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METRIC NON-LINEAR CONNECTIONS ON THE PROLONGATION OF A LIE ALGEBROID TO ITS DUAL BUNDLE

BY

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Abstract. In the present paper the problem of compatibility between a nonlinear connection and other geometric structures on Lie algebroids is studied. The notion of dynamical covariant derivative is introduced and a metric nonlinear connection is found. We prove that the nonlinear connection induced by a regular Hamiltonian on a Lie algebroid is the unique connection which is compatible with the metric and symplectic structures.

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1. Introduction

The notions of Lie algebroid (see MACKENZIE [9]) and its prolongations over the vector bundle projection (see HIGGINS AND MACKENZIE [6]) generalize the concepts of tangent and cotangent bundles. Using the geometry of Lie algebroids, WEINSTEIN [19] shows that is possible to give a common description of the most interesting classical mechanical systems. In the last years the problems raised by Weinstein and related topics have been investigated in many papers (see for instance [1, 5, 7, 8, 10, 13, 14, 15, 17]).

In this paper we study the problem of compatibility between a nonlinear connection and some other geometric structures on a Lie algebroid and its prolongation over the vector bundle projection of the dual bundle. The paper is organized as follows. The second section contains the preliminary results on Lie algebroids. In section three the compatibility between a nonlinear connection and a pseudo-Riemannian metric is studied. Using the notions of adapted tangent structure and $\mathcal{J}$-regular section [7] we can

introduce the notions of dynamical covariant derivative and metric nonlinear connection. We obtain the expression of the Jacobi endomorphism on Lie algebroids and the relation with the curvature of the nonlinear connection. We prove that the canonical nonlinear connection induced by a regular Hamiltonian is the unique connection which is metric and compatible with the symplectic structure. For the particular cases of tangent and cotangent bundles see [2, 3, 4, 11, 12, 16].

2. Preliminaries on Lie algebroids

Let M be a real, C^∞-differentiable, n-dimensional manifold and (TM, π_M, M) its tangent bundle. A *Lie algebroid* over a manifold M is a triple $(E, [\cdot,\cdot]_E, \sigma)$, where (E, π, M) is a vector bundle of rank m over M, satisfying the following conditions:

a) the $C^\infty(M)$-module of sections $\Gamma(E)$ has a Lie algebra structure $[\cdot,\cdot]_E$.

b) $\sigma : E \to TM$ is a bundle map (called the anchor) which induces a Lie algebra homomorphism (also denoted σ) from the Lie algebra of sections $(\Gamma(E), [\cdot,\cdot]_E)$ to the Lie algebra of vector fields $(\chi(M), [\cdot,\cdot])$ satisfying the Leibniz rule

$$[s_1, fs_2]_E = f[s_1, s_2]_E + (\sigma(s_1)f)s_2, \ \forall s_1, s_2 \in \Gamma(E), \ f \in C^\infty(M).$$

For $\omega \in \bigwedge^k(E^*)$ the *exterior derivative* $d^E\omega \in \bigwedge^{k+1}(E^*)$ is given by

$$\begin{aligned} d^E\omega(s_1, \ldots, s_{k+1}) = &\sum_{i=1}^{k+1}(-1)^{i+1}\sigma(s_i)\omega(s_1, \ldots, \hat{s}_i, \ldots, s_{k+1}) \\ &+ \sum_{1\le i<j\le k+1}(-1)^{i+j}\omega([s_i, s_j]_E, s_1, \ldots, \hat{s}_i, \ldots, \hat{s}_j, \ldots, s_{k+1}) \end{aligned}$$

where $s_i \in \Gamma(E)$, $i = \overline{1, k+1}$, and it follows that $(d^E)^2 = 0$. For $\xi \in \Gamma(E)$ the *Lie derivative* with respect to ξ is given by $\mathcal{L}_\xi = i_\xi \circ d^E + d^E \circ i_\xi$, where i_ξ is the contraction with ξ. If we take the local coordinates (q^i) on an open subset $U \subset M$, a local basis $\{s_\alpha\}$ of the sections of the bundle $\pi^{-1}(U) \to U$ generates local coordinates (q^i, y^α) on E. The local functions $\sigma^i_\alpha(q)$, $L^\gamma_{\alpha\beta}(q)$ on M given by

$$\sigma(s_\alpha) = \sigma^i_\alpha \frac{\partial}{\partial q^i}, \quad [s_\alpha, s_\beta]_E = L^\gamma_{\alpha\beta}s_\gamma, \quad i = \overline{1, n}, \quad \alpha, \beta, \gamma = \overline{1, m},$$

are called the *structure functions* of the Lie algebroid.

2.1. The prolongation of Lie algebroid over the vector bundle projection of the dual bundle

Let $\tau : E^* \to M$ be the dual bundle of $\pi : E \to M$ and $(E, [\cdot,\cdot]_E, \sigma)$ a Lie algebroid structure over M. One can construct a Lie algebroid structure over E^*, by taking the prolongation over $\tau : E^* \to M$ (see [6], [8], [7]). The associated vector bundle is $(\mathcal{T}E^*, \tau_1, E^*)$ where $\mathcal{T}E^* = \bigcup_{u^* \in E^*} \mathcal{T}_{u^*}E^*$, and $\mathcal{T}_{u^*}E^* = \{(u_x, v_{u^*}) \in E_x \times T_{u^*}E^* | \sigma(u_x) = T_{u^*}\tau(v_{u^*}), \tau(u^*) = x \in M\}$, and the projection $\tau_1 : \mathcal{T}E^* \to E^*$, $\tau_1(u_x, v_{u^*}) = u^*$. The anchor is the projection $\sigma^1 : \mathcal{T}E^* \to TE^*$, $\sigma^1(u, v) = v$. Notice that if $\mathcal{T}\tau : \mathcal{T}E^* \to E$, $\mathcal{T}\tau(u, v) = u$ then $(V\mathcal{T}E^*, \tau_{1|V\mathcal{T}E^*}, E^*)$ with $V\mathcal{T}E^* = Ker\mathcal{T}\tau$ is a subbundle of $(\mathcal{T}E^*, \tau_1, E^*)$, called the *vertical subbundle*. If (q^i, μ_α) are local coordinates on E^* at u^* and $\{s_\alpha\}$ is a local basis of sections of $\pi : E \to M$ then a local basis of $\Gamma(\mathcal{T}E^*)$ is $\{\mathcal{Q}_\alpha, \mathcal{P}^\alpha\}$ where [8]

$$\mathcal{Q}_\alpha(u^*) = \left(s_\alpha(\tau(u^*)), \sigma_\alpha^i \frac{\partial}{\partial q^i}|_{u^*}\right), \quad \mathcal{P}^\alpha(u^*) = \left(0, \frac{\partial}{\partial \mu_\alpha}|_{u^*}\right).$$

The structure functions on $\mathcal{T}E^*$ are given by the following formulas

$$\begin{aligned}
&\sigma^1(\mathcal{Q}_\alpha) = \sigma_\alpha^i \frac{\partial}{\partial q^i}, \quad \sigma^1(\mathcal{P}^\alpha) = \frac{\partial}{\partial \mu_\alpha},\\
&[\mathcal{Q}_\alpha, \mathcal{Q}_\beta]_{\mathcal{T}E^*} = L^\gamma_{\alpha\beta}\mathcal{Q}_\gamma, \quad [\mathcal{Q}_\alpha, \mathcal{P}^\alpha]_{\mathcal{T}E^*=0}, \quad [\mathcal{P}^\alpha, \mathcal{P}^\beta]_{\mathcal{T}E^*=0},\\
&d^E\mathcal{Q}^\gamma = -\frac{1}{2}L^\gamma_{\alpha\beta}\mathcal{Q}^\alpha \wedge \mathcal{Q}^\beta, \quad d^E\mathcal{P}_\alpha = 0, \quad d^E q^i = \sigma^i_\alpha \mathcal{Q}^\alpha, \quad d^E\mu_\alpha = \mathcal{P}_\alpha,
\end{aligned}$$

where $\{\mathcal{Q}^\alpha, \mathcal{P}_\alpha\}$ is the dual basis of $\{\mathcal{Q}_\alpha, \mathcal{P}^\alpha\}$. In local coordinates the *Liouville section* is given by $\theta_E = \mu_\alpha \mathcal{Q}^\alpha$. The *canonical symplectic structure* ω_E is defined by $\omega_E = -d^E\theta_E$. ω_E is non degenerate, $d^E\omega_E = 0$ and we obtain $\omega_E = \mathcal{Q}^\alpha \wedge \mathcal{P}_\alpha + \frac{1}{2}\mu_\alpha L^\alpha_{\beta\gamma}\mathcal{Q}^\beta \wedge \mathcal{Q}^\gamma$. The *Liouville-Hamilton section* $\mathcal{C}$ has local expression $\mathcal{C} = \mu_\alpha \mathcal{P}^\alpha$. We remark that $V\mathcal{T}E^*$ is Lagrangian for ω_E, i.e. $\omega_E(\rho_1, \rho_2) = 0$, for every vertical sections $\rho_1, \rho_2 \in \Gamma(V\mathcal{T}E^*)$.

3. Metric nonlinear connections on Lie algebroids

A nonlinear connection on $\mathcal{T}E^*$ is an almost product structure $\mathcal{N}$ on $\tau_1 : \mathcal{T}E^* \to E^*$(i.e. a bundle morphism $\mathcal{N} : \mathcal{T}E^* \to \mathcal{T}E^*$, such that $\mathcal{N}^2 = id$) smooth on $\mathcal{T}E^* \backslash \{0\}$ such that $V\mathcal{T}E^* = \ker(id + \mathcal{N})$. If $\mathcal{N}$ is a connection on $\mathcal{T}E^*$ then $H\mathcal{T}E^* = \ker(id - \mathcal{N})$ is the horizontal distribution associated to $\mathcal{N}$ and $\mathcal{T}E^* = V\mathcal{T}E^* \oplus H\mathcal{T}E^*$. A connection $\mathcal{N}$ on $\mathcal{T}E^*$

induces two projectors $\mathrm{h}, \mathrm{v} : \mathcal{T}E^* \to \mathcal{T}E^*$ such that $\mathrm{h}(\rho) = \rho^{\mathrm{h}}$ and $\mathrm{v}(\rho) = \rho^{\mathrm{v}}$ for every $\rho \in \Gamma(\mathcal{T}E^*)$, given by $\mathrm{h} = \frac{1}{2}(id + \mathcal{N})$ and $\mathrm{v} = \frac{1}{2}(id - \mathcal{N})$. The local sections $\{\mathcal{P}^\alpha\}_{\alpha=\overline{1,m}}$ define a local frame of $V\mathcal{T}E^*$, and the sections $\delta^*_\alpha = (\mathcal{Q}_\alpha)^{\mathrm{h}} = \mathcal{Q}_\alpha + \mathcal{N}_{\alpha\beta}\mathcal{P}^\beta$, generate a local frame of $H\mathcal{T}E^*$. The frame $\{\delta^*_\alpha, \mathcal{P}^\alpha\}$ is a local basis of $\mathcal{T}E^*$ called *Berwald basis*. The dual basis is $\{\mathcal{Q}^\alpha, \delta\mathcal{P}_\alpha\}$ where $\delta\mathcal{P}_\alpha = \mathcal{P}_\alpha - \mathcal{N}_{\alpha\beta}\mathcal{Q}^\beta$. A connection $\mathcal{N}$ is called symmetric, or compatible with the symplectic structure, if $H\mathcal{T}E^*$ is Lagrangian for ω_E, i.e. $\omega_E(\mathrm{h}X, \mathrm{h}Y) = 0$, for $\forall X, Y \in \Gamma(\mathcal{T}E^*)$ and it follows that $\mathcal{N}$ is symmetric if and only if

$$\mathcal{N}_{\alpha\beta} - \mathcal{N}_{\beta\alpha} = \mu_\gamma L^\gamma_{\alpha\beta}. \tag{3.1}$$

The Lie brackets of the adapted basis $\{\delta^*_\alpha, \mathcal{P}^\alpha\}$ are [7]

$$[\delta^*_\alpha, \delta^*_\beta]_{\mathcal{T}E^*} = L^\gamma_{\alpha\beta}\delta^*_\gamma + \mathcal{R}_{\alpha\beta\gamma}\mathcal{P}^\gamma,\ [\delta^*_\alpha, \mathcal{P}^\beta]_{\mathcal{T}E^*} = -\frac{\partial \mathcal{N}_{\alpha\gamma}}{\partial \mu_\beta}\mathcal{P}^\gamma,\ [\mathcal{P}^\alpha, \mathcal{P}^\beta]_{\mathcal{T}E^*} = 0,$$

$$\mathcal{R}_{\alpha\beta\gamma} = \delta^*_\alpha(\mathcal{N}_{\beta\gamma}) - \delta^*_\beta(\mathcal{N}_{\alpha\gamma}) - L^\varepsilon_{\alpha\beta}\mathcal{N}_{\varepsilon\gamma}. \tag{3.2}$$

The curvature of a connection $\mathcal{N}$ on $\mathcal{T}E^*$ is given by $\Omega = -\mathrm{N}_{\mathrm{h}}$ where h is horizontal projector and N_{h} is the Nijenhuis tensor of h. In local coordinates we get $\Omega = -\frac{1}{2}\mathcal{R}_{\alpha\beta\gamma}\mathcal{Q}^\alpha \wedge \mathcal{Q}^\beta \otimes \mathcal{P}^\gamma$, where $\mathcal{R}_{\alpha\beta\gamma}$ is given by (3.2) and is called the *curvature tensor* of $\mathcal{N}$.

Definition 1. An almost tangent structure $\mathcal{J}$ on $\mathcal{T}E^*$ is a bundle morphism $\mathcal{J} : \mathcal{T}E^* \to \mathcal{T}E^*$ of $\tau_1 : \mathcal{T}E^* \to E^*$ of rank m, such that $\mathcal{J}^2 = 0$. An almost tangent structure $\mathcal{J}$ on $\mathcal{T}E^*$ is called *adapted* if $im\mathcal{J} = \ker \mathcal{J} = V\mathcal{T}E^*$.

Locally, an adapted almost tangent structure has the following form $\mathcal{J} = t_{\alpha\beta}\mathcal{Q}^\alpha \otimes \mathcal{P}^\beta$, where the matrix $(t_{\alpha\beta}(q, \mu))$ is nondegenerate. It follows that $\mathcal{J}$ is an integrable structure (i.e. the Nijenhuis tensor associated to $\mathcal{J}$ vanishes) if and only if [7]

$$\frac{\partial t^{\alpha\gamma}}{\partial \mu_\beta} = \frac{\partial t^{\beta\gamma}}{\partial \mu_\alpha}, \tag{3.3}$$

where $t^{\alpha\gamma}t_{\gamma\beta} = \delta^\alpha_\beta$. An adapted almost tangent structure $\mathcal{J}$ on $\mathcal{T}E^*$ is called symmetric if $\omega_E(\mathcal{J}\rho_1, \rho_2) = \omega_E(\mathcal{J}\rho_2, \rho_1)$, for $\forall \rho_1, \rho_2 \in \Gamma(\mathcal{T}E^*)$. Locally, this requires the symmetry of $t_{\alpha\beta}$. If g is a pseudo-Riemannian metric on the vertical subbundle $V\mathcal{T}E^*$ (i.e. a $(0,2)$-type symmetric tensor

$g = g^{\alpha\beta}(q,\mu)\mathcal{P}_\alpha \otimes \mathcal{P}_\beta$ of rank m on $\mathcal{T}E^*$) then there exists a unique symmetric adapted almost tangent structure on $\mathcal{T}E^*$ such that

$$g(\mathcal{J}\rho, \mathcal{J}\upsilon) = -\omega_E(\mathcal{J}\rho, \upsilon), \quad \forall \rho, \upsilon \in \Gamma(\mathcal{T}E^*), \tag{3.4}$$

and we say that $\mathcal{J}$ is induced by g. Locally, (3.4) implies $t^{\alpha\beta} = g^{\alpha\beta}$.

Definition 2. Let $\mathcal{J}$ be an adapted almost tangent structure on $\mathcal{T}E^*$. A section ρ of $\mathcal{T}E^*$ is called $\mathcal{J}-$regular if

$$\mathcal{J}[\rho, \mathcal{J}\nu]_{\mathcal{T}E^*} = -\mathcal{J}\nu, \quad \forall \nu \in \Gamma(\mathcal{T}E^*). \tag{3.5}$$

Locally, the section $\rho = \xi^\alpha \mathcal{Q}_\alpha + \rho_\beta \mathcal{P}^\beta$ is $\mathcal{J}-$regular if and only if $t^{\alpha\beta} = \frac{\partial \xi^\beta}{\partial \mu_\alpha}$ where $t^{\alpha\beta}t_{\alpha\gamma} = \delta^\beta_\gamma$. If the equation (3.5) is satisfied for any $\nu \in \Gamma(\mathcal{T}E^*)$ with $rank[t^{\alpha\beta}] = m$, then $\mathcal{J}$ is an integrable structure. Indeed, we have $\frac{\partial t^{\alpha\beta}}{\partial \mu_\gamma} = \frac{\partial^2 \xi^\beta}{\partial \mu_\gamma \partial \mu_\alpha} = \frac{\partial^2 \xi^\beta}{\partial \mu_\alpha \partial \mu_\gamma} = \frac{\partial t^{\gamma\beta}}{\partial \mu_\alpha}$, and we obtain (3.3).

Definition 3. A map $\nabla : \mathfrak{T}(\mathcal{T}E^*\backslash\{0\}) \to \mathfrak{T}(\mathcal{T}E^*\backslash\{0\})$ is said to be a tensor derivation on $\mathcal{T}E^*\backslash\{0\}$ if the following conditions are satisfied:

i) ∇ is $\mathbb{R}$-linear;

ii) ∇ is type preserving, i.e. $\nabla\mathfrak{T}^r_s(\mathcal{T}E^*\backslash\{0\}) \subset \mathfrak{T}^r_s(\mathcal{T}E^*\backslash\{0\})$, for each $(r,s) \in \mathbb{N}\times\mathbb{N}$;

iii) ∇ obeys the Leibnitz rule $\nabla(P\otimes S) = \nabla P \otimes S + P \otimes \nabla S$ for any tensors P, S on $\mathcal{T}E^*\backslash\{0\}$;

iv) ∇ commutes with any contractions.

We consider the $\mathbb{R}$-linear map $\nabla_\rho : \Gamma(\mathcal{T}E^*\backslash\{0\}) \to \Gamma(\mathcal{T}E^*\backslash\{0\})$ by

$$\nabla_\rho X = \mathrm{h}[\rho, \mathrm{h}X]_{\mathcal{T}E^*} + \mathrm{v}[\rho, \mathrm{v}X]_{\mathcal{T}E^*}, \quad \forall X \in \Gamma(\mathcal{T}E^*\backslash\{0\})$$

where ρ is a $\mathcal{J}-$regular section and it follows that

$$\nabla_\rho(fX) = \rho(f)X + f\nabla_\rho X, \quad \forall f \in C^\infty(\mathcal{T}E^*\backslash\{0\}),\ X \in \Gamma(\mathcal{T}E^*\backslash\{0\}).$$

Any tensor derivation on $\mathcal{T}E^*\backslash\{0\}$ is completely determined by its actions on smooth functions and sections on $\mathcal{T}E^*\backslash\{0\}$ (see [18] generalized Willmore's theorem, p. 1217). Therefore, there exists a unique tensor derivation ∇_ρ on $\mathcal{T}E^*\backslash\{0\}$ such that $\nabla_\rho \mid_{C^\infty(\mathcal{T}E^*\backslash\{0\})} = \rho$, $\nabla_\rho \mid_{\Gamma(\mathcal{T}E^*\backslash\{0\})} = \nabla_\rho$. We will call the tensor derivation ∇_ρ, the *dynamical covariant derivative* induced by the $\mathcal{J}$-regular section ρ and a nonlinear connection $\mathcal{N}$.

Proposition 1. *The following equations hold*

$$[\rho, \mathcal{P}^\beta]_{\mathcal{T}E^*} = -t^{\alpha\beta}\delta^*_\alpha + \left(t^{\alpha\beta}\mathcal{N}_{\alpha\gamma} - \frac{\partial \rho_\gamma}{\partial \mu_\beta}\right)\mathcal{P}^\gamma, \tag{3.6}$$

$$[\rho, \delta^*_\beta]_{\mathcal{T}E^*} = -\left(\delta^*_\beta(\xi^\alpha) + \xi^\varepsilon L^\alpha_{\varepsilon\beta}\right)\delta^*_\alpha + \mathcal{R}_{\beta\gamma}\mathcal{P}^\gamma, \tag{3.7}$$

$$\mathcal{R}_{\beta\gamma} = \delta^*_\beta(\xi^\alpha)\mathcal{N}_{\alpha\gamma} + \rho(\mathcal{N}_{\beta\gamma}) - \delta^*_\beta(\rho_\gamma) - \xi^\varepsilon L^\alpha_{\varepsilon\beta}\mathcal{N}_{\alpha\gamma}. \tag{3.8}$$

The action of ∇_ρ on the Berwald basis has the form

$$\nabla_\rho \mathcal{P}^\beta = \mathrm{v}[\rho, \mathcal{P}^\beta]_{\mathcal{T}E^*} = \left(t^{\alpha\beta}\mathcal{N}_{\alpha\gamma} - \frac{\partial \rho_\gamma}{\partial \mu_\beta}\right)\mathcal{P}^\gamma,$$

$$\nabla_\rho \delta^*_\beta = \mathrm{h}[\rho, \delta^*_\beta]_{\mathcal{T}E^*} = -\left(\delta^*_\beta(\xi^\alpha) + \xi^\varepsilon L^\alpha_{\varepsilon\beta}\right)\delta^*_\alpha.$$

For a pseudo-Riemannian metric g on $\mathcal{T}E^*$ the action of ∇_ρ is given by $\nabla_\rho g(X,Y) = \rho(g(X,Y)) - g(\nabla_\rho X, Y) - g(X, \nabla_\rho Y)$, which in local coordinates leads to

$$g^{\alpha\beta}_{/} := \nabla_\rho g\left(\mathcal{P}^\alpha, \mathcal{P}^\beta\right) = \rho(g^{\alpha\beta}) - g^{\varepsilon\beta}\left(t^{\alpha\gamma}\mathcal{N}_{\gamma\varepsilon} - \frac{\partial \rho_\varepsilon}{\partial \mu_\alpha}\right) - g^{\varepsilon\alpha}\left(t^{\beta\gamma}\mathcal{N}_{\gamma\varepsilon} - \frac{\partial \rho_\varepsilon}{\partial \mu_\beta}\right).$$

Definition 4. A nonlinear connection is called metric or compatible with the metric tensor g if $\nabla_\rho g = 0$, for all $\mathcal{J}$-regular sections ρ, that is

$$\rho(g(X,Y)) = g(\nabla_\rho X, Y) + g(X, \nabla_\rho Y), \quad \forall X, Y \in \Gamma(V\mathcal{T}E^*).$$

Theorem 1. *The connection $\widetilde{\mathcal{N}}$ with the coefficients*

$$\widetilde{\mathcal{N}_{\alpha\beta}} = \mathcal{N}_{\alpha\beta} + \frac{1}{2}t_{\alpha\varepsilon}g_{\beta\gamma}g^{\varepsilon\gamma}_{/}, \tag{3.9}$$

is a metric nonlinear connection.

Proof. Let us consider the dynamical covariant derivative induced by ρ and $\widetilde{\mathcal{N}}$ given by

$$\nabla_\rho g\left(\mathcal{P}^\alpha, \mathcal{P}^\beta\right) = \rho(g^{\alpha\beta}) - g^{\varepsilon\beta}\left(t^{\alpha\gamma}\widetilde{\mathcal{N}}_{\gamma\varepsilon} - \frac{\partial \rho_\varepsilon}{\partial \mu_\alpha}\right) - g^{\varepsilon\alpha}\left(t^{\beta\gamma}\widetilde{\mathcal{N}}_{\gamma\varepsilon} - \frac{\partial \rho_\varepsilon}{\partial \mu_\beta}\right),$$

and using (3.9) it follows

$$\nabla_\rho g(\mathcal{P}^\alpha, \mathcal{P}^\beta) = g^{\alpha\beta}_{/} - \frac{1}{2}g^{\varepsilon\beta}t^{\alpha\gamma}t_{\gamma r}g_{\varepsilon p}g^{rp}_{/} - \frac{1}{2}g^{\alpha\varepsilon}t^{\beta\gamma}t_{\gamma r}g_{\varepsilon p}g^{rp}_{/} = 0,$$

that is $\widetilde{\mathcal{N}}$ is a metric nonlinear connection. □

3.1. Nonlinear connection induced by a $\mathcal{J}$-regular section

If $\mathcal{J}$ is an adapted tangent structure and ρ is a $\mathcal{J}$- regular section then [7], $\mathcal{N} = -\mathcal{L}_\rho \mathcal{J}$, is a nonlinear connection on $\mathcal{T}E^*$ with local coefficients given by

$$\mathcal{N}_{\alpha\beta} = \frac{1}{2}\left(t_{\alpha\gamma}\frac{\partial \rho_\beta}{\partial \mu_\gamma} - \sigma^i_\alpha t_{\gamma\beta}\frac{\partial \xi^\gamma}{\partial q^i} - \rho(t_{\alpha\beta}) + \xi^\gamma t_{\varepsilon\beta} L^\varepsilon_{\gamma\alpha}\right). \tag{3.10}$$

Definition 5. The Jacobi endomorphism ψ is given by $\psi = \mathrm{v}[\rho, \mathrm{h}X]_{\mathcal{T}E^*}$.

Locally, from (3.7) we obtain that $\psi = \mathcal{R}_{\alpha\beta}\mathcal{Q}^\alpha \otimes \mathcal{P}^\beta$, where $\mathcal{R}_{\alpha\beta}$ is given by (3.8); $\mathcal{R}_{\alpha\beta}$ are the local coefficients of the Jacobi endomorphism.

Proposition 2. *The following result holds* $\psi = i_\rho \Omega + \mathrm{v}[\mathrm{v}\rho, \mathrm{h}X]_{\mathcal{T}E^*}$.

Proof. Indeed, $\psi(X) = \mathrm{v}[\rho, \mathrm{h}X]_{\mathcal{T}E^*} = \mathrm{v}[\mathrm{h}\rho, \mathrm{h}X]_{\mathcal{T}E^*} + \mathrm{v}[\mathrm{v}\rho, \mathrm{h}X]_{\mathcal{T}E^*}$ and $\Omega(\rho, X) = \mathrm{v}[\mathrm{h}\rho, \mathrm{h}X]_{\mathcal{T}E^*}$, that is $\psi(X) = \Omega(\rho, X) + \mathrm{v}[\mathrm{v}\rho, \mathrm{h}X]_{\mathcal{T}E^*}$. □

Remark 1. If ρ is a horizontal section $\rho = \mathrm{h}\rho$, then we obtain $\psi = i_\rho \Omega$ and locally it follows $\rho_\gamma = \xi^\alpha N_{\alpha\gamma}$, which yields $\mathcal{R}_{\alpha\beta} = \mathcal{R}_{\varepsilon\alpha\beta}\xi^\varepsilon$.

In what follows, we consider a regular Hamiltonian $\mathcal{H} : E^* \to \mathbb{R}$, that is the matrix

$$g^{\alpha\beta}(q, \mu) = \frac{\partial^2 \mathcal{H}}{\partial \mu_\alpha \partial \mu_\beta}, \tag{3.11}$$

is nondegenerate. Any regular Hamiltonian $\mathcal{H}$ on E^* induces a pseudo-Riemannian metric on $V\mathcal{T}E^*$ with the metric tensor (3.11). Moreover, it induces a unique symmetric adapted tangent structure (denoted $\mathcal{J}_\mathcal{H}$) such that (3.3) is verified and a $\mathcal{J}-$regular section given by $\rho = \frac{\partial \mathcal{H}}{\partial \mu_\alpha}\mathcal{Q}_\alpha + \rho_\alpha \mathcal{P}^\alpha$. There exists a unique section $\rho_\mathcal{H} \in \Gamma(\mathcal{T}E^*)$ such that $i_{\rho_\mathcal{H}}\omega_E = d\mathcal{H}$ and with respect to the local basis $\{\mathcal{Q}_\alpha, \mathcal{P}^\alpha\}$, the expression of $\rho_\mathcal{H}$ is [8]

$$\rho_\mathcal{H} = \frac{\partial \mathcal{H}}{\partial \mu_\alpha}\mathcal{Q}_\alpha - \left(\sigma^i_\alpha \frac{\partial \mathcal{H}}{\partial q^i} + \mu_\gamma L^\gamma_{\alpha\beta}\frac{\partial \mathcal{H}}{\partial \mu_\beta}\right)\mathcal{P}^\alpha. \tag{3.12}$$

Corollary 1. *The canonical nonlinear connection* $\mathcal{N}=-\mathcal{L}_{\rho_\mathcal{H}}\mathcal{J}_\mathcal{H}$ *has the coefficients given by*

$$\begin{aligned}\mathcal{N}_{\alpha\beta} = \frac{1}{2}\Bigg(& \sigma^i_\gamma\{g_{\alpha\beta}, \mathcal{H}\} - \frac{\partial^2 \mathcal{H}}{\partial q^i \partial \mu_\varepsilon}(\sigma^i_\beta g_{\alpha\varepsilon} + \sigma^i_\alpha g_{\beta\varepsilon}) \\ & + \mu_\gamma L^\gamma_{\varepsilon\kappa}\frac{\partial \mathcal{H}}{\partial \mu_\varepsilon}\frac{\partial g_{\alpha\beta}}{\partial \mu_\kappa} + \mu_\gamma L^\gamma_{\alpha\beta} + \frac{\partial \mathcal{H}}{\partial \mu_\delta}(g_{\alpha\varepsilon}L^\varepsilon_{\delta\beta} + g_{\beta\varepsilon}L^\varepsilon_{\delta\alpha})\Bigg),\end{aligned} \tag{3.13}$$

where the Poisson bracket is

$$\{g_{\alpha\beta}, \mathcal{H}\} = \frac{\partial g_{\alpha\beta}}{\partial \mu_\gamma}\frac{\partial \mathcal{H}}{\partial q^i} - \frac{\partial g_{\alpha\beta}}{\partial q^i}\frac{\partial \mathcal{H}}{\partial \mu_\gamma}.$$

Theorem 2. *The canonical nonlinear connection induced by a regular Hamiltonian $\mathcal{H}$ is a metric nonlinear connection.*

Proof. Introducing the coefficients (3.13) into the expression of the dynamical covariant derivative and using (3.12) we obtain

$$\begin{aligned}
\nabla_\rho g\left(\mathcal{P}^\alpha, \mathcal{P}^\beta\right) &= \sigma^i_\varepsilon \frac{\partial g^{\alpha\beta}}{\partial q^i}\frac{\partial \mathcal{H}}{\partial \mu_\varepsilon} - \sigma^i_\varepsilon \frac{\partial \mathcal{H}}{\partial q^i}\frac{\partial g^{\alpha\beta}}{\partial \mu_\varepsilon} - \mu_\gamma L^\gamma_{\varepsilon\delta}\frac{\partial \mathcal{H}}{\partial \mu_\delta}\frac{\partial g^{\alpha\beta}}{\partial \mu_\varepsilon} \\
&- \frac{1}{2}\left(g^{\varepsilon\beta}g^{\alpha\gamma} + g^{\varepsilon\alpha}g^{\beta\gamma}\right)\left[\sigma^i_l\left(\frac{\partial g_{\gamma\varepsilon}}{\partial \mu_l}\frac{\partial \mathcal{H}}{\partial q^i} - \frac{\partial g_{\gamma\varepsilon}}{\partial q^i}\frac{\partial \mathcal{H}}{\partial \mu_l}\right) - \sigma^i_\varepsilon \frac{\partial^2 \mathcal{H}}{\partial q^i \partial \mu_l}g_{\gamma l}\right. \\
&\left. -\sigma^i_\gamma \frac{\partial^2 \mathcal{H}}{\partial q^i \partial \mu_l}g_{\varepsilon l} + \mu_l L^l_{rs}\frac{\partial \mathcal{H}}{\partial \mu_r}\frac{\partial g_{\gamma\varepsilon}}{\partial \mu_s} + \mu_l L^l_{\gamma\varepsilon} + \frac{\partial \mathcal{H}}{\partial \mu_\delta}\left(g_{\gamma l}L^l_{\delta\varepsilon} + g_{\varepsilon l}L^l_{\delta\gamma}\right)\right] \\
&- g^{\varepsilon\beta}\sigma^i_\varepsilon \frac{\partial^2 \mathcal{H}}{\partial \mu_\alpha \partial q^i} - g^{\varepsilon\beta}L^\alpha_{\varepsilon\delta}\frac{\partial \mathcal{H}}{\partial \mu_\delta} - g^{\varepsilon\beta}g^{\alpha\delta}\mu_\gamma L^\gamma_{\varepsilon\delta} \\
&- g^{\varepsilon\alpha}\sigma^i_\varepsilon \frac{\partial^2 \mathcal{H}}{\partial \mu_\beta \partial q^i} - g^{\varepsilon\alpha}L^\beta_{\varepsilon\delta}\frac{\partial \mathcal{H}}{\partial \mu_\delta} - g^{\varepsilon\alpha}g^{\beta\delta}\mu_\gamma L^\gamma_{\varepsilon\delta}.
\end{aligned}$$

From the equalities

$$g^{\varepsilon\beta}g^{\alpha\gamma}\frac{\partial g_{\gamma\varepsilon}}{\partial \mu_l} = -g^{\varepsilon\beta}g_{\gamma\varepsilon}\frac{\partial g^{\alpha\gamma}}{\partial \mu_l} = -\frac{\partial g^{\alpha\beta}}{\partial \mu_l}, \tag{3.14}$$

$$g^{\varepsilon\beta}g^{\alpha\gamma}\frac{\partial g_{\gamma\varepsilon}}{\partial q^i} = -g^{\varepsilon\beta}g_{\gamma\varepsilon}\frac{\partial g^{\alpha\gamma}}{\partial q^i} = -\frac{\partial g^{\alpha\beta}}{\partial q^i}, \quad L^\gamma_{\varepsilon\delta} = -L^\gamma_{\delta\varepsilon} \tag{3.15}$$

by direct computation, it follows that $\nabla_\rho g\left(\mathcal{P}^\alpha, \mathcal{P}^\beta\right) = 0$, which ends the proof. □

Theorem 3. *The canonical nonlinear connection induced by a regular Hamiltonian is the unique metric and symmetric nonlinear connection.*

Proof. Let us consider a metric and symmetric nonlinear connection $\mathcal{N}$ with the coefficients $\mathcal{N}_{\gamma\varepsilon}$. Then we have

$$\rho_{\mathcal{H}}(g^{\alpha\beta}) = g^{\varepsilon\beta}\left(g^{\alpha\gamma}\mathcal{N}_{\gamma\varepsilon} - \frac{\partial \rho_\varepsilon}{\partial \mu_\alpha}\right) + g^{\varepsilon\alpha}\left(g^{\beta\gamma}\mathcal{N}_{\gamma\varepsilon} - \frac{\partial \rho_\varepsilon}{\partial \mu_\beta}\right)$$

and using (3.12) we obtain

$$\frac{\partial\mathcal{H}}{\partial\mu_l}\sigma_l^i\frac{\partial g^{\alpha\beta}}{\partial q^i}-\left(\sigma_l^i\frac{\partial\mathcal{H}}{\partial q^i}+\mu_\gamma L_{l\varepsilon}^\gamma\frac{\partial\mathcal{H}}{\partial\mu_\varepsilon}\right)\frac{\partial g^{\alpha\beta}}{\partial\mu_l}$$
$$=g^{\varepsilon\beta}g^{\alpha\gamma}\mathcal{N}_{\gamma\varepsilon}+g^{\varepsilon\alpha}g^{\beta\gamma}\mathcal{N}_{\gamma\varepsilon}+g^{\gamma\beta}\sigma_\gamma^i\frac{\partial^2\mathcal{H}}{\partial\mu_\alpha\partial q^i}+g^{\gamma\beta}L_{\gamma\tau}^\alpha\frac{\partial\mathcal{H}}{\partial\mu_\tau}$$
$$+g^{\gamma\beta}\mu_l L_{\gamma\tau}^l\frac{\partial^2\mathcal{H}}{\partial\mu_\alpha\partial\mu_\tau}+g^{\varepsilon\alpha}\sigma_\varepsilon^i\frac{\partial^2\mathcal{H}}{\partial\mu_\beta\partial q^i}+g^{\varepsilon\alpha}L_{\varepsilon\tau}^\beta\frac{\partial\mathcal{H}}{\partial\mu_\tau}+g^{\varepsilon\alpha}\mu_l L_{\varepsilon\tau}^l\frac{\partial^2\mathcal{H}}{\partial\mu_\beta\partial\mu_\tau}.$$

But the connection is symmetric (3.1) and using (3.14), (3.15) we get

$$\sigma_l^i\frac{\partial\mathcal{H}}{\partial q^i}\frac{\partial g_{\varepsilon\gamma}}{\partial\mu_l}-\sigma_l^i\frac{\partial\mathcal{H}}{\partial\mu_l}\frac{\partial g_{\varepsilon\gamma}}{\partial q^i}+\mu_\tau L_{l\alpha}^\tau\frac{\partial\mathcal{H}}{\partial\mu_\alpha}\frac{\partial g_{\varepsilon\gamma}}{\partial\mu_l}=2N_{\varepsilon\gamma}+\mu_\tau L_{\gamma\varepsilon}^\tau$$
$$-g_{\alpha\varepsilon}\sigma_\gamma^i\frac{\partial^2\mathcal{H}}{\partial\mu_\alpha\partial q^i}-g_{\alpha\gamma}\sigma_\varepsilon^i\frac{\partial^2\mathcal{H}}{\partial\mu_\alpha\partial q^i}-g_{\alpha\varepsilon}L_{\gamma l}^\alpha\frac{\partial\mathcal{H}}{\partial\mu_l}-g_{\alpha\gamma}L_{\varepsilon l}^\alpha\frac{\partial\mathcal{H}}{\partial\mu_l},$$

and we get the coefficients (3.13), which ends the proof. □

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