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GEOMETRICAL OBJECTS AND STRUCTURES ASSOCIATED TO SOME DYNAMICAL SYSTEMS (II)

BY

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Abstract. In the paper we continue and develop our studies on dynamical systems. We consider dynamical systems of first order partial differential equations, which describe the evolution of some deformable continuous media. To such a system, using the coefficients of the corresponding variational forms, we can associate, canonically, a geometrical structure of metrical nature and a connection, a covariant derivative, a parallel transport and autoparallel manifolds such that the trajectories of the given system are autoparallels with respect to this structure.

A second geometrical structure corresponds to the adjoint linear system of variational forms.

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1. d-metrics and connections

Let M be a smooth manifold of dimension m and of local coordinates (u^a). Let $J^1(\mathbf{R}^n, M)$ be the first order jet space and $J^1_n(M)$ the phases space. A point in $J^1(\mathbf{R}^n, M)$ is given in a local chart of M and the canonical chart of $\mathbf{R}^n$, by its coordinates in this chart: (x^i, u^a, u^a_i). On $J^1_n(M)$ we consider a natural atlas, induced by the atlas of M, where a change of local charts is given by

$$\bar{x}^i = x^i, \quad \bar{u}^a = \bar{u}^a(u^\alpha), \quad \bar{u}^a_i = \bar{u}^a_i(u^\alpha, u^\alpha_j) = \frac{\partial \bar{u}^a}{\partial u^\alpha} u^\alpha_i.$$

A geometrical structure defined on $J^1(\mathbf{R}^n, M)$ is given by a set of n

non-degenerated two times covariant tensors of components

$$(1.1)\quad a^i_{ab} = a^i_{ab}(x^h, u^c, u^c_h),\ (det(a^i_{ab}) \neq 0,\ i = \overline{1,n}),\ \bar{a}^i_{ab} = \frac{\partial u^\alpha}{\partial \bar{u}^a}\frac{\partial u^\beta}{\partial \bar{u}^b} a^i_{\alpha\beta},$$

called "*d-metrics*", and a set of m^2 local functions (a_{ab}), which change, on a change of local chart, by the rules

$$(1.2)\qquad \bar{a}_{ab} = \frac{\partial u^\alpha}{\partial \bar{u}^a}\left(\frac{\partial u^\beta}{\partial \bar{u}^b} a_{\alpha\beta} + \frac{\partial u^\beta_i}{\partial \bar{u}^b} a^i_{\alpha\beta}\right),$$

called *Christoffel symbols of first kind* of a connection N ([4]).

We construct the geometric objects (n^2 mixed tensors) of components

$$(1.3)\qquad \varepsilon^{aj}_{ib} = \frac{1}{n} a^{ac}_i a^j_{cb},\quad \bar{\varepsilon}^{aj}_{ib} = \frac{\partial \bar{u}^a}{\partial u^\alpha}\frac{\partial u^\beta}{\partial \bar{u}^b}\varepsilon^{\alpha j}_{i\beta},\ (\forall i, j = \overline{1,n}),$$

where (a^{ab}_i) is the inverse matrix of (a^i_{ab}), for any $i = \overline{1,n}$.

We define the functions

$$(1.4)\qquad \underset{i}{M}{}^a_b = \frac{1}{n} a^{ac}_i a_{cb},\quad \overline{\underset{i}{M}}{}^a_b = \frac{\partial \bar{u}^a}{\partial u^\alpha}\left(\frac{\partial u^\beta}{\partial \bar{u}^b}\underset{i}{M}{}^\alpha_\beta + \frac{\partial u^\beta_j}{\partial \bar{u}^b}\varepsilon^{\alpha j}_{i\beta}\right),$$

called *Christoffel symbols of second kind* or *coefficients of connection.*

Remarks. $\mathbf{1}^o$ Given a connection N by its coefficients $\underset{i}{M}{}^a_b$, if we add a set of mixed second order d-tensors, we also obtain a connection. Both connections are constructed from the same set of metrics.

Conversely, by substracting two connections based on the same set of metrics, we obtain a set of mixed second order tensors.

Given a system of metrics, there is a bijection between the set of associated connections and the sets of n mixed second order tensor fields.

$\mathbf{2}^o$ Let us consider an arbitrary number p of connections $\overset{1}{\underset{i}{M}}, \overset{2}{\underset{i}{M}}, \ldots, \overset{p}{\underset{i}{M}}$. Then $\underset{i}{M}{}^a_b = \frac{1}{p}\sum_{q=1}^p \overset{q}{\underset{i}{M}}{}^a_b$ is a connection, too.

$\mathbf{3}^o$ In the case that all the d-metrics a^i_{ab} $(i = \overline{1,n})$ coincide (and, in particular, if $n = 1$), the formulas (1.4) become the rules of change of the components of a nonlinear connection ([1], [2]). We have $\varepsilon^{aj}_{ib} = \delta^a_b$, $\varepsilon^{a1}_{1b} = \delta^a_b$, respectively.

2. Partial covariant differential operators associated to a d-connection N

In this paragraph we generalize to deformable continuous media, some results obtained in [3] and [5] for the case of ordinary differential systems.

Let us consider a d-connection N on the manifold M. We define a system of n partial differential operators D_i, by the conditions:

1) $D_i c = 0$, if $c \in \mathbf{R}^n$,

2) $D_i : X \in \mathcal{X}_d(M) \to \mathcal{X}_d(M)$ is a linear application,

3) $D_i : \omega \in \Lambda^1_d(M) \to \Lambda^1_d(M)$ is a linear application,

4) The action of the operators D_i are extended to the algebra of d-tensor fields by the rule $D_i(U \otimes V) = D_i(U) \otimes V + U \otimes D_i(V)$, where U and V are d-tensor fields.

2.1. The construction of the partial differential operators

1°. Let $X = X^a \frac{\partial}{\partial u^a}$ be a d-vector field on M. We define, locally,

$$D_i\left(\frac{\partial}{\partial u^a}\right) = \underset{i}{M}{}^{b}_{a} \frac{\partial}{\partial u^b},$$

where the functions $\underset{i}{M}{}^{a}_{b}$ change, on a change of local chart, by the rules (1.4). Thus, it follows $D_i X = (D_i X^a + \underset{i}{M}{}^{a}_{b} X^b)\frac{\partial}{\partial u^a}$. The functions $\frac{\Delta X^a}{\Delta x^i} = D_i X^a + \underset{i}{M}{}^{a}_{b} X^b$ are called *partial covariant derivatives* of the components X^a, in the chosen chart. On a change of local chart, it is necessary to have

$$\frac{\Delta \overline{X}^a}{\Delta x^i} = \frac{\partial \bar{u}^a}{\partial u^\alpha}\frac{\Delta X^\alpha}{\Delta x^i}. \tag{2.1}$$

Proposition 1. *The necessary and sufficient condition such that the formulas* (2.1) *to hold is that the operators* D_i *to satisfy the relations*

$$D_i\left(\frac{\partial \bar{u}^a}{\partial u^\gamma}\right) + \frac{\partial \bar{u}^a}{\partial u^\alpha}\frac{\partial \bar{u}^b}{\partial u^\gamma}\frac{\partial u^\beta_j}{\partial \bar{u}^b}\varepsilon^{\alpha j}_{i\beta} = 0, \quad (i = \overline{1,n}). \tag{2.2}$$

Proposition 2. *The relations* (2.2) *are equivalent with*

$$D_i\left(\frac{\partial u^\alpha}{\partial \bar{u}^c}\right) = \frac{\partial u^\beta_j}{\partial \bar{u}^c}\varepsilon^{\alpha j}_{i\beta}, \quad (i = \overline{1,n}). \tag{2.2'}$$

Conversely, given a system of n operators D_i satisfying the formulas (2.2) or (2.2′), they determine a connection on M. We have:

Proposition 3. *The necessary and sufficient condition such that a system of local functions* $\underset{i}{M}{}^a_b$ *to represent a connection on the manifold* M (*to satisfy, on a change of local chart, the formulas* (1.4)) *is that the partial differential operators to satisfy the conditions* (2.2) *or* (2.2′).

Definition 1. A vector field is transported by parallelism (on a submanifold $u^a = u^a(x^h)$), if all partial covariant derivatives of it vanish. The parallelism equations are $D_i X^a + \underset{i}{M}{}^a_b X^b = 0$.

A family of submanifolds, given by the equations $u^a = u^a(x^h)$, is called autoparallel if $D_i(\frac{\partial u^a}{\partial x^h}) + \underset{i}{M}{}^a_b \frac{\partial u^b}{\partial x^h} = 0$, $\forall h = \overline{1,n}$.

2°. We extend the action of the differential operators D_i on distinguished 1-forms, by

$$D_i(du^a) = \underset{i}{\overset{*}{M}}{}^a_b du^b.$$

By the compatibility condition it follows $\underset{i}{\overset{*}{M}}{}^a_b + \underset{i}{M}{}^a_b = 0$.

If ω is locally written $\omega = Y_a du^a$, we have $D_i(\omega) = (D_i Y_a - Y_b \underset{i}{M}{}^b_a) du^a$.

We denote $\frac{\Delta Y_a}{\Delta x^i} = D_i Y_a + Y_b \underset{i}{\overset{*}{M}}{}^b_a$, or $\frac{\Delta Y_a}{\Delta x^i} = D_i Y_a - Y_b \underset{i}{M}{}^b_a$. These functions are called *partial covariant derivatives* of the components Y_a of the form ω, in the given chart. On a change of local chart, due to the relations (2.2′), we have $\frac{\Delta \overline{Y}_a}{\Delta x^i} = \frac{\partial u^\alpha}{\partial \bar{u}^a} \frac{\Delta Y_\alpha}{\Delta x^i}$.

3°. Let us now consider a $p+q$ order d-tensor T, p times contravariant and q times covariant, locally written $T = T^{a_1 \dots a_p}_{b_1 \dots b_q} \frac{\partial}{\partial u^{a_1}} \otimes \dots \otimes \frac{\partial}{\partial u^{a_p}} \otimes du^{b_1} \otimes \dots \otimes du^{b_q}$.

For any i, the action of the operator D_i on T lead us to the functions

$$\frac{\Delta T^{a_1 \dots a_p}_{b_1 \dots b_q}}{\Delta x^i} = D_i T^{a_1 \dots a_p}_{b_1 \dots b_q} + \sum_{h=1}^{p} T^{a_1 \dots a_{c_h} \dots a_p}_{b_1 \dots \dots \dots b_q} \underset{i}{M}{}^{a_h}_{c_h} + \sum_{k=1}^{q} T^{a_1 \dots \dots \dots a_p}_{b_1 \dots b_{c_k} \dots b_q} \underset{i}{\overset{*}{M}}{}^{c_k}_{b_k},$$

called *partial covariant derivatives* of the components of the tensor T. They change by the rules

$$\frac{\Delta \overline{T}^{a_1 \dots a_p}_{b_1 \dots b_q}}{\Delta x^i} = \frac{\partial \bar{u}^{a_1}}{\partial u^{\alpha_1}} \cdots \frac{\partial \bar{u}^{a_p}}{\partial u^{\alpha_p}} \frac{\partial u^{\beta_1}}{\partial \bar{u}^{b_1}} \cdots \frac{\partial u^{\beta_q}}{\partial \bar{u}^{b_q}} \frac{\Delta T^{\alpha_1 \dots \alpha_p}_{\beta_1 \dots \beta_q}}{\Delta x^i}, \quad (\forall i = \overline{1,n}).$$

2.2. The calculus of the partial covariant differential operators

1°. The values of the operators D_i on the components X^a of a vector field X are given by $D_i X^a = \varepsilon^{aj}_{ib} \frac{dX^b}{dx^j}$. Indeed, it holds:

Proposition 4. *The relations* $(2.2')$ *define the operators* D_i *on fields.*

Proof. For an index c ($c = \overline{1,m}$) fixed, $\bar{u}^c$ can be considered as parameter on a curve $u^\alpha = u^\alpha(\bar{u}^c)$ and $\frac{\partial u^\alpha}{\partial \bar{u}^c}$ as the tangent vector field to this curve. Conversely, any vector field is a tangent field to a congruence of integral curves in another chart.

By this property and $(2.2')$ it follows the above relation. □

Thus, we have $D_i(X) = [\varepsilon^{\alpha j}_{i\beta} \frac{dX^\beta}{dx^j} + \underset{i}{M}{}^{\alpha}_{\beta} X^\beta] \frac{\partial}{\partial u^\alpha} = \frac{\Delta X^\alpha}{\Delta x^i} \frac{\partial}{\partial u^\alpha}$.

2°. We extend the operators D_i on d-1-forms $\omega = Y_a du^a$. By the above calculus, we have $D_i(Y_a) = \varepsilon^{bj}_{ia} \frac{dY_b}{dx^j}$. Thus, the partial covariant derivatives of the components Y_a of a d-1-form are $\frac{\Delta Y_a}{\Delta x^i} = \varepsilon^{bj}_{ia} \frac{dY_b}{dx^j} - Y_b \underset{i}{M}{}^{b}_{a}$.

3°. The extension of the operators D_i on functions is defined implicitly by the relations $D_i(f)\delta^a_b = \varepsilon^{aj}_{ib} \frac{df}{dx^j}$.

2.3. Simplifications of the writing

In what follows, we replace the coefficients $\underset{i}{M}{}^{a}_{b}$ with other coefficients $\underset{i}{N}{}^{a}_{b}$, given by the formulas $\underset{i}{M}{}^{a}_{b} = \varepsilon^{aj}_{ic} \underset{j}{N}{}^{c}_{b}$, and we express the partial covariant derivatives with their help.

1°. The partial covariant derivatives of the components X^a of a vector field X become $\frac{\Delta X^a}{\Delta x^i} = \varepsilon^{aj}_{ib} \frac{dX^b}{dx^j} + \underset{i}{M}{}^{a}_{b} X^b = \varepsilon^{aj}_{ic} (\frac{dX^c}{dx^j} + \underset{j}{N}{}^{c}_{b} X^b) := \varepsilon^{aj}_{ic} \frac{\delta X^c}{\delta x^j}$. Let us consider the functions $\frac{\delta X^c}{\delta x^j}$. We call them *short partial covariant derivatives* of the components of the vector field X. On a change of local chart, they change by the rules $\frac{\delta \overline{X}^a}{\delta x^i} = \frac{\partial \bar{u}^a}{\partial u^\alpha} \frac{\delta X^\alpha}{\delta x^i}$.

Proposition 5. *A vector field X is transported by parallelism if its components satisfy the relations* $\frac{dX^c}{dx^j} + \underset{j}{N}{}^{c}_{b} X^b = 0$, $(\forall i = \overline{1,n})$. *A submanifold* $u^a = u^a(x^h)$ *is autoparallel if* $\frac{\partial^2 u^a}{\partial x^i \partial x^j} + \underset{i}{N}{}^{a}_{b} \frac{\partial u^b}{\partial x^j} = 0$, $(\forall i, j = \overline{1,n})$.

2°. The partial covariant derivatives of the components of a d-1-form ω are written as $\frac{\Delta Y_a}{\Delta x^i} = \varepsilon^{bj}_{ia} (\frac{dY_b}{dx^j} - \underset{i}{N}{}^{c}_{b} Y_c) = \varepsilon^{bj}_{ia} \frac{\delta Y_b}{\delta x^j}$. We call $\frac{\delta Y_b}{\delta x^j}$ *short partial* *covariant derivatives* of the components of the form ω. On a change of local chart, they change by the rules $\frac{\delta \overline{Y}_a}{\delta x^i} = \frac{\partial u^\alpha}{\partial \bar{u}^a} \frac{\delta Y_\alpha}{\delta x^i}$.

Proposition 6. *By the rules of change of the coefficients* $\underset{i}{M}{}^{b}_{a}$, *the functions* $\underset{i}{N}{}^{b}_{a}$ *change by the rules*

$$\underset{i}{\overline{N}}{}^{a}_{b} = \frac{\partial \bar{u}^a}{\partial u^\alpha} \left(\frac{\partial u^\beta}{\partial \bar{u}^b} \underset{i}{N}{}^{\alpha}_{\beta} + \frac{\partial u^\beta_i}{\partial \bar{u}^b} \right)$$

and conversely.

3°. We can consider short partial covariant derivatives of the components of a tensor T of order $p+q$, p times contravariant and q times covariant, locally written $T = T^{a_1 \ldots a_p}_{b_1 \ldots b_q} \frac{\partial}{\partial u^{a_1}} \otimes \cdots \otimes \frac{\partial}{\partial u^{a_p}} \otimes du^{b_1} \otimes \cdots \otimes du^{b_q}$, by the formulas

$$\frac{\delta T^{a_1 \ldots a_p}_{b_1 \ldots b_q}}{\delta x^i} = \frac{d T^{a_1 \ldots a_p}_{b_1 \ldots b_q}}{d x^i} + \sum_{h=1}^{p} T^{a_1 \ldots a_{c_h} \ldots a_p}_{b_1 \ldots \ldots \ldots b_q} \underset{i}{N}{}^{a_h}_{c_h} - \sum_{k=1}^{q} T^{a_1 \ldots \ldots \ldots a_p}_{b_1 \ldots b_{c_k} \ldots b_q} \underset{i}{N}{}^{c_k}_{b_k}.$$

On a change of local chart, they change by the rules

$$\frac{\delta \overline{T}^{a_1 \ldots a_p}_{b_1 \ldots b_q}}{\delta x^i} = \frac{\partial \bar{u}^{a_1}}{\partial u^{\alpha_1}} \cdots \frac{\partial \bar{u}^{a_p}}{\partial u^{\alpha_p}} \frac{\partial u^{\beta_1}}{\partial \bar{u}^{b_1}} \cdots \frac{\partial u^{\beta_q}}{\partial \bar{u}^{b_q}} \frac{\delta T^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q}}{\delta x^i}, \quad (\forall i = \overline{1, n}).$$

3. Systems of partial differential equations

3.1. Variational forms

Let us consider a system of implicit first order partial differential equations

$$F_a(x^i, u^b, u^b_i) = 0, \quad a = \overline{1, m}, \tag{3.1}$$

where F_a are the components of a covector. The system is non-degenerated if $\det(\frac{\partial F_a}{\partial u^b_i}) \neq 0$, $\forall i = \overline{1, n}$.

A system of functions $u^a = u^a(x^i)$ is called a *solution* of the system (3.1) if the identity $F_a(x^i, u^b(x^h), \frac{\partial u^b(x^h)}{\partial x^i}) \equiv 0$, $\forall (x^i) \in \mathbf{R}^n$, holds.

The equations (3.1) define on $J^1(\mathbf{R}^n, M)$ a submanifold K, the manifold of the solutions of the system.

A vector field $X = \xi^a(u^b)\frac{\partial}{\partial u^a}$ on M, with its extension $\hat{X} = \xi^a\frac{\partial}{\partial u^a} + \frac{d\xi^a}{dx^i}\frac{\partial}{\partial u^a_i}$, generates on $J^1(\mathbf{R}^n, M)$ a pseudogroup of transformations with one parameter $\bar{x}^i = x^i$, $\bar{u}^a = u^a + \varepsilon\xi^a$, $\bar{u}^a_i = u^a_i + \varepsilon\frac{d\xi^a}{dx^i}$, and, thus, a system of variational forms

$$M_a^X = a^i_{ab}\frac{d\xi^b}{dx^i} + a_{ab}\xi^b, \tag{3.2}$$

where $a^i_{ab} = \frac{\partial F_a}{\partial u^b_i}$, $a_{ab} = \frac{\partial F_a}{\partial u^b}$, $\det(a^i_{ab}) \neq 0$, $\forall i = \overline{1, n}$.

A first geometric object associated to the given system is a set of n "metrics" given by the functions (a^i_{ab}). Indeed, these functions change by rules of type (1.1).

A second geometric object is given by the functions (a_{ab}). On the manifold K of the solutions of the system $(F_a = 0)$, they change by the rules (1.2). They represent the Christoffel symbols of first kind of a connection.

We can construct the coefficients $\varepsilon^{aj}_{ib} = \frac{1}{n}a^{ac}_i a^j_{cb}$ and $\underset{i}{M}{}^a_b = \frac{1}{n}a^{ac}_i a_{cb}$ as in (1.3) and (1.4) respectively, where (a^{ab}_i) is the inverse matrix of (a^i_{ab}), $\forall i = \overline{1, n}$.

Remark. On the solutions manifold K the coefficients $\underset{i}{M}{}^a_b$ change by the rules (1.4), thus, they define a connection on K, a reason to call them Christoffel symbols of second kind.

Let us consider a scleronomic dynamical system given by the equations $F_a(u^b, u^b_i) = 0$. The following property holds ([4]):

Theorem 1. *The solutions of a scleronomic dynamical system are autoparallel submanifolds with respect to the associated geometrical structure and conversely, for initial conditions situated on the solutions manifold, any autoparallel submanifold is a solution* (*it is entirely included in this manifold*).

3.2. Adjoint forms

The system of variational forms (3.2) is associated to the system (3.1) and to the vector field X, but its coefficients depend only on the system (3.1). Let $Y = \eta^a\frac{\partial}{\partial u^a}$ be another vector field and we denote by $\widetilde{M}_a^Y = \widetilde{a}^i_{ab}\frac{d\eta^b}{dx^i} + \widetilde{a}_{ab}\eta^b$ a system of forms.

Definition 2. A system $\widetilde{M}_a^Y$ is called adjoint to the system M_a^X if there exists a set of functions J^i such that, for any vector fields X and Y, the relation

$$M^X(Y) - \widetilde{M}(X) = \frac{dJ^i}{dx^i}$$

holds ([6]).

Proposition 7. *Given a system of variational forms, there exists an adjoint system. The adjointness relations are*

$$\widetilde{a}^i_{ab} + a^i_{ba} = 0,\ \widetilde{a}_{ab} - a_{ba} = -\frac{da^i_{ba}}{dx^i}. \tag{3.3}$$

Remark. The relation of adjointness is involutive.

The coefficients of the adjoint form change, respectively, by the same rules as the coefficients of the variational form.

We obtain a set of n non-degenerated distinguished tensors of components $\widetilde{a}^i_{ab}$, $(i = \overline{1,n})$. The matrices (a^i_{ab}), $(\widetilde{a}^i_{ab})$ are, respectively, antitransposed one to the other, they are non-degenerated and not symmetric.

For any i, let $(\widetilde{a}^{ab}_i)$ be the inverse matrix of $(\widetilde{a}^i_{ab})$. We define the functions

$$\widetilde{\varepsilon}^{aj}_{ib} = \frac{1}{n}\widetilde{a}^{ac}_i\widetilde{a}^j_{cb}, \quad \underset{i}{\widetilde{M}}{}^a_b = \frac{1}{n}\widetilde{a}^{ac}_i\widetilde{a}_{cb}.$$

The coefficients $\underset{i}{\widetilde{M}}{}^a_b$ define a new connection on K.

A second geometrical structure corresponds to the adjoint linear system of variational forms.

Proposition 8. *The relations between the components* ε^{aj}_{ib} *and* $\widetilde{\varepsilon}^{aj}_{ib}$ *are given by* $(\widetilde{\varepsilon}^{aj}_{ib}) = (\varepsilon^{aj}_{ib})^t$, $\forall i,j = \overline{1,n}$.

Indeed, we have $n(\widetilde{\varepsilon}^{aj}_{ib}) = (\widetilde{a}^{ac}_i\widetilde{a}^j_{cb}) = (a^{ca}_i a^j_{bc}) = (a^{ac}_i a^j_{cb})^t = n(\varepsilon^{aj}_{ib})^t$.

Proposition 9. *The coefficients* $\underset{i}{M}{}^a_b$ *and* $\underset{i}{\widetilde{M}}{}^a_b$ *are related by the formulas* $(\underset{i}{\widetilde{M}}{}^a_b) = (\underset{i}{\widetilde{M}}{}^a_b)^t + (a^{ca}_i \frac{da^j_{bc}}{dx^j})$.

The connection $\widetilde{N}$ is not in the class of connections $N + T$ (it is not built by the same set of metrics as N).

We consider now both structures at the same time. The connections N and $\widetilde{N}$ are defined by the coefficients $\underset{i}{M}{}^a_b$ and $\underset{i}{\widetilde{M}}{}^a_b$ respectively. The coefficients $\underset{i}{N}{}^{a}_{b}$ and $\underset{i}{\widetilde{N}}{}^{a}_{b}$ are defined by the relations: $\underset{i}{M}{}^{a}_{b} = \varepsilon^{aj}_{ic}\underset{j}{N}{}^{c}_{b}$ and $\underset{i}{\widetilde{M}}{}^{a}_{b} = \widetilde{\varepsilon}^{aj}_{ic}\underset{j}{\widetilde{N}}{}^{c}_{b}$, which can be written as $a_{ab} = a^{j}_{ac}\underset{j}{N}{}^{c}_{b}$, $\widetilde{a}_{ab} = \widetilde{a}^{j}_{ac}\underset{j}{\widetilde{N}}{}^{c}_{b}$.

Let k be a submanifold on M defined by the equations $u^a = u^a(x^i)$. Let $X = X^a\frac{\partial}{\partial u^a}$ and $Y = Y^a\frac{\partial}{\partial u^a}$ be two vector fields, which are transported by parallelism, the first with respect to the connection N, the second field with respect to the connection $\widetilde{N}$.

We consider the metrics a^i_{ab} and the "invariant" $a^i_{ab}X^bY^a$ with the semnification of scalar product or "angle" between the vector fields X and Y. We calculate $\frac{d}{dx^i}(a^i_{ab}X^bY^a)$. By the above conditions and the adjointness relations (3.3), we have

$$\left(\frac{da^i_{ab}}{dx^i} - a^i_{ac}\underset{i}{N}{}^{c}_{b} - a^i_{cb}\underset{i}{\widetilde{N}}{}^{c}_{a}\right)X^bY^a = \left(\frac{da^i_{ab}}{dx^i} - a^i_{ac}\underset{i}{N}{}^{c}_{b} + \widetilde{a}^i_{bc}\underset{i}{\widetilde{N}}{}^{c}_{a}\right)X^bY^a$$
$$= \left(\frac{da^i_{ab}}{dx^i} - a_{ab} + \widetilde{a}_{ba}\right)X^bY^a = 0.$$

That leads us to the following result:

Theorem 2. *If two vector fields X and Y are transported by parallelism on a submanifold k, the first field with respect to the connection N given by the initial system, the second field with respect to the connection $\widetilde{N}$ given by the adjoint system of forms, then the "angle" made by X and Y* (*in this order*) *is constant with respect to any of the metrics* (a^i_{ab}) *and* $(\widetilde{a}^i_{ab})$.

Definition 3. Two systems of partial differential equations are equivalent if they admit the same solutions. The properties which are common to all the equivalent systems are called dynamical properties.

Theorem 3. *The notions of metrics and of Christoffel symbols of first kind, expressed by the functions a^i_{ab}, $\widetilde{a}^i_{ab}$ and a_{ab}, $\widetilde{a}_{ab}$ respectively, have geometrical character, they change when we change the representant of the dynamical system. The notions of: Christofels symbols of second kind, defined and expressed by the coefficients $\underset{i}{M}{}^{a}_{b}$ or $\underset{i}{N}{}^{a}_{b}$, $\underset{i}{\widetilde{M}}{}^{a}_{b}$ or $\underset{i}{\widetilde{N}}{}^{a}_{b}$, differential operators, covariant derivative operators, parallel transport and autoparallel manifolds with respect to any of the two connections, have dynamical character.*

3.3. Selfadjoint systems

Definition 4. A system of variational forms (and also the system of partial differential equations to which these forms are associated) is called *selfadjoint* if it coincides with its adjoint system, that means if the relations:

$$a^i_{ab} + a^i_{ba} = 0,\ a_{ab} - a_{ba} = -\frac{da^i_{ba}}{dx^i} \tag{3.4}$$

hold.

The first set of relations (3.4) and $det(a^i_{ab}) \neq 0$ lead us to the restrictive condition $m = 2p$.

By the second set of relations (3.4) follows that a selfadjoint first order system has to be written in main form:

$$F_a = B^i_{ab} u^b_i + E_a = 0 \tag{3.5}$$

with $B^i_{ab}, E_a \in \mathcal{C}^\infty(J^0(\mathbf{R}^n, M))$, $det(B^i_{ab}) \neq 0$, $\forall i = \overline{1,n}$. Indeed, the equations $a_{ab} - a_{ba} = -\frac{da^i_{ba}}{dx^i}$ are linear with respect to u^c_{ij}, thus, there coefficients $\frac{\partial^2 F_a}{\partial u^b_i \partial u^c_j}$ have to vanish.

The self-adjointness conditions (3.4) become:

$$B^i_{ab} + B^i_{ba} = 0,\ \sum_{(a,b,c)} \frac{\partial B^i_{ab}}{\partial u^c} = 0,\ \frac{\partial E_b}{\partial u^a} - \frac{\partial E_a}{\partial u^b} = 0. \tag{3.6}$$

On a change of local chart, the coefficients of the equations (3.5) change by the rules:

$$\overline{B}^i_{ab} = \frac{\partial u^\alpha}{\partial \bar{u}^a}\frac{\partial u^\beta}{\partial \bar{u}^b} B^i_{\alpha\beta}, \quad \overline{E}_a = \frac{\partial u^\alpha}{\partial \bar{u}^a} E_\alpha.$$

For any $i = \overline{1,n}$, the coefficients $a^i_{ab} = B^i_{ab}$ are the components of a skew-symmetric non-degenerated distinguished tensor on M.

The coefficients of the connection are:

$$\underset{i}{M}{}^a_b = \frac{1}{n} B^{ac}_i \left(\frac{\partial B^j_{cd}}{\partial u^b} u^d_j + \frac{\partial E_c}{\partial u^b} \right).$$

In the case of selfadjoint systems the connections N and $\widetilde{N}$ coincide.

REFERENCES

1. Miron, R. – *Dynamical systems of Lagrangian and Hamiltonian mechanical systems,* Finsler geometry, Sapporo 2005–in memory of Makoto Matsumoto, 309–340, Adv. Stud. Pure Math., 48, Math. Soc. Japan, Tokyo, 2007.
2. Miron, R.; Anastasiei, M. – *The Geometry of Lagrange Spaces: Theory and Applications,* Fundamental Theories of Physics, 59, Kluwer Academic Publishers Group, Dordrecht, 1994.
3. Obădeanu, V. – *Sisteme Dinamice Diferenţiale* (II)*,* Tip. Univ. Timişoara, 2006.
4. Obădeanu, V.; Ciobanu, M. – *Geometrical objects and structures associated to some dynamical systems,* An. Ştiinţ. Univ. "Al. I. Cuza" Iaşi, Mat. (N.S.), 53 (2007), suppl. 1, 287–296.
5. Obădeanu, V.; Ciobanu, M. – *Geometrical structures associated to superfields,* Tensor (N.S.), 71 (2009), 97–108.
6. Santilli, R.M. – *Foundations of Theoretical Mechanics. I. The Inverse Problem in Newtonian Mechanics,* Texts and Monographs in Physics, Springer-Verlag New York, New York-Heidelberg, 1978.

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