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A VANISHING THEOREM FOR VERTICAL TENSOR FIELDS ON COMPLEX FINSLER BUNDLES

BY

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Abstract. Using the Bochner technique for the vertical Laplacian associated to a complex Finsler bundle (E, L) over a hermitian manifold (M, g), we obtain a vanishing theorem for holomorphic vertical tensor fields with compact support on the total space of (E, L).

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1. Introduction and preliminaries notions

The importance of Bochner technique, initiated in [4, 5, 6], in geometrical study both in Riemannian and Kählerian manifolds, is without question, and for instance, were discussed in details in [8, 12, 16, 19, 20, 22]. The study of existence of holomorphic sections on complex Finsler bundles was initiated in [13]. There is obtained an important theorem which states that a holomorphic vector bundle E with a strongly pseudoconvex complex Finsler metric F over a compact hermitian manifold (M, g) admits no holomorphic sections if a mean curvature is negative, (see also [3], [10] and [14]). Different from [13, 14], using the Bochner technique for the horizontal Laplacian associated to a strongly Kähler-Finsler manifold, recently in [24], is obtained a vanishing theorem of holomorphic forms on strongly Kähler-Finsler manifolds. This theorem generalizes the classical vanishing theorems of holomorphic tensor fields on Kähler manifolds [5, 6]. Also, in [21], are obtained some vanishing theorems of Bochner and Kodaira type for some horizontal forms on strongly Kähler-Finsler manifolds.

In a previous paper [11], we have proved that the vertical Laplacian associated to a complex Finsler bundle (E, L) over a hermitian manifold (M, g) has similar properties as the horizontal Laplacian associated to a strongly Kähler-Finsler manifold. Thus, following the same ideas from [23, 24], in the present paper we obtain a vanishing theorem of Bochner type for holomorphic vertical tensor fields with compact support on the total space of (E, L).

Let us begin our study with a short review of the geometry of the total space of a complex Finsler bundle. For more, see [1, 2, 3, 17].

Let $\pi : E \to M$ be a holomorphic vector bundle of rank $m \geq 2$ over a complex manifold M of $\dim_{\mathbb{C}} M = n$. Let us consider $\mathcal{U} = \{U_\alpha\}$ an open covering set of M and (z^k), $k = 1, \ldots, n$ the local complex coordinates in a chart (U, φ) and $s_U = \{s_a\}$, $a = 1, \ldots, m$ a local holomorphic frame for the sections of E over U. The covering $\{U, s_U\}$, $U \in \mathcal{U}$ induces a complex coordinate system $u = (z^k, \eta^a)$ on $\pi^{-1}(U)$, where $s = \eta^a s_a$ is a holomorphic section on $E_z = \pi^{-1}(z)$. If we denote by $g_{UV} : U \cap V \to GL(m, \mathbb{C})$ the transition functions, then in $z \in U \cap V$, $g_{UV}(z)$ has a local representation by the complex matrix $M^a_b(z)$ and if (z'^k, η'^a) are the complex coordinates in $\pi^{-1}(V)$, the transition laws of these coordinates are:

$$z'^k = z'^k(z) \ , \ \eta'^a = M^a_b(z)\, \eta^b, \tag{1.1}$$

where $M^a_b(z)$, $a, b = 1, \ldots, m$ are holomorphic functions and $\det M^a_b \neq 0$.

As we already know, the total space E has a structure of $(m + n)$-dimensional complex manifold because the transition functions $M^a_b(z)$ are holomorphic. Consider the complexified tangent bundle $T_{\mathbb{C}}E = T'E \oplus T''E$, where $T'E$ and $T''E = \overline{T'E}$ are the holomorphic and antiholomorphic tangent bundles. The vertical holomorphic tangent bundle $VE = \ker \pi_*$ is the relative tangent bundle of the holomorphic projection π and a local frame field on V_uE is $\{\frac{\partial}{\partial \eta^a}\}$, $a = 1, \ldots, m$. The vertical distribution V_uE is isomorphic to the sections module of E in u.

A subbundle HE of $T'E$ complementary to VE, i.e. $T'E = VE \oplus HE$, is called a *complex nonlinear connection* on E, briefly c.n.c. A local base for the horizontal distribution H_uE, called adapted for the c.n.c. is $\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N^a_k \frac{\partial}{\partial \eta^a}\}$, $k = 1, \ldots, n$, where $N^a_k(z, \eta)$ are the coefficients of the c.n.c.

As in the case of general theory of vector bundles see [15], we notice that the existence of a c.n.c. is an important ingredient in the "linearization" of geometry of the total space of a holomorphic vector bundle.

In the following we consider the abreviate notations: $\partial_k = \frac{\partial}{\partial z^k}$, $\delta_k = \frac{\delta}{\delta z^k}$ and $\dot{\partial}_a = \frac{\partial}{\partial \eta^a}$. The adapted frames denoted by $\{\delta_{\overline{k}} := \frac{\delta}{\delta \overline{z}^k}\}$ and $\{\dot{\partial}_{\overline{a}} := \frac{\partial}{\partial \overline{\eta}^a}\}$, for the distributions $\overline{HE}$ and $\overline{VE}$ are obtained respectively by conjugation. The adapted coframes are locally given by $\{dz^k\}$, $\{\delta\eta^a = d\eta^a + N^a_k dz^k\}$, $\{d\overline{z}^k\}$ and $\{\delta\overline{\eta}^a = d\overline{\eta}^a + N^{\overline{a}}_{\overline{k}} d\overline{z}^k\}$, respectively.

A strictly pseudoconvex complex Finsler structure on E, is a positive real valued smooth function $F^2 = L : E \to \mathbb{R}_+ \cup \{0\}$, which satisfies the following conditions:

(i) L is smooth on $E - \{\text{zero section}\}$;

(ii) $L(z,\eta) \geq 0$ and $L(z,\eta) = 0$ if and only if $\eta = 0$;

(iii) $L(z,\lambda\eta) = |\lambda|^2 L(z,\eta)$ for any $\lambda \in \mathbb{C}$;

(iv) $(h_{a\overline{b}}) = (\dot{\partial}_a \dot{\partial}_{\overline{b}}(L))$ (the complex Levi matrix) is positive defined and determines a hermitian metric tensor on the fibers of vertical bundle VE.

Definition 1.1 ([13])**.** The pair (E,L) is called a complex Finsler bundle.

According to [2], [17], a c.n.c. related only to the complex Finsler structure L is the Chern-Finsler c.n.c., locally given by

$$\overset{CF}{N^a_k} = h^{\overline{c}a} \partial_k \dot{\partial}_{\overline{c}}(L). \tag{1.2}$$

We also identify the holomorphic local frame fields $s = \{s_1, \ldots, s_m\}$ of E, with the one of the pull-back bundle $\widetilde{E} := \pi^* E$ which is isomorphic to VE by $\dot{\partial}_a \leftrightarrow \pi^* s_a := s_a$. Then, $\widetilde{E}$ admits a hermitian metric h_L defined by

$$h_L(Z,W) = h_{a\overline{b}} Z^a \overline{W}^b \text{ for any } Z = Z^a s_a,\ W^a s_a \in \Gamma(\widetilde{E}). \tag{1.3}$$

Let $\nabla : \Gamma(\widetilde{E}) \to A^1(\widetilde{E})$ be the hermitian connection of the bundle $(\widetilde{E}, h_L)$, i.e. $\nabla = \nabla' + \nabla''$ is the unique connection on the bundle $(\widetilde{E}, h_L)$ satisfying the conditions

$$\nabla'' = d'' \,;\ dh_L(Z,W) = h_L(\nabla Z, W) + h_L(Z, \nabla W),\ \forall Z, W \in \Gamma(\widetilde{E}). \tag{1.4}$$

Definition 1.2 ([1])**.** The hermitian connection ∇ on $(\widetilde{E}, h_L)$ is called the Chern-Finsler linear connection of the bundle (E,L).

The $(1,0)$-connection form $\omega = (\omega_b^a)$ of ∇, with respect to a holomorphic frame $s = \{s_a\}$, $a = 1, \ldots, m$, is defined by

$$\nabla s_b = \omega_b^a \otimes s_a \, , \; \omega_b^a = \Gamma_{bk}^a dz^k + C_{bc}^a d\eta^c, \tag{1.5}$$

where the local coefficients of the connection are given by

$$\Gamma_{bk}^a = h^{\bar{c}a} \partial_k (h_{b\bar{c}}) \, , \; C_{bc}^a = h^{\bar{d}a} \; \dot{\partial}_c \; (h_{b\bar{d}}). \tag{1.6}$$

Using the adapted frames and coframes with respect to the Chern-Finsler c.n.c., the $(1,0)$-connection form ω_b^a can be rewritten by

$$\omega_b^a = L_{bk}^a dz^k + C_{bc}^a \delta\eta^c \, , \; L_{bk}^a = h^{\bar{c}a} \delta_k (h_{b\bar{c}}). \tag{1.7}$$

We notice that the vertical coefficients C_{bc}^a satisfy the symmetry relation $C_{bc}^a = C_{cb}^a$ and it is easy to check that $C_{ab}^a = \dot{\partial}_b \, (\ln h)$ where $h = \det(h_{a\bar{b}})$.

The $(1,1)$-curvature form $R = (R_b^a)$ of ∇ is locally given by

$$R_b^a = d^{''} \omega_b^a. \tag{1.8}$$

Using the decomposition $d^{''} = d^{''h} + d^{''v} + \partial_1 + \partial_2$, see [9], [18], a straightforward calculus in (1.8) leads to the following decomposition of the curvature

$$R = R^{h\bar{h}} + R^{h\bar{v}} + R^{v\bar{h}} + R^{v\bar{v}}, \tag{1.9}$$

i.e. in a horizontal component $R^{h\bar{h}}$, in two mixed components $R^{h\bar{v}}$, $R^{v\bar{h}}$ and a vertical component $R^{v\bar{v}}$. Also, we notice that with respect to a holomorphic local frame field $s = \{s_a\}$ the vertical part of the curvature is given by

$$R^{v\bar{v}} s_b = (S_{b,c\bar{d}}^a \delta\eta^c \wedge \delta\overline{\eta}^d) s_a, \tag{1.10}$$

where $S_{b,c\bar{d}}^a = - \; \dot{\partial}_{\bar{d}} \; (C_{bc}^a)$ and it will be important in our study.

2. Vertical Ricci identities

In this section, we briefly recall the vertical covariant derivatives with respect to the Chern-Finsler connection for contravariant and covariant vertical tensor fields. Also, some vertical Ricci identities are established.

If $T^{A_p\overline{B_q}}(z,\eta)$ are the components of a contravariant vertical complex tensor field of type (p,q) on E, where $A_p=(a_1\dots a_p)$ and $B_q=(b_1\dots b_q)$, then its vertical covariant derivatives with respect to the Chern-Finsler connection are given by

$$\nabla_{\dot{\partial}_a}T^{A_p\overline{B_q}}=\dot{\partial}_a\,(T^{A_p\overline{B_q}})+\sum_{k=1}^{p}T^{a_1\dots a_{k-1}\lambda a_{k+1}\dots a_p\overline{B_q}}C^{a_k}_{\lambda a},\tag{2.1}$$

$$\nabla_{\dot{\partial}_{\overline{a}}}T^{A_p\overline{B_q}}=\dot{\partial}_{\overline{a}}\,(T^{A_p\overline{B_q}})+\sum_{k=1}^{q}T^{A_p\overline{b_1}\dots\overline{b_{k-1}}\,\overline{\lambda}\,\overline{b_{k+1}}\dots\overline{b_q}}C^{\overline{b_k}}_{\overline{\lambda}\,\overline{a}}.\tag{2.2}$$

If $T_{A_p\overline{B_q}}(z,\eta)$ are the components of a covariant vertical complex tensor field of type (p,q) on E, then its vertical covariant derivatives with respect to the Chern-Finsler connection are given by

$$\nabla_{\dot{\partial}_a}T_{A_p\overline{B_q}}=\dot{\partial}_a\,(T_{A_p\overline{B_q}})-\sum_{k=1}^{p}T_{a_1\dots a_{k-1}\lambda a_{k+1}\dots a_p\overline{B_q}}C^{\lambda}_{a_k a},\tag{2.3}$$

$$\nabla_{\dot{\partial}_{\overline{a}}}T_{A_p\overline{B_q}}=\dot{\partial}_{\overline{a}}\,(T_{A_p\overline{B_q}})-\sum_{k=1}^{q}T_{A_p\overline{b_1}\dots\overline{b_{k-1}}\,\overline{\lambda}\,\overline{b_{k+1}}\dots\overline{b_q}}C^{\overline{\lambda}}_{\overline{b_k}\,\overline{a}}.\tag{2.4}$$

We remark that if we combine the formulas above, we get the vertical covariant derivatives for mixed contravariant and covariant vertical complex tensor fields. Also, it is easy to check

$$\nabla_{\dot{\partial}_c}h_{a\bar{b}}=\nabla_{\dot{\partial}_{\bar{c}}}h_{a\bar{b}}=\nabla_{\dot{\partial}_c}h^{\bar{b}a}=\nabla_{\dot{\partial}_{\bar{c}}}h^{\bar{b}a}=0.\tag{2.5}$$

Now, by using the above vertical covariant derivatives with respect to Chern-Finsler connection, a straightforward calculus leads to

Proposition 2.1. *Let (E,L) be a complex Finsler bundle, X^a the components of a contravariant vertical complex tensor field X on E and φ_a the components of a covariant vertical complex tensor field φ on E. Then*

$$[\nabla_{\dot{\partial}_b},\nabla_{\dot{\partial}_{\bar{c}}}]X^a=X^dS^a_{d,b\bar{c}}\ ,\ [\nabla_{\dot{\partial}_b},\nabla_{\dot{\partial}_{\bar{c}}}]X^{\overline{a}}=-X^{\overline{d}}S^{\overline{a}}_{\overline{d},b\bar{c}}\,,$$
$$[\nabla_{\dot{\partial}_b},\nabla_{\dot{\partial}_{\bar{c}}}]\varphi_a=-\varphi_dS^d_{a,b\bar{c}}\ ,\ [\nabla_{\dot{\partial}_b},\nabla_{\dot{\partial}_{\bar{c}}}]\varphi_{\overline{a}}=\varphi_{\overline{d}}S^{\overline{d}}_{\overline{a},b\bar{c}}.$$

If we denote by $S_{a\bar{b},c\bar{d}}=h_{e\bar{b}}S^e_{a,c\bar{d}}$ then, by using the homogeneity conditions of complex Finsler structure L, namely

$$\dot{\partial}_c\,(h_{a\bar{b}})\eta^c=0\,,\ \dot{\partial}_c\,(h_{a\bar{b}})\eta^a=0\,,\ \dot{\partial}_{\bar{c}}\,(h_{a\bar{b}})\overline{\eta}^b=0\,,\ \dot{\partial}_{\bar{c}}\,(h_{a\bar{b}})\overline{\eta}^c=0,$$

we obtain

Proposition 2.2. *Let (E,L) be a complex Finsler bundle. Then*

$$(S_{a\bar{b},c\bar{d}} - S_{\bar{b}a,\bar{d}c})\eta^a\overline{\eta}^b = 0, \tag{2.6}$$

$$S_{a\bar{b},c\bar{d}} - S_{c\bar{b},a\bar{d}} = 0, \tag{2.7}$$

$$(S_{a\bar{b},c\bar{d}} - S_{c\bar{d},a\bar{b}})\eta^a\overline{\eta}^b\eta^c\overline{\eta}^d = 0. \tag{2.8}$$

Definition 2.1. The vertical generalized Ricci tensor field of (E,L) is locally defined by

$$S_{a\bar{d}} = S^b_{a,b\bar{d}} = -\,\dot{\partial}_{\bar{d}}\,(C^b_{ab}). \tag{2.9}$$

We also have

$$S_{a\bar{d}} = h^{\bar{c}b}S_{a\bar{c},b\bar{d}} = -\,\dot{\partial}_{\bar{d}}\dot{\partial}_a\,(\ln h). \tag{2.10}$$

3. A vanishing theorem

Let us suppose that the base manifold of the complex Finsler bundle (E,L) is a hermitian manifold (M,g). Then, due to [7], in natural manner we can consider the following hermitian metric structure of Sasaki type on E

$$G = g_{j\bar{k}}(z)dz^j \otimes d\overline{z}^k + h_{a\bar{b}}(z,\eta)\delta\eta^a \otimes \delta\overline{\eta}^b, \tag{3.1}$$

where $g_{j\bar{k}}(z)$ is the hermitian metric on the base manifold M, $h_{a\bar{b}}(z,\eta)$ is the fundamental metric tensor defined by the complex Finsler structure L and the adapted coframes are considered with respect to the Chern-Finsler c.n.c.

In [11], is proved that with respect to the hermitian metric structure from (3.1) the vertical complex Laplacians for smooth functions on E are given by

$$\Box'^{v} f = \frac{1}{h}\,\dot{\partial}_a\,(hh^{\bar{b}a}\,\dot{\partial}_{\bar{b}}\,f) \text{ and } \Box''^{v} f = \frac{1}{h}\,\dot{\partial}_{\bar{a}}\,(hh^{\bar{a}b}\,\dot{\partial}_b\,f). \tag{3.2}$$

Remark 3.1. In terms of the vertical covariant derivatives with respect to the Chern-Finsler linear connection, we have

$$\Box'^{v} f = h^{\bar{b}a}\nabla_{\dot{\partial}_a}\nabla_{\dot{\partial}_{\bar{b}}}f. \tag{3.3}$$

Also, if f is a smooth function with compact support on E then, we have

$$\int_E \Box'^v f dV_E = 0, \tag{3.4}$$

where dV_E is the volume form associated to the hermitian structure G on E, see for details [11].

Now, let $X^{A_p}_{B_q}(z,\eta)$ be the components of a mixed vertical complex tensor field X with compact support on E of contravariant valency p and of covariant valency q. In the following we denote by $||X||^2$ the local scalar product of this tensor with itself with respect to the natural inner product induced by the hermitian metric structure G on E, that is

$$||X||^2 = h_{A_p\overline{C_p}} h^{\overline{D_q}B_q} X^{A_p}_{B_q} X^{\overline{C_p}}_{\overline{D_q}}, \tag{3.5}$$

where $h_{A_p\overline{C_p}} = h_{a_1\overline{c_1}} \dots h_{a_p\overline{c_p}}$ and $h^{D_q\overline{B_q}} = h^{\overline{d_1}b_1} \dots h^{\overline{d_q}b_q}$.

If $X^{A_p}_{B_q}(z,\eta)$ are holomorphic functions of the fiber coordinates (η^a), then $\nabla_{\dot\partial_{\overline{c}}} X^{A_p}_{B_q} = 0$. We have

Proposition 3.1. *Let (E,L) be a complex Finsler bundle over a hermitian manifold (M,g). Then*

$$\Box'^v ||X||^2 = ||\nabla'^v X||^2 + \mathcal{G}^v(X), \tag{3.6}$$

where we have denoted $||\nabla'^v X||^2 = h^{\bar b a} h_{A_p\overline{C_p}} h^{\overline{D_q}B_q} (\nabla_{\dot\partial_a} X^{A_p}_{B_q})(\nabla_{\dot\partial_{\overline{b}}} X^{\overline{C_p}}_{\overline{D_q}})$,

$$\mathcal{G}^v(X) = -\sum_{k=1}^{p} S^{\overline{c_k}}_{\overline{\lambda_k}} X^{\overline{D_q}}_{\overline{C_p}} X^{\overline{c_1}\dots\overline{c_{k-1}}\,\overline{\lambda_k}\,\overline{c_{k+1}}\dots\overline{c_p}}_{\overline{D_q}} + \sum_{k=1}^{q} S^{\overline{\lambda_k}}_{\overline{d_k}} X^{\overline{D_q}}_{\overline{C_p}} X^{\overline{C_p}}_{\overline{d_1}\dots\overline{d_{k-1}}\,\overline{\lambda_k}\,\overline{d_{k+1}}\dots\overline{d_q}},$$

and $X^{\overline{D_q}}_{\overline{C_p}} = h_{A_p\overline{C_p}} h^{\overline{D_q}B_q} X^{A_p}_{B_q}$, $S^{\overline{\lambda_k}}_{\overline{d_k}} = h^{\bar b a} S^{\overline{\lambda_k}}_{\overline{d_k},\bar b a}$.

Proof. We have

$$\begin{aligned}
\Box'^v ||X||^2 &= h^{\bar b a} \nabla_{\dot\partial_a} \nabla_{\dot\partial_{\overline{b}}} (h_{A_p\overline{C_p}} h^{\overline{D_q}B_q} X^{A_p}_{B_q} X^{\overline{C_p}}_{\overline{D_q}}) \\
&= h^{\bar b a} h_{A_p\overline{C_p}} h^{\overline{D_q}B_q} \nabla_{\dot\partial_a} (X^{A_p}_{B_q} \nabla_{\dot\partial_{\overline{b}}} X^{\overline{C_p}}_{\overline{D_q}}) \\
&= h^{\bar b a} h_{A_p\overline{C_p}} h^{\overline{D_q}B_q} [\nabla_{\dot\partial_a} X^{A_p}_{B_q} \nabla_{\dot\partial_{\overline{b}}} X^{\overline{C_p}}_{\overline{D_q}} + X^{A_p}_{B_q} \nabla_{\dot\partial_a} (\dot\partial_{\overline{b}}\, X^{\overline{C_p}}_{\overline{D_q}} \\
&+ \sum_{k=1}^{p} X^{\overline{c_1}\dots\overline{c_{k-1}}\,\overline{\lambda_k}\,\overline{c_{k+1}}\dots\overline{c_p}}_{\overline{D_q}} C^{\overline{c_k}}_{\overline{\lambda_k}\,\overline{b}} - \sum_{k=1}^{q} X^{\overline{C_p}}_{\overline{d_1}\dots\overline{d_{k-1}}\,\overline{\lambda_k}\,\overline{d_{k+1}}\dots\overline{d_q}} C^{\overline{\lambda_k}}_{\overline{d_k}\,\overline{b}})]
\end{aligned}$$

$$= h^{\bar{b}a}h_{A_p\overline{C_p}}h^{\overline{D_q}B_q}[\nabla_{\dot{\partial}_a}X^{A_p}_{B_q}\nabla_{\dot{\partial}_{\bar{b}}}X^{\overline{C_p}}_{\overline{D_q}} + X^{A_p}_{B_q}\sum_{k=1}^{p}X^{\overline{c_1}\ldots\overline{c_{k-1}}\,\overline{\lambda_k}\,\overline{c_{k+1}}\ldots\overline{c_p}}_{\overline{D_q}}\,\dot{\partial}_a\,(C^{\overline{c_k}}_{\overline{\lambda_k}\,\bar{b}})$$

$$- X^{A_p}_{B_q}\sum_{k=1}^{q}X^{\overline{C_p}}_{\overline{d_1}\ldots\overline{d_{k-1}}\,\overline{\lambda_k}\,\overline{d_{k+1}}\ldots\overline{d_q}}\,\dot{\partial}_a\,(C^{\overline{\lambda_k}}_{\overline{d_k}\,\bar{b}})] = ||\nabla'^{v}X||^2 + \mathcal{G}^v(X)$$

which completes the proof. □

By (3.4) and the Proposition 3.1., one gets

Theorem 3.1. *Let (E, L) be a complex Finsler bundle over a hermitian manifold (M, g) and $X^{A_p}_{B_q}(z, \eta)$ be the components of a mixed vertical complex tensor field X with compact support on E. If $X^{A_p}_{B_q}$ are holomorphic functions of fiber coordinates (η^a) and satisfy the inequality $\mathrm{Re}\mathcal{G}^v(X) \geq 0$ then $\mathcal{G}^v(X) = 0$ and $\nabla_{\dot{\partial}_a}X^{A_p}_{B_q} = 0$ at every point $(z, \eta) \in E$.*

Finally, we give some particularizations of the above theorem.

Proposition 3.2. *Let (E, L) be a complex Finsler bundle over a hermitian manifold (M, g) and $X^a(z, \eta)$ the components of a contravariant vertical complex tensor field with compact support on E. Then*

(i) If X^a are holomorphic functions of (η^a) coordinates and $\mathrm{Re}S_{a\bar{b}}X^aX^{\bar{b}} \leq 0$ then $\nabla_{\dot{\partial}_b}X^a = 0$;

(ii) If X^a are holomorphic functions of (η^a) coordinates and $\mathrm{Re}S_{a\bar{b}}X^aX^{\bar{b}} < 0$ then $X^a = 0$.

Proposition 3.3. *Let (E, L) be a complex Finsler bundle over a hermitian manifold (M, g) and $\varphi_a(z, \eta)$ the components of a covariant vertical complex tensor field with compact support on E. If we denote by $S^{a\bar{b}} = h^{\bar{c}d}h^{\bar{e}a}S^{\bar{b}}_{\bar{e},\bar{c}d}$, then*

(i) If φ_a are holomorphic functions of (η^a) coordinates and $\mathrm{Re}S^{a\bar{b}}\varphi_a\varphi_{\bar{b}} \geq 0$ then $\nabla_{\dot{\partial}_b}\varphi_a = 0$;

(ii) If φ_a are holomorphic functions of (η^a) coordinates and $\mathrm{Re}S^{a\bar{b}}\varphi_a\varphi_{\bar{b}} > 0$ then $\varphi_a = 0$.

Remark 3.2. If the complex Finsler structure comes from a hermitian structure on E, that is $L(z, \eta) = h_{a\bar{b}}(z)\eta^a\overline{\eta}^b$, then $C^a_{bc} = 0$ and so $\mathcal{G}^v(X) = 0$. In this case, if $X^{A_p}_{B_q}(z, \eta)$ are the components of a mixed vertical complex tensor field with compact support on E and if $X^{A_p}_{B_q}$ are holomorphic functions of (η^a) coordinates, then $\nabla_{\dot{\partial}_b} X^{A_p}_{B_q} = \dot{\partial}_b\, X^{A_p}_{B_q} = 0$, so $X^{A_p}_{B_q} = X^{A_p}_{B_q}(z)$. But on the other hand, if this happens, then obviously $X^{A_p}_{B_q}$ doesn't have compact in E. This contradicts the assumption that $X^{A_p}_{B_q}$ are compactly supported in E, so $X^{A_p}_{B_q} = 0$.

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REFERENCES

1. Abate, M.; Patrizio, G. – *Finsler Metrics-A global Approach. With Applications to Geometric Function Theory,* Lecture Notes in Mathematics, 1591, Springer-Verlag, Berlin, 1994.
2. Aikou, T. – *Finsler geometry on complex vector bundles. A sampler of Riemann-Finsler geometry,* 83-105, Math. Sci. Res. Inst. Publ., 50, Cambridge Univ. Press, Cambridge, 2004.
3. Aikou, T. – *The Chern-Finsler connection and Finsler-Kähler manifolds*, Finsler geometry, Sapporo 2005-in memory of Makoto Matsumoto, 343-373, Adv. Stud. Pure Math., 48, Math. Soc. Japan, Tokyo, 2007.
4. Bochner, S. – *Curvature in Hermitian metric,* Bull. Amer. Math. Soc., 53 (1947), 179–195.
5. Bochner, S. – *Vector fields and Ricci curvature,* Bull. Amer. Math. Soc., 52 (1946), 776–797.
6. Bochner, S. – *Tensor fields and Ricci curvature in Hermitian metric,* Proc. Nat. Acad. Sci. U.S.A., 37 (1951), 704–706.
7. Cao, J.-K.; Wong, P.-M. – *Finsler geometry of projectivized vector bundles,* J. Math. Kyoto Univ., 43 (2003), 369–410.
8. Griffiths, P.; Harris, J. – *Principles of Algebraic Geometry,* Pure and Applied Mathematics, Wiley-Interscience [John Wiley & Sons], New York, 1978.
9. Ida, C. – *V and H cohomology of complex Finsler bundles,* An. Ştiinţ. Univ. "Al. I. Cuza" Iaşi, Mat. (N.S.), 53 (2007), suppl. 1, 213–222.
10. Ida, C. – *Weitzenböck type formulas and a vanishing theorem on complex Finsler bundles*, Tensor (N.S.), 71 (2009), 61–68

11. Ida, C. – *Vertical Laplacian on complex Finsler bundles*, Acta Math. Acad. Paedagog. Nyházi. (N.S.), 26 (2010), 313–327.
12. Kobayashi, S. – *Differential Geometry of Complex Vector Bundles,* Publications of the Mathematical Society of Japan, 15, Kanô Memorial Lectures, 5, Princeton University Press, Princeton, NJ, Iwanami Shoten, Tokyo, 1987.
13. Kobayashi, S. – *Negative vector bundles and complex Finsler structures,* Nagoya Math. J., 57 (1975), 153–166.
14. Kobayashi, S. – *Complex Finsler vector bundles,* Finsler geometry (Seattle, WA, 1995), 145-153, Contemp. Math., 196, Amer. Math. Soc., Providence, RI, 1996.
15. Miron, R.; Anastasiei, M. – *The Geometry of Lagrange Spaces: Theory and Applications,* Fundamental Theories of Physics, 59, Kluwer Academic Publishers Group, Dordrecht, 1994.
16. Morrow, J.; Kodaira, K. – *Complex Manifolds,* Holt, Rinehart and Winston, Inc., New York-Montreal, Que.-London, 1971.
17. Munteanu, G. – *Complex Spaces in Finsler, Lagrange and Hamilton Geometries,* Fundamental Theories of Physics, 141, Kluwer Academic Publishers, Dordrecht, 2004.
18. Pitiş, G.; Munteanu, G. – *V-cohomology of complex Finsler manifolds,* Studia Univ. Babeş-Bolyai Math., 43 (1998), 75–82.
19. Vaisman, I. – *Cohomology and Differential Forms,* Pure and Applied Mathematics, 21, Marcel Dekker, Inc., New York, 1973.
20. Wu, H. – *The Bochner technique in differential geometry,* Math. Rep., 3 (1988), 289–538.
21. Xiao, J.; Zhong, T.; Qiu, C. – *Bochner technique on strong Kähler-Finsler manifolds*, Acta Math. Sci. Ser. B Engl. Ed., 30 (2010), 89–106.
22. Yano, K.; Bochner, S. – *Curvature and Betti Numbers*, Annals of Mathematics Studies, Princeton University Press, Princeton, N.J., 1953.
23. Zhong, C. – *Some results on complex Finsler manifolds*, July, 2007, Preprint.
24. Zhong, C. – *A vanishing theorem on Kaehler Finsler manifolds*, Differential Geom. Appl., 27 (2009), 551–565.
25. Zhong, C. – *Laplacians on the holomorphic tangent bundle of a Kaehler manifold*, Sci. China Ser. A, 52 (2009), 2841–2854.

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