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ON MAPPINGS BETWEEN MANIFOLDS OF A LINEAR SPACE BY PARALLELISM OF TANGENT MANIFOLDS

BY

ION CREANGĂ

Alexandru Myller [1] had studied correspondences by parallel tangent planes between surfaces of the Euclidean space, E_3, by considering some researches initiated by K. M. Peterson [2].

In this paper, we try to extend these researches to the study of the correspondences between manifolds of a linear space by using a concept of parallelism of tangent manifolds.

1. Let L and A be a linear space and a subspace of it. The set of vectors

$$\mathcal{A}=x_0+A=\{x_0+a \mid a\in A\},\quad x_0\in L \text{ (fixed)} \tag{1}$$

forms a linear variety of L. A variety $\mathcal{A}$ (where $x_1\in L$ and B is a subspace of L) is called *parallel* to $\mathcal{B}=x_1+B(\ldots)$, if $A\subseteq B$. The parallelism of linear varieties of L is a reflexive, transitive relation, without being symmetric. The set of all maximal subspaces of L (with respect to set-inclusion) defines the *directions of parallelism* of L. If $\mathcal{A}$ is parallel to $\mathcal{B}$ and $x_0\in B$, then $\mathcal{A}\subseteq\mathcal{B}$ and conversely. If $\mathcal{A}\cap\mathcal{B}=\varnothing$ and $A\cap B=\{0\}$, then we call $\mathcal{A}$ and $\mathcal{B}$ *skew* to each other. The dimension of A dim $\mathcal{A}$, will be called dimension of $\mathcal{A}$).

2. Let U and L be two linear spaces over the same field. The injective mappings

$$f: U \to L \tag{2}$$

define the *manifolds* of L by U. If f is linear, then the linear varieties of U are transformed into linear varieties of L.

If U and L are finite dimensional linear spaces over R — real field — namely dim $U=r$, dim $L=n$, $r<n$, then, by taking the bases $\{g_i \mid i=1,\ldots,r\}$ and $\{e_j \mid j=1,2,\ldots,n\}$, respectively, in U and L, we have

$$x=f(u)=\xi^j(u^1,u^2,\ldots,u^r)\,e_j, \tag{3}$$

where $u=u^i g_i\in U$ and $x=\xi^j e_j\in L$ (here and everywhere in what follows, we denote by $u^i g_i$ and $\xi^j e_j$ the sums $\sum_{i=1}^{r} u^i g_i$ and $\sum_{j=1}^{n}\xi^j e_j$). U is called the *space of parameters* of the manifold defined by (2) and (3). If the functions $\xi^j(u^1, \ldots, u^r)$ are differentiable in R, we define

$$dx = d\xi^j e_j = \left(\frac{\partial \xi^j}{\partial u^k} du^k\right) e_j \; (j = 1, \ldots, n;\ k = 1, \ldots, r), \tag{4}$$

and

$$\frac{\partial x}{\partial u^k} = \frac{\partial \xi^j}{\partial u^k} e_j ; \tag{5}$$

the subspace generated by the vectors $\frac{\partial x}{\partial u^k}$ is the space of tangent vectors of the manifold (3) in the point u, — we denote it by $T(u)$, — and the linear variety

$$x = x(u) + T(u) \tag{6}$$

is the *tangent manifold* of (3) in u. If the vectors $\frac{\partial x}{\partial u^k}$ are linearly independent in L, then the dimension of the variety (6) is r. The necessary and sufficient condition for (6) to have dimension r is that the matrix

$$\left(\frac{\partial \xi^j}{\partial u^k}\right) \tag{7}$$

has the rank r.

3. Now consider the space L of dimension n and the base $\{e_i \mid i = 1, \ldots, n\}$ over R. Consider the operator :

$$F : L \to L \tag{8}$$

defined on L. If $x = \xi^i e_i \in L$, we have $F(x) = \eta^j(\xi^1, \ldots, \xi^n) e_j$ $(j = 1, \ldots, n)$. This operator maps a manifold $U \xrightarrow{f} L$ of L into another, $U \xrightarrow{F \circ f} L$, the image of (3) by F, namely, for any $x = f(u)$, we have

$$y = F(x) = F(f(u)) = \eta^j(\xi^1(u^1, \ldots, u^r), \ldots, \xi^n(u^1, \ldots, u^r)) e_j = \tag{9}$$
$$= \tau^j(u^1, \ldots, u^r) e_j \qquad (j = 1, \ldots, n).$$

Denote by $T_1(u)$ the tangent subspace of L in y given by (9) generated by $\left\{\frac{\partial y}{\partial u^h} \mid h = 1, \ldots, r\right\}$. The tangent manifold in u of the manifold (9) is expressed by

$$y = y(u) + T_1(u). \tag{10}$$

If (6) is parallel to (10), then $T(u) \subseteq T_1(u)$, i.e. the following equalities hold :

$$\frac{\partial x}{\partial u^h} = \alpha_h^s \frac{\partial y}{\partial u^s} \quad (s, h = 1, \ldots, r). \tag{11}$$

By writing (11) for the coordinates of the vectors, we obtain

$$\frac{\partial \xi^i}{\partial u^h} = \alpha_h^s \frac{\partial \eta^i}{\partial \xi^j} \cdot \frac{\partial \xi^j}{\partial u^s} . \tag{12}$$

Using the matrix notation :

$$X = \begin{pmatrix} \frac{\partial \xi^1}{\partial u^1} & \frac{\partial \xi^2}{\partial u^1} & \cdots & \frac{\partial \xi^n}{\partial u^1} \\ \cdot & \cdot & \cdots & \cdot \\ \frac{\partial \xi^1}{\partial u^r} & \frac{\partial \xi^2}{\partial u^r} & \cdots & \frac{\partial \xi^n}{\partial u^r} \end{pmatrix}, \quad Y = \begin{pmatrix} \alpha_1^1 & \alpha_1^2 & \ldots & \alpha_1^r \\ \cdot & \cdot & \cdots & \cdot \\ \alpha_r^1 & \alpha_r^2 & \ldots & \alpha_r^r \end{pmatrix}, \tag{13}$$

$$W = \begin{pmatrix} \frac{\partial \eta^1}{\partial \xi^1} & \frac{\partial \eta^2}{\partial \xi^1} & \cdots & \frac{\partial \eta^n}{\partial \xi^1} \\ \cdot & \cdot & \cdots & \cdot \\ \frac{\partial \eta^1}{\partial \xi^n} & \frac{\partial \eta^2}{\partial \xi^n} & \cdots & \frac{\partial \eta^n}{\partial \xi^n} \end{pmatrix},$$

the equations (12) turn into the matrix equation

$$X = YXW, \tag{14}$$

where W is the Jacobian of the functions η^j with respect to ξ^i, $j, i = 1, \ldots, n$. If $\det W \neq 0$, i. e. the functions η^j are locally independent, we obtain the equation $XW^{-1} = YX$, which was studied in [3], [4].

Given the operator F, the manifolds of L by U which are carried by F in manifolds having tangent manifolds parallel in corresponding points are obtained by (12) or by (13). Considering in (14) the matrices X and Y unknown, to determine them in terms of W, one must solve the systems of partial derivatives : $\frac{\partial \xi^i}{\partial u^h} = \varphi_h^i(\xi^1, \ldots, \xi^n)$ and $\alpha_h^s = \psi_h^s(\xi^1, \ldots, \xi^n)$. Using some results of [3] and denoting by X_0, Y_0 a solution of (14), for the first system we obtain the following statements :

a) If V is a matrix and X_0, Y_0 is a solution of (14) permutable with W, then $X = X_0 V$, $Y = Y_0$ is also a solution of (14).

b) If M is a matrix permutable with Y_0, then MX_0, Y_0 is also a solution of (14).

c) If S is a matrix similar to W^{-1}, hence $S = P^{-1}W^{-1}P$, then $X_0 P$, Y_0 is a solution of the equation $XS = YX$.

d) $Y_0^s X_0$, Y_0 and $Y_0^s X_0 W^{-p}$, Y_0 are solutions of (14) (s and p are natural numbers). X_0, Y_0^m is also a solution of the equation $XW^{-m} = YX$.

e) If Y is given, then the set of all solutions X of the equation (14) is a linear space. For example, take Y as a diagonal matrix $Y = \operatorname{diag}\{\alpha_1^1, \ldots, \alpha_r^r\}$, with $\alpha_h^h \neq 0$, $h = 1, \ldots, r$. The system (12) becomes

$$\frac{\partial \eta^i}{\partial \xi^1}\frac{\partial \xi^1}{\partial u^h} + \frac{\partial \eta^i}{\partial \xi^2}\frac{\partial \xi^2}{\partial u^h} + \ldots + \left(\frac{\partial \eta^i}{\partial \xi^i} - \frac{1}{\alpha_h^h}\right)\frac{\partial \xi^i}{\partial u^h} + \ldots + \frac{\partial \eta^i}{\partial \xi^n}\frac{\partial \xi^n}{\partial u^h} = 0, \tag{15}$$

for $i=1,\dots,n$, a homogenous linear system in $\dfrac{\partial \xi^i}{\partial u^h}$, h — fixed. This system has nontrivial solutions if

$$\det\left(W - \frac{1}{\alpha_h^h} E\right) = 0, \tag{16}$$

hence $1/\alpha_h^h$ are eigenvalues of the equation (16). The derivatives $\partial\xi^j/\partial u^h$ can be determined by (15).

5. Consider now the case of an operator F of L carrying a manifold of L into another, such that the corresponding points have parallel tangent manifolds. Let $x = x(u)$ be a point of the given manifold, $x = x(u) + T(u)$ the tangent manifold in x. $T(u)$ is a subspace generated by $\left\{\dfrac{\partial x}{\partial u^h} \mid h = 1, \dots, r\right\}$. If $y = y(u) = F(x(u))$ and $y = y(u) + T_1(u)$ are the corresponding point and tangent manifold in it, then the condition determining F is

$$T_1(u) \subseteq T(u), \tag{17}$$

hence

$$\frac{\partial y}{\partial u^k} = \beta_k^t \frac{\partial x}{\partial u^t} \qquad (t = 1, \dots, r;\ k = 1, \dots, r) \tag{18}$$

or, for coordinates,

$$\frac{\partial \eta_i^t}{\partial \xi^j} \frac{\partial \xi^j}{\partial u^k} = \beta_k^t \frac{\partial \xi^t}{\partial u^t} \qquad (i = 1, \dots, n). \tag{19}$$

With the notations in 4. and $Z = (\beta_k^t)$, we can write

$$XW = ZX. \tag{20}$$

If X is known, we obtain a case studied also in [3], [4]. If Y admits an inverse matrix, then (20) is just (14). For coordinates we have

$$\frac{\partial \eta_i^t}{\partial \xi^j} = \Phi_j^t (u^1, \dots, u^r;\ \xi^1, \dots, \xi^n), \tag{21}$$

which, by integrating, give

$$\eta_i^t = \eta_i^t (u^1, \dots, u^r;\ \xi^1, \dots, \xi^n). \tag{22}$$

These formulas determine the operator F in terms of the parameters u^t and some constants.

6. Consider the space of parameters U as a direct sum of two subspaces

$$U = U_1 \oplus U_2. \tag{23}$$

Take the injective mappings $f_i : U_i \to L$, $i = 1, 2$ and define $f : U \to L$ by $f(u) = f(u_1 + u_2) = f_1(u_1) + f_2(u_2)$. Then f is injective, if $f_1(U_1) \cap f_2(U_2) =$

$\{0\}$. In this case, the manifold $x = f(u) = f_1(u_1) + f_2(u_2)$ is called a *direct sum* of the two given manifolds. In the finite case, there exist manifolds of the form

$$x(u) = (\xi_1^i (u^1, \dots, u^h) + \xi_2^i (u^{h+1}, \dots, u^r))\, e_i, \tag{24}$$

where $\xi_1^i e_i$ and $\xi_2^i e_i$ define two manifolds of L with respect to the same U_1 and U_2 as above, respectively of the dimension h and $r - h$, subspaces of U. A special case is given by the manifolds for which $f_2 : U_2 \to L$ are linear mappings; this case generalises the case of ruled surfaces in the affine 3-dimensional space. If U_2 is of dimension 1 and $n = r$, then the manifold is called a *congruence*.

7. Assume that $L = L_1 \oplus L_2$, $U = U_1 \oplus U_2$ and, for any $x = x_1 + x_2 \in L$, $u = u_1 + u_2 \in U$, $x_i \in L_i$, $u_i \in U_i$, $i = 1, 2$, the decomposition

$$x(u) = x_1(u_1) + x_2(u_2) \tag{25}$$

holds.

Consider the operators F for which

$$F(x) = F_1(x_1) + F_2(x_2), \tag{26}$$

where $F_i : L_i \to L_i$, $i = 1, 2$. For the finite dimensional spaces, let $\{e_i \mid i = 1, \dots, n\}$ be a base of L, where $\{e_j \mid j = 1, \dots, s\}$ and $\{e_{s+k} \mid k = 1, \dots, n-s\}$ are bases in L_1 and L_2, respectively. We have then $y = F(x) = \eta_i^i(\xi^1, \dots, \xi^n) e_i = \eta_i^j(\xi^1, \dots, \xi^s) e_j + \eta_i^{s+k}(\xi^{s+1}, \dots, \xi^n) e_{s+k}$ $(i = 1, \dots, n;\ j = 1, \dots, s;\ k = 1, \dots, n-s)$ and $x(u) = x(u_1) + x(u_2) = (\xi^i(u^1, \dots, u^h) + \xi^i(u^{h+1}, \dots, u^r))\, e_i$. Therefore

$$\begin{aligned} F(x(u)) = {} & (\eta_i^j(\xi^1(u^1, \dots, u^h), \dots, \xi^s(u^1, \dots, u^h)) + \eta_i^j(\xi^1(u^{h+1}, \dots \\ & \dots, u^r), \dots, \xi^n(u^{h+1}, \dots, u^r)))\, e_j + (\eta_i^{s+k}(\xi^{s+1}(u^1, \dots, u^h), \dots, \xi^n(u^1, \dots \\ & \dots, u^h)) + \eta_i^{s+k}(\xi^{s+1}(u^{h+1}, \dots, u^r), \dots, \xi^n(u^{h+1}, \dots, u^r)))\, e_{s+k} \end{aligned} \tag{27}$$

and (27) represents the image of the manifold $x(u)$ by the considered operators F.

8. A manifold

$$x(u) = x_0(u^1, \dots, u^h) + u^{h+1} a_{h+1} + \dots + u^r a_r \qquad (r < n), \tag{28}$$

where a_{h+j} are constant vectors of L is called a *ruled* (or semilinear) *manifold*. The elements of the subspace generated by a_{h+j}, $j = 1, \dots, r-h$, are the generators of the given ruled manifold. Its tangent space in a point is of the form

$$x = x_0(u^1, \dots, u^h) + \lambda^i \frac{\partial x_0}{\partial u^i} + u^{h+j} a_{h+j}, \qquad i = 1, \dots, h, \quad j = 1, \dots, r-h. \tag{29}$$

Let F be an operator of L such that $y(u) = F(x_0(u^1, \dots, u^h)) + u^{h+j} F(a_{h+j})$ $(j = 1, \dots, r-h)$ or

$$y(u) = y_0(u) + u^{h+j} b_{h+j}, \tag{30}$$

where $y_0(u) = F(x_0(u^1, \dots, u^h))$ and $b_{h+j} = F(a_{h+j})$. Then (30) is also a ruled manifold, the image of (28) by F. F realises a correspondence by parallelism of generators, if

(31) $$\{u^{h+t}\, a_{h+t}\} \subseteq \{u^{h+j}\, b_{h+j}\}.$$

This relation implies

(32) $$a_{h+t} = \beta_{h+t}^{h+s} F(a_{h+s}) \quad (t, s = 1, \dots, r-h).$$

Conversely, the image of (28) corresponds to it by parallelism of generators, if

(33) $$F(a_{h+t}) = \gamma_{h+t}^{h+t} a_{h+t}.$$

The tangent manifold in a point u of the manifold (30) is:

(34) $$y = F(x_0(u^1, \dots, u^h)) + \rho^i \frac{\partial F(x_0(u^1, \dots, u^h))}{\partial u^i} + \sigma^{h+j} F(a_{h+j}),$$

$$(i = 1, \dots, h; \quad j = 1, \dots, r-h).$$

The tangent manifold (29) is parallel to (34), if

(35) $$\begin{cases} \dfrac{\partial x_0}{\partial u^i} = \varphi^i \dfrac{\partial F(x_0(u^1, \dots, u^h))}{\partial u^i} + \psi^{h+j} F(a_{h+j}), \\[2ex] a_{h+j} = \sigma^i \dfrac{\partial F(x_0(u^1, \dots, u^h))}{\partial u^i} + \tau^{h+j} F(a_{h+j}). \end{cases}$$

9. In Euclidean linear spaces, many concepts and researches of Al. M y l l e r in [1] for E_3 could be extended; for example we may consider the conic curvature of a curve on a surface, cylindric lines and so on.

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Faculty of Mathematics
University „Al. I. Cuza", Iași
R.S. România

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CONNEXIONS FINSLERIENNES GÉNÉRALES PRESQUE SYMPLECTIQUES

PAR

GHEORGHE ATANASIU et IOAN GHINEA

Récemment, dans [2], on a déterminé la forme générale des connexions finsleriennes compatibles avec une structure presque symplectique et respectivement conforme presque symplectique. Ce travail, utilise la définition de la connexion finslerienne dans le fibré Finsler d'une variété différentiable M, donnée par le prof. M. M a t s u m o t o [3], et une méthode mise en valeur par le prof. R. M i r o n [4] et utilisée avec succès dans [1], [5], [10], [2], [9]. La méthode a été pour la première fois appliquée par le prof. Radu Miron dans la détermination de toutes les connexions finsleriennes métriques [6]. Des résultats inédits concernant les connexions Finsler métriques vont paraître dans [7].

Dans la note ci-présente nous allons prouver qu'il existe une connexion finslerienne, définie sur M, de telle manière que la h-dérivée covariante et la v-dérivée covariante du champ finslerien du type (0,2) disons $g_{ij}(x, y)$, antisymétrique, nondégénéré, qui détermine la structure presque symplectique sur M, soient, respectivement, deux champs finsleriens K_{ijk}, Q_{ijk}, du type (0,3), qui remplissent les conditions d'antisymétrie dans les deux premiers indices.

1. Soit M_{2n} une variété C^∞-différentiable, de dimension $2n$.

Une connexion finslerienne [3], [7] sur M_{2n} est un triplet $(N_j^i, F_{jk}^i, C_{jk}^i)$ qui à un changement de coordonnées et d'élément d'appui

(1.1) $$x^{i'} = x^{i'}(x^1, \dots, x^{2n}), \qquad y^{i'} = \frac{\delta x^{i'}}{\partial x^j} y^j,$$

se transforme selon les formules

$$N_{j'}^{i'}(x', y') = \frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^j}{\partial x^{j'}} N_j^i + \frac{\partial x^{i'}}{\partial x^i} \frac{\partial^2 x^i}{\partial x^{j'} \partial x^{k'}} \frac{\partial x^{k'}}{\partial x^k} y^k,$$

(1.2) $$F_{j'k'}^{i'}(x', y') = \frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^j}{\partial x^{j'}} \frac{\partial x^k}{\partial x^{k'}} F_{jk}^i + \frac{\partial x^{i'}}{\partial x^i} \frac{\partial^2 x^i}{\partial x^{j'} \partial x^{k'}},$$

$$C_{j'k'}^{i'}(x', y') = \frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^j}{\partial x^{j'}} \frac{\partial x^k}{\partial x^{k'}} C_{jk}^i.$$