^2d_1)}\dot\nabla_j R_{ilkr}y^l y^r y^h,\\
THHT^h_{kij}\partial^T_h &= (\partial^T_i C^h_{jk} + A^h_{il}C^l_{jk} - C^h_{jl}B^l_{ik} - \dot\nabla_j R^h_{lik}y^l)\partial^T_h,\\
THTH^h_{kij} &= \partial^T_i B^h_{kj} + B^l_{kj}B^h_{il} - A^l_{ik}B^h_{lj},\ THTT^h_{kij} = -\dot\nabla_j A^h_{ik} = 0.
\end{aligned}
$$

We mention that from the final values of $HHTT^h_{kij}\partial^T_h$ and $THHT^h_{kij}\partial^T_h$, obtained by replacing the expressions of $A^h_{ij}, B^h_{ij}, C^h_{ij}$ given in Proposition 3.2, we shall eliminate the terms containing $y^h\partial^T_h = 0$.

4. The holomorphic φ-sectional curvature

In this section we shall study the condition under which the Sasaki manifold $(T_rM, \varphi, \xi, \eta, G)$, with φ, ξ, η, and G given respectively by (3.5), (3.6), and (3.7), is a Sasaki space form, i.e. it has constant holomorphic φ-sectional curvature. The tensor field corresponding to the curvature of the Sasakian space form $(T_rM, \varphi, \xi, \eta, G)$ is given by:

$$
\begin{aligned}
K_0(X,Y)Z = \frac{k+3}{4}&[G(Y,Z)X - G(X,Z)Y]\\
+ \frac{k-1}{4}&[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\\
&+ G(X,Z)\eta(Y)\xi - G(Y,Z)\eta(X)\xi + G(Z,\varphi Y)\varphi X\\
&- G(Z,\varphi X)\varphi Y + 2G(X,\varphi Y)\varphi Z],
\end{aligned}
$$

where k is a constant.

With respect to the generator system $\{\delta_i, \partial_j^T\}_{i,j=1,\dots,n}$, the expressions are

$$K_0(\partial_i^T, \partial_j^T)\partial_k^T = TTTT_{0kij}^{\ \ h}\partial_h^T,\ K_0(\partial_i^T, \partial_j^T)\delta_k = TTHH_{0kij}^{\ \ h}\delta_h,$$
$$K_0(\partial_i^T, \delta_j)\partial_k^T = THTH_{0kij}^{\ \ h}\delta_h,\ K_0(\partial_i^T, \delta_j)\delta_k = THHT_{0kij}^{\ \ h}\partial_h^T,$$
$$K_0(\delta_i, \delta_j)\partial_k^T = HHTT_{0kij}^{\ \ h}\partial_h^T,\ K_0(\delta_i, \delta_j)\delta_k = HHHH_{0kij}^{\ \ h}\delta_h,$$

where the M-tensor fields involved as coefficients have the expressions:

$$\begin{aligned}
TTTT_{0kij}^{\ \ h} &= \frac{k+3}{4}\alpha(G'^{(2)}_{kj}\delta_i^h - G'^{(2)}_{ki}\delta_j^h),\\
TTHH_{0kij}^{\ \ h} &= a_2^2\frac{k-1}{4}\alpha G'^{(1)}_{kl}\Big\{\frac{1}{r^2}\Big[g_{0i}(\delta_j^h y^l - \delta_j^l y^h) - g_{0j}(\delta_i^h y^l - \delta_i^l y^h)\Big]\\
&\quad + \delta_j^l\delta_i^h - \delta_i^l\delta_j^h\Big\},\\
THTH_{0kij}^{\ \ h} &= -\frac{k+3}{4}\alpha G'^{(2)}_{ki}\delta_j^h + \frac{k-1}{4}\alpha\Big[\frac{1}{r^2}G'^{(2)}_{ki}g_{0j}y^h\\
&\quad + G'^{(2)}_{kj}\Big(\frac{1}{r^2}g_{0i}y^h - \delta_i^h\Big) + 2G'^{(2)}_{ij}\Big(\frac{1}{r^2}g_{0k}y^h - \delta_k^h\Big)\Big],\\
THHT_{0kij}^{\ \ h} &= \frac{k+3}{4}\alpha G'^{(1)}_{kj}\delta_i^h + \frac{k-1}{4}\alpha\Big[-4\lambda^2\alpha g_{0j}g_{0k}\delta_i^h\\
&\quad - \Big(\frac{1}{r^2}g_{0i}y^l - \delta_i^l\Big)G'^{(1)}_{kl}\delta_j^h + 2a_1^2 G'^{(2)}_{ij}\delta_k^h\Big],\\
HHTT_{0kij}^{\ \ h} &= \frac{k-1}{4}\alpha a_1^2(G'^{(2)}_{kj}\delta_i^h - G'^{(2)}_{ki}\delta_j^h),\\
HHHH_{0kij}^{\ \ h} &= \frac{k+3}{4}\alpha(G'^{(1)}_{kj}\delta_i^h - G'^{(1)}_{ki}\delta_j^h)\\
&\quad + \frac{k-1}{4}\alpha[4\lambda^2\alpha g_{0k}(g_{0i}\delta_j^h - g_{0j}\delta_i^h) + \frac{1}{r^2}(G'^{(1)}_{ki}g_{0j} - G'^{(1)}_{kj}g_{0i})y^h].
\end{aligned}$$

In the sequel we shall prove the main theorem of this paper.

Theorem 4.1. *The Sasakian manifold $(T_rM, \varphi, \xi, \eta, G)$, with φ, ξ, η, and G given respectively by* (3.5)*,* (3.6)*, and* (3.7)*, is never a Sasaki space form.*

Proof. We have to study the vanishing conditions for all the components of the difference $K - K_0$ with respect to the generator system $\{\delta_i, \partial_j^T\}_{i,j=1,\dots,n}$.

The component $THHT^h_{kij} - THHT0^h_{kij}$ contains $-\frac{c_1(k+3)(c_1+r^2d_1)}{16\lambda^2r^2}g_{jk}\delta^h_i$ and $THTH^h_{kij} - THTH0^h_{kij}$ contains the term $\frac{c_2(k+3)(c_1+r^2d_1)}{16\lambda^2r^2}(g_{ik}+g_{0i}g_{0k})\delta^h_j$. Hence a necessary vanishing condition is $k=-3$.

If $(T_rM,\varphi,\xi,\eta,G)$ is Sasakian, the base manifold has constant sectional curvature c. Taking this into account, and replacing the value -3 of k into the expression of the component $HHHH^h_{kij} - HHHH0^h_{kij}$, this will contain the terms $c(\delta^h_i g_{jk} - \delta^h_j g_{ik})$, which vanish if and only if $c=0$ (i.e. the base manifold is flat). In this case, some components of the difference $K-K_0$ vanish under certain very restrictive conditions, but the component $TTTT^h_{kij} - TTTT0^h_{kij}$ is never vanishing, since it takes the form

$$\frac{1}{r^2}\Big[\delta^h_i(g_{jk} - \frac{1}{r^2}g_{0j}g_{0k}) - \delta^h_j(g_{ik} - \frac{1}{r^2}g_{0i}g_{0k})\Big] \neq 0.$$

Thus the theorem is proved. $\square$

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